Blow 'Em Up!

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History of the Resolution Problem

Problem: Resolve the singularities of an algebraic variety over $\mathbb{R}/\mathbb{C}$.

Resolution of curve singularities in the 19th siecle.

Resolution of surface singularities by Walker 1935.

Resolution of 3-fold singularities by Zariski 1944.

Resolution of singularities by Hironaka 1964.

Constructive Proofs by Villamayor, Bierstone/Milman 1989-1997.

Variations/Simplifications by Encinas/Villamayor, Hauser, Włodarczyk.

Implementations by Bodnár/Schicho, Frühbis-Krüger/Pfister.

A Game Modeling Resolution Problems

“Blow ’Em Up” is a game between two players. Every instance of a resolution problem uniquely determines all moves of the first player.

The resolution problem can be solved if the second player can win this game.

In order to prove that every singularity can be resolved, it suffices to show that the second player has a winning strategy.

Spaces

A space is a finite partially ordered set $(W, <)$ with a unique maximum t, together with a ranking $\dim : W \to \mathbb{N}$ such that $x < y \implies \dim(x) < \dim(y)$. The dimension of the space is defined as $\dim(t)$.

A downward closed set is a subset $V \subset W$ such that $(x \in V \wedge y < x) \implies y \in V$.

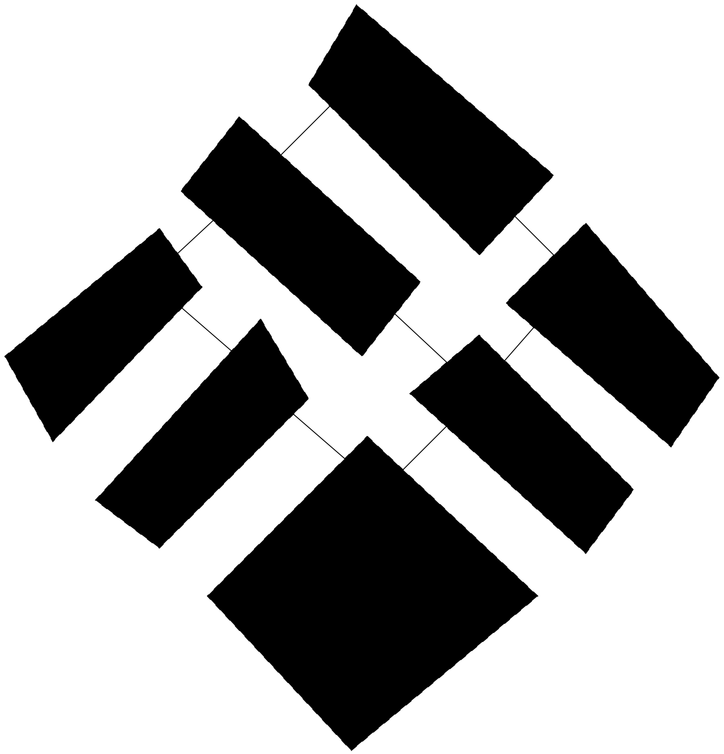
A subset $V \subset W$ is called pure iff $(x, y \in V \wedge x < z < y) \implies z \in V$ (midward closed) and all maximal elements of V have the same dimension, which is called $\dim(V)$.

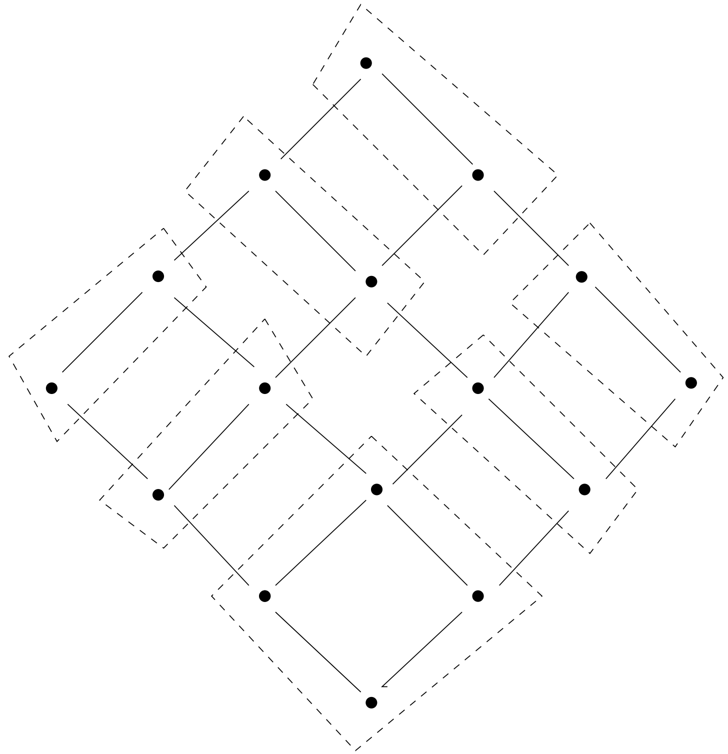
Refinements

A refinement of W is a space W' of the same dimension and a surjective map $\phi : W' \to W$ with the following properties.

1. The fibers of ϕ are non-empty pure subsets of W'.
2. For $x \in W$, we have $\dim(x) = \dim(\phi^{-1}(x))$.
3. If $x' \leq y'$, then $\phi(x') \leq \phi(y')$.
4. If $x < y$, then there exist $x' \in \phi^{-1}(x)$ and $y' \in \phi^{-1}(y)$ such that $x' \leq y'$.

Example: Refinement





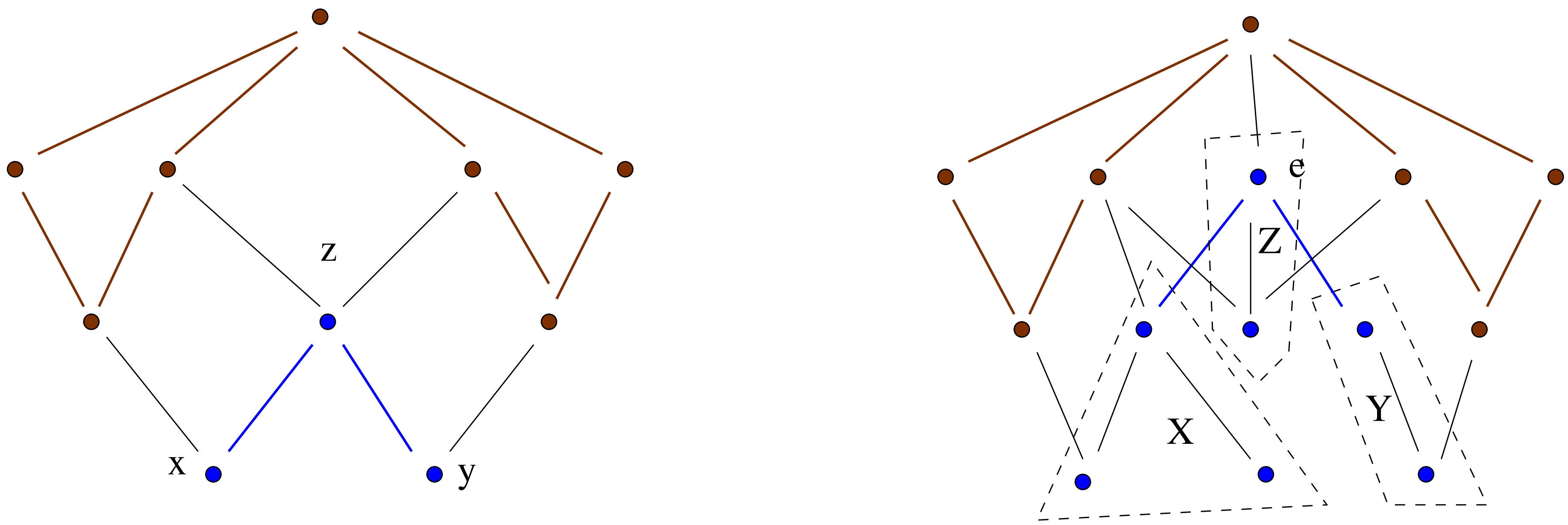
Blowups

Let $z \in W$ be a node. A "blowup of z" is a new space $\widetilde{W}$ and a function $\pi : \widetilde{W} \to W$ that satisfies the refinement rules (1), (3), and (4'). The subset $\pi^{-1}(z)$ has a unique maximal element, called the exceptional node e. The refinement rule (2) is replaced by the following.

- If $x \not\leq z$, then there is a unique preimage x' of x, and $\dim(x') = \dim(x)$.

- For $x \leq z$, we have $\dim(\pi^{-1}(x)) = \dim(x) - \dim(z) + n - 1$. In particular, $\dim(e) = n - 1$.

Example: Blowup

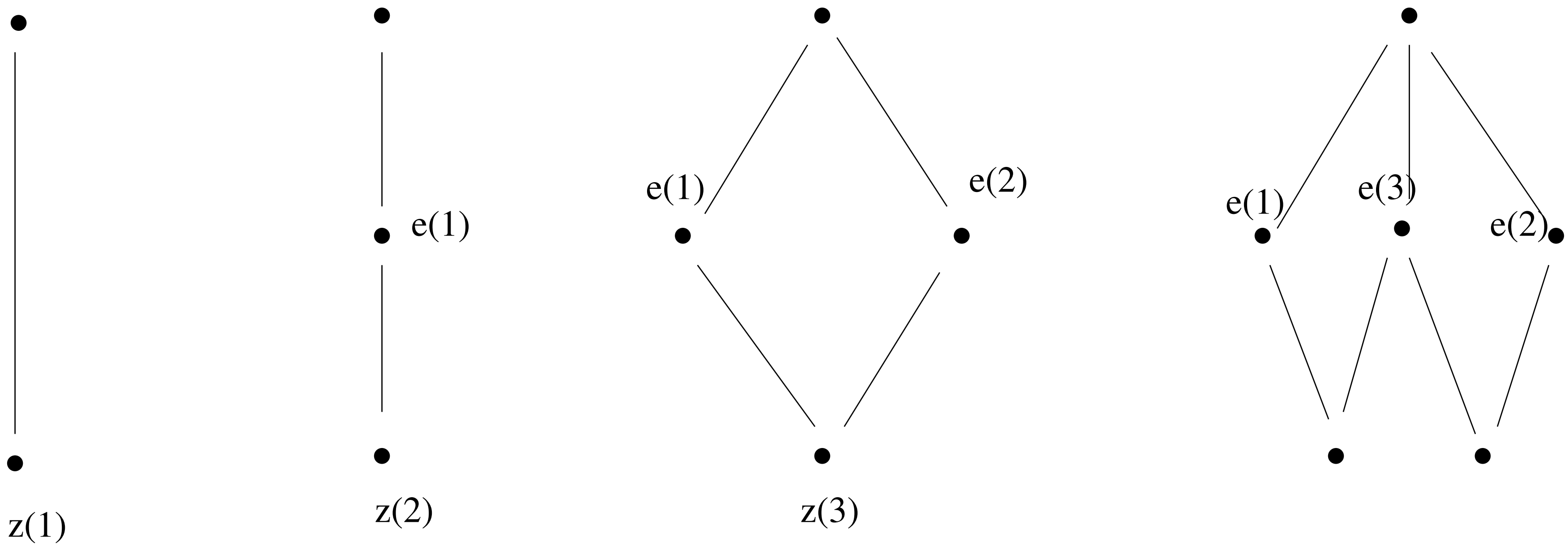


The Handicap

The playground of “Blow ’Em Up” consists of two components: a space W and a handicap U, which is a finite sequence of nodes dimension $n-1$, called “handies”. The handicup puts restrictions on the choice of possible blowup centers.

After blowup, the new handicap $\widetilde{U}$ is the sequence of all unique pre-images of e_i, $e_i \in U$, together with the exceptional node e. (If e_i is blown up, then it is removed from the handicup.) In some sense, the handicap keeps track of the history of blowups.

Example: Playground History



Mephisto

Der Herr: Hast du mir weiter nichts zu sagen?
Kommst du nur immer anzuklagen?
Ist auf der Erde ewig dir nichts recht?
Mephisto: Nein, Herr! Ich find es dort, wie immer, herzlich schlecht.

He provides three things.

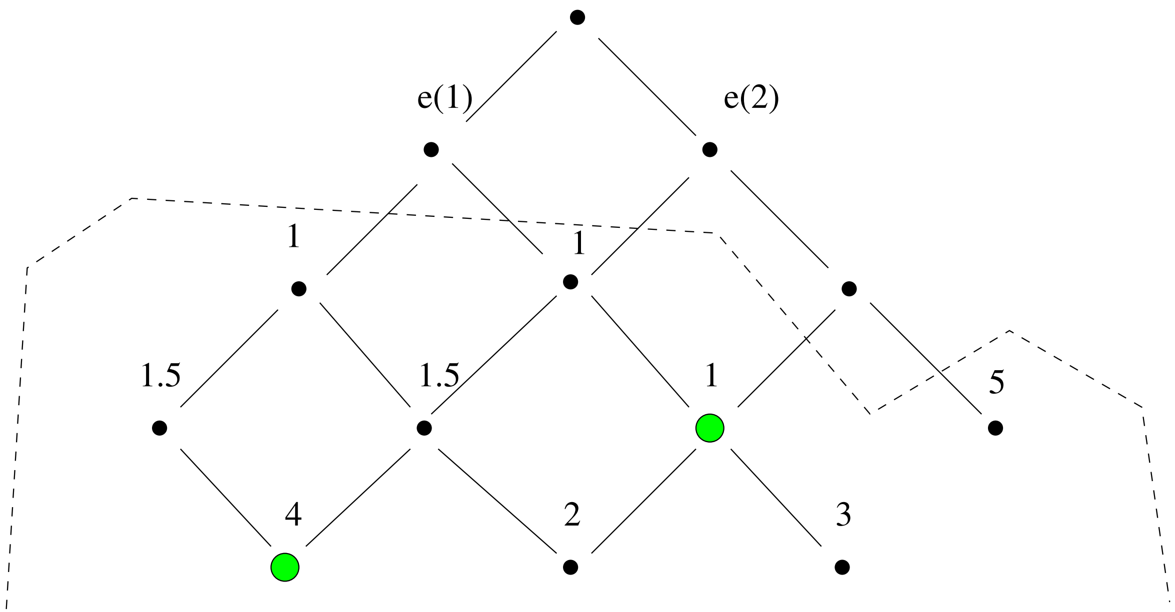
1. a downward closed set $D \subset W$ not containing t, called the "singular set". If D is empty, then the game is over, and Mephisto loses.

Mephisto

2. a monotonically decreasing function $u : D \rightarrow \mathbb{Q}$, the “order”. Its values are ≥ 1, and integral multiples of some positive rational number, the stepsize.

3. a subset $G \subset W$. Its elements are called “green nodes”.

Example: Mephisto's Data



Dynamite

The second player nominates a green node $z \in G \cap D$.

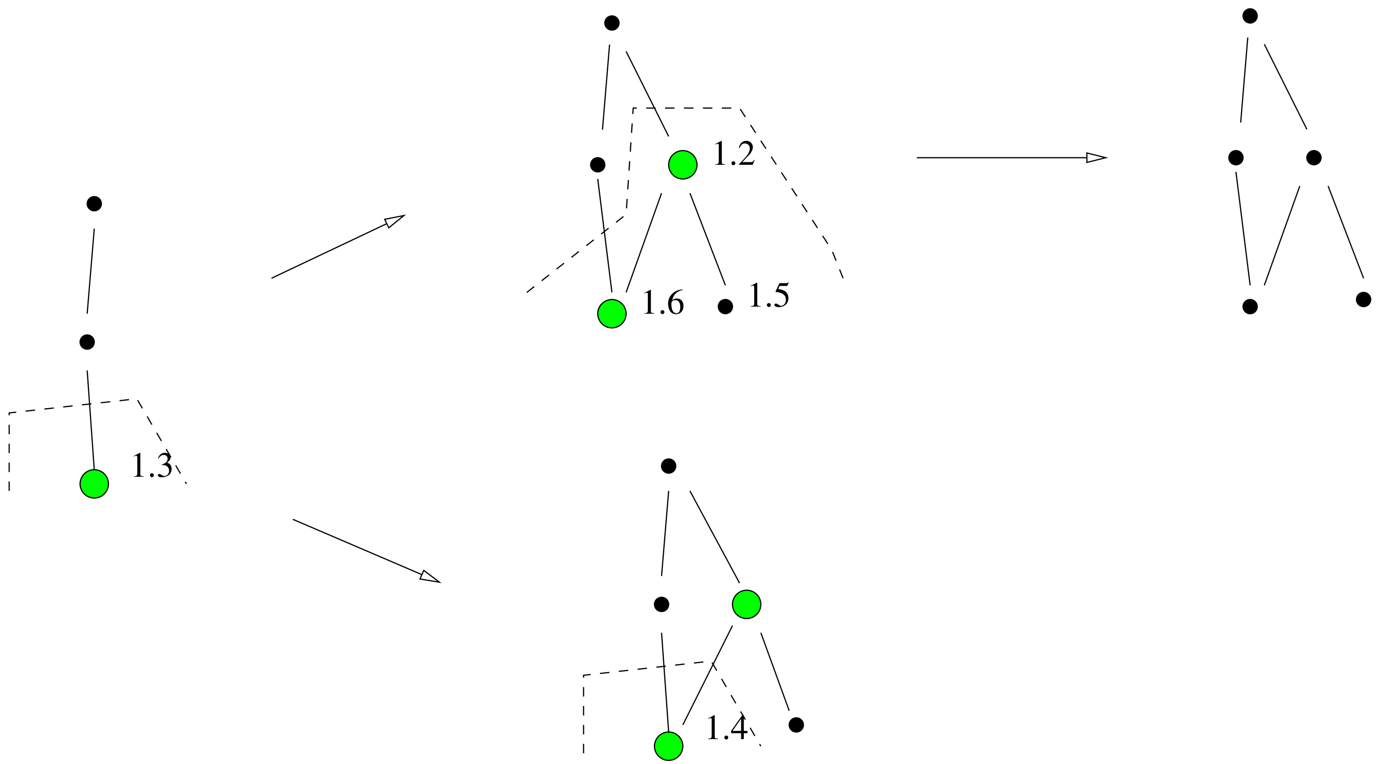
Then it is again Mephisto's turn to provide a blowup of z, a new singular set $\widetilde{D}$, new order function $\widetilde{u} : D \to \mathbb{Q}$, and new set $\widetilde{G}$ of green nodes.

For nodes that are not below e, nothing is changed. For nodes less than or equal to e, the new order is bound by $2 \cdot$(old order)$- 1$ (rule **CG**). If z has maximal dimension $n - 1$, then the new order drops by 1 (rule **FI**).

Any maximal node in the fiber of a green node is green (rule **PG**).

Example Games

Games with $\dim(W) = 1$ are easy to win, by the rule **FI**.



Rules for Refinement

In certain situations, Mephisto needs to refine W and update the singular set, order function, and green set.

Then the singular set and order function is determined by ϕ. Any maximal node in the fiber of a green node is green (rule **PG**). But it is also possible that new green nodes appear.

Quests

A quest is a game inside the game. Dynamite may open a new quest $\mathfrak{f}$. Then Mephisto provides a singular set $D(f)$, order function $f : D(f) \to \mathbb{Q}$, and green nodes $G(f)$, maybe after a refinement.

Any quest has a dimension. The number $\dim(\mathfrak{f}) - 1$ is also an upper bound for the dimension of any node in $D(f)$.

Any quest has a "bin", which is a finite list of integers. At any time the quest is open, the handicap of the quest is

$$U(f) := (e_i \in U \mid i \notin \mathrm{bin}(\mathfrak{f})).$$

Quests

If $D(f)$ is empty, the quest is resolved.

If Dynamite chooses to blowup a center $z \in D(f)$, then Mephisto has to provide a new singular set, order function, and green nodes for the quest $\mathfrak{f}$.

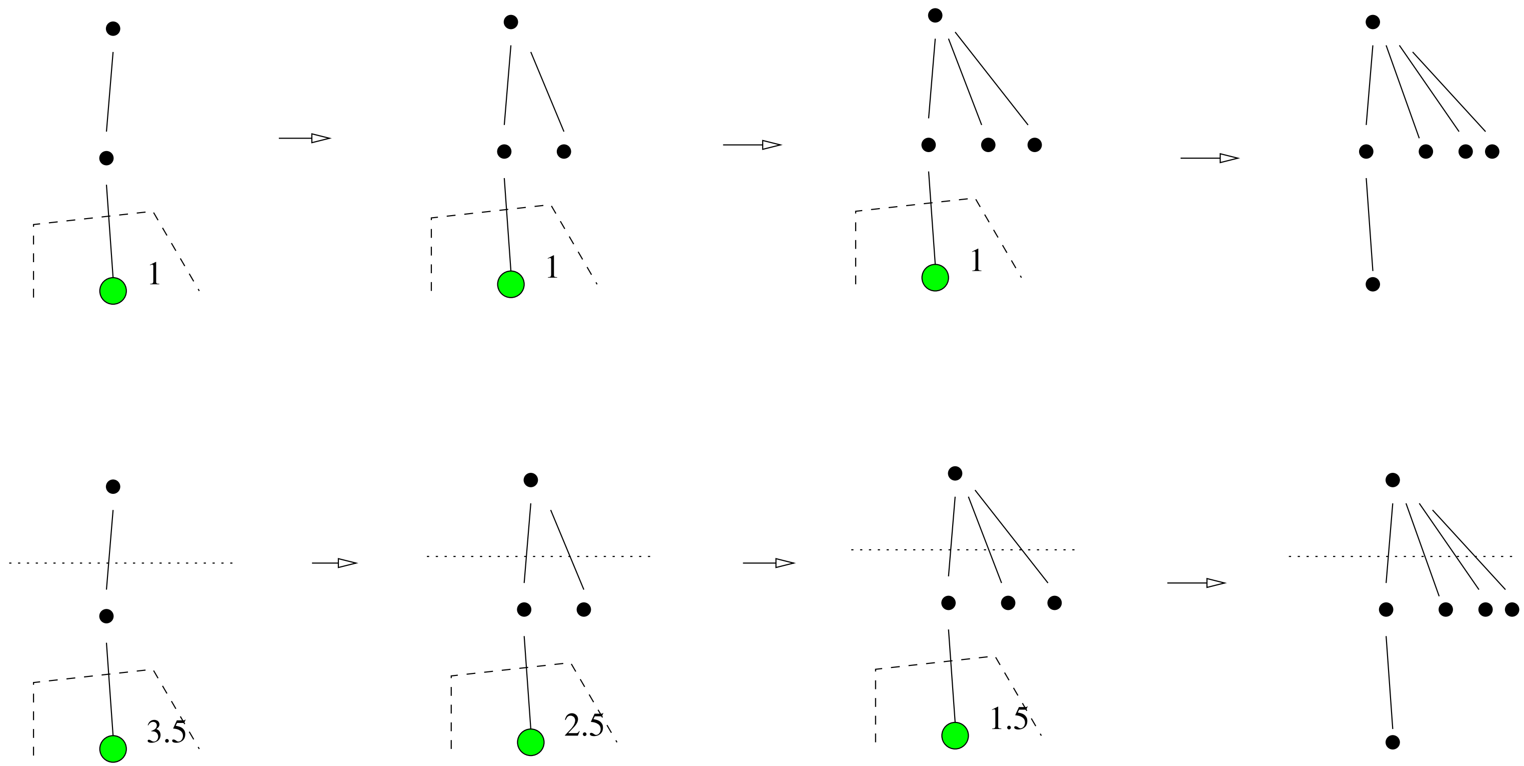
If Dynamite chooses to blowup a center not contained in $D(f)$, then the quest is lost.

Descent Quests

A quest may become "flat", meaning that its order function is everywhere 1. Once it is flat, it remains flat – this is a consequence of the **CG** rule.

Let $\mathfrak{f}$ be a flat quest of dimension m and bin b. A "descent quest" for $\mathfrak{f}$ is a quest $\mathfrak{g}$ of dimension $m-1$ and bin b such that whenever $\mathfrak{f}$ is open, we have $D(g) = D(f)$ and $G(g) = G(f)$. The new information Mephisto needs to provide is the order function $g : D(f) \to \mathbb{Q}$.

Example: Use of Descent Quest



Flat Quests without Handicap

Let $\mathfrak{f}$ be a flat quest of dimension m such that $U(f)$ is empty. We distinguish two cases.

If the maximal dimension of nodes in $D(f)$ is less than $m-1$, then Dynamite may create a descent quest (rule **FS**).

If $z \in D(f)$ is a node of dimension $m-1$, then there cannot be a descent query because the nodes in quests of dimension $m-1$ cannot have dimension $m-1$. But then rule **FB** guarantees that z is green, and rule **FI** implies that something good (for Dynamite) happens when we blow it up.

Forgetful Quests

Imagine that the set G of green nodes is empty. Then Dynamite has to define a quest in order to snatch a green node from some refinement Mephisto will define. New green nodes are more likely when the handicap is smaller.

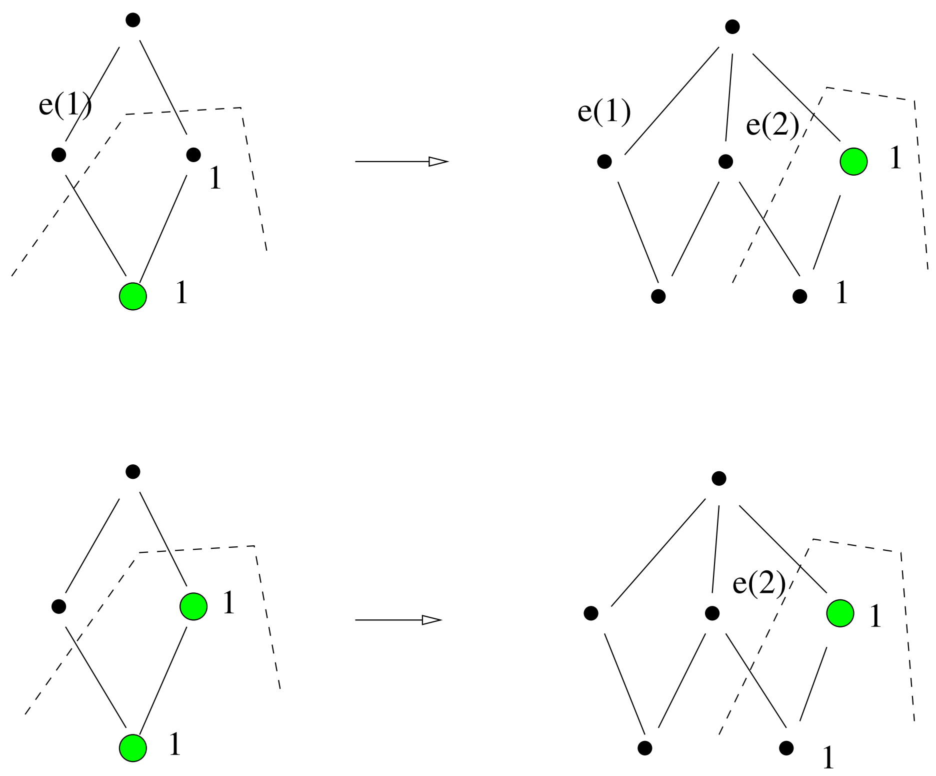
If $\mathfrak{f}$ is a quest and e_i is the i-th node in the handicap $U(f)$, then it is possible to define a quest with $\mathfrak{f}(\not{i})$ such that $U(f(\not{i})) = U(f) - \{e_i\}$, one just puts i into the bin. It is open as long as $\mathfrak{f}$ is open, and the order function is also the same. The new information Mephisto needs to provide is the set of green nodes.

Forgetful Quests

Any node which is green for f is also green for $f(\not{i})$.

Conversely, if z is green for $f(\not{i})$, and either $z \leq e_i$ or there is no y such that $y < z$ and $y < e_i$, then z is green for f.

Example: Use of Forgetful Quest



Bridges

Let $\mathfrak{f}$ be a quest with order function f and handicap $U(f) = \{e_1, \ldots, e_n\}$. Let $\alpha, \alpha_1, \ldots, \alpha_n \geq 0$ be rational numbers. Then Dynamite may build the bridge

$$a = [\alpha, \alpha_1, \ldots, \alpha_n].$$

It is also allowed to form new bridges from old bridges by positive linear combination.

Bridge Functions

Every bridge has a bridge function $D(f) \to \mathbb{Q}$, monotonically decreasing with non-negative values. If z is less or equal than $e_1, \ldots, e_m$, but not less than other handies, then

$$a(z) = \alpha f(z) + \alpha_1 + \cdots + \alpha_m.$$

If the handies above z are not the first handies in the handicup, then a similar formula holds.

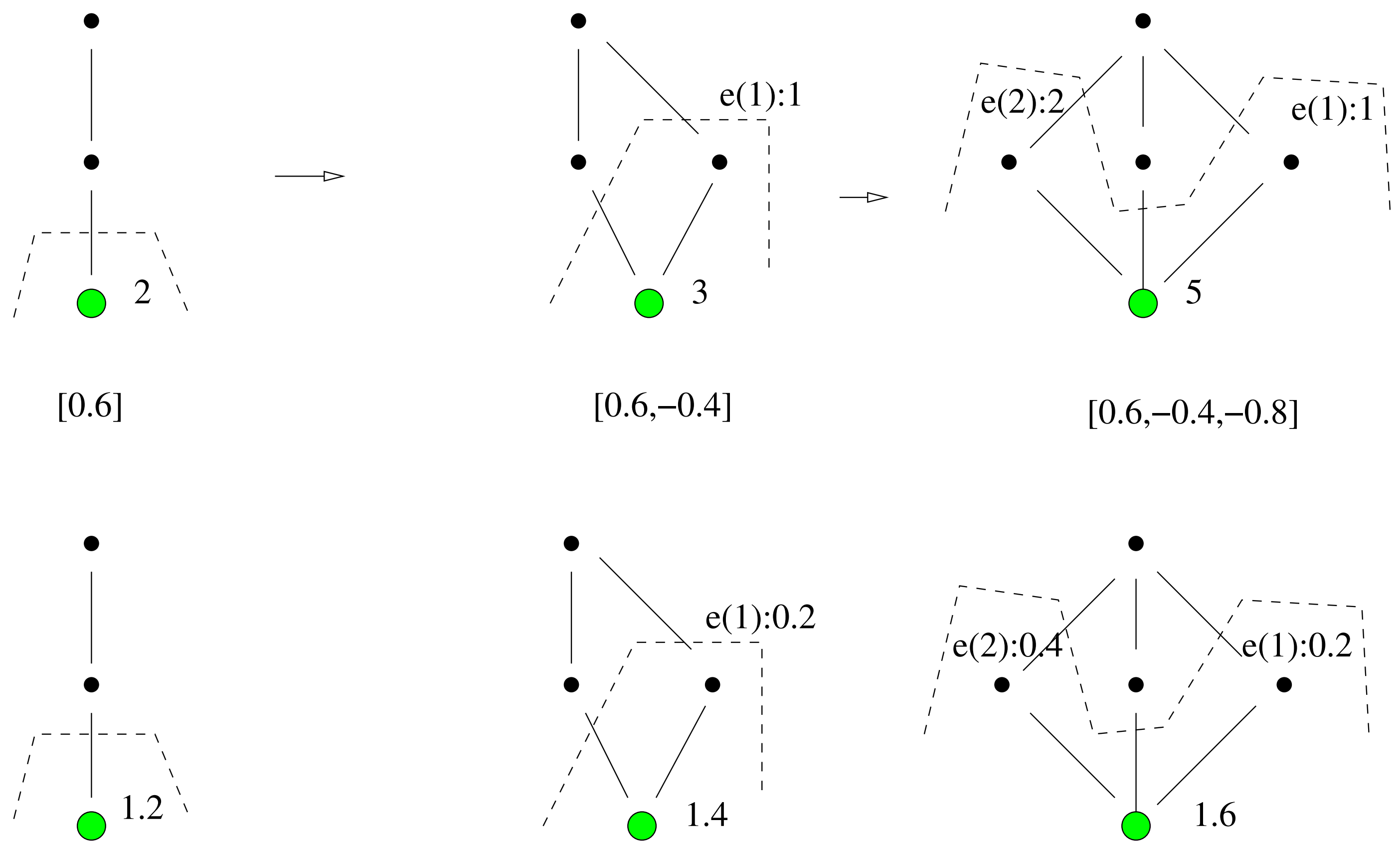
Blowup of Bridges

Assume that z is less or equal than $e_1, \ldots, e_m$, but not less than other handies. If $a(z) \geq 1$, then the blowup of the bridge a is defined as

$$\widetilde{a} = [\alpha, \alpha_1, \ldots, \alpha_n, \alpha + \alpha_1 + \cdots + \alpha_m - 1].$$

It is possible to get a bridge with a negative coefficient. Still, the bridge function $\widetilde{a} : D(\widetilde{f}) \to \mathbb{Q}$ must be non-negative and montonically decreasing. This is an additional constraint for Mephisto to choose $\widetilde{f}$.

Example: Bridge History

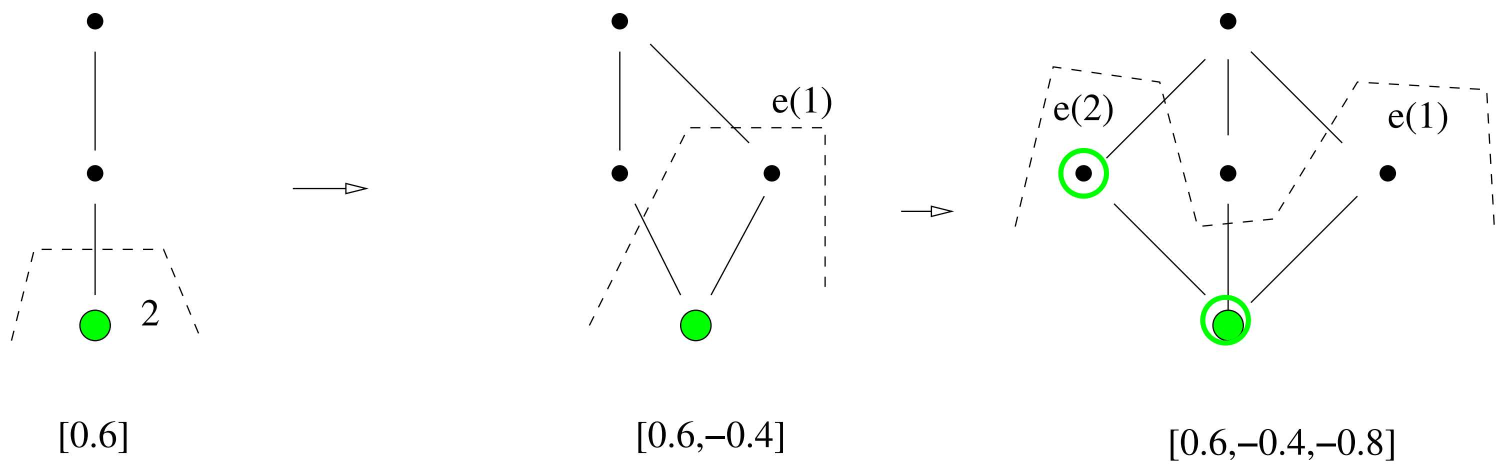


Green Nodes from Bridges

Let $e_1, \ldots, e_m \in \mathcal{U}(f)$ be handies. Let $\alpha_1, \ldots, \alpha_m \geq 0$ such that $\alpha_1 + \cdots + \alpha_m \geq 1$. Assume that the bridge $[1, -\alpha_1, \ldots, -\alpha_m, 0, \ldots, 0]$ exists. Then any maximal node of $D(f)$ less than or equal to $e_1, \ldots, e_m$ is green.

After blowing up all these green nodes – by the way these green nodes do not have a common lower node, so that the blowups do not interfere with each other – there is no y which is smaller than $\widetilde{e_i}$ for $i = 1, \ldots, m$ (rule **BS**).

Example: Green Nodes from Bridges



Quests from Bridges

Let $\mathfrak{f}$ be a quest and let a be a bridge for $\mathfrak{f}$. Then Dynamite can define the "bridge quest" $\mathfrak{b}$, with same dimension and bin as $\mathfrak{f}$.

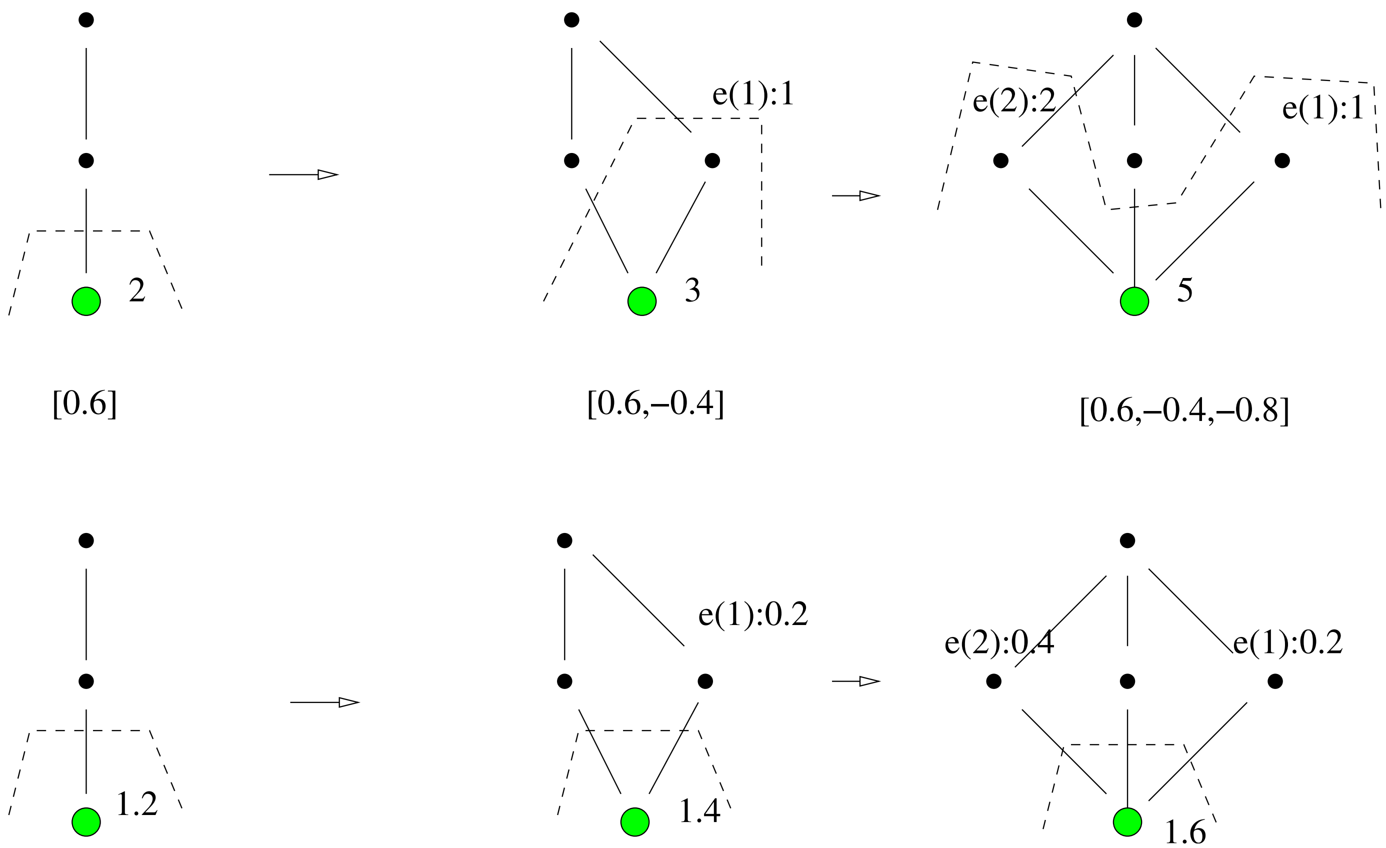
Mephisto does not need to provide any information for a bridge quest:

1. $D(b) = \{x \mid a(x) \geq 1\}$.

2. $b : D(b) \to \mathbb{Q}$, $x \mapsto \min(f(x), a(x))$.

3. $G(b) = G(f)$.

Example: Bridge Quest



The Challenge

There exists a winning strategy for Dynamite (Villamayor's algorithm).

Can you find a winning strategy without knowing one of the algorithms for resolution of singularities?

If yes, this would make our point clear: resolution may be complicated, but the most complex part does not require any algebra.

Prize: a bottle of wine.

Perspective: write a joint paper together with H. Hauser and me, where your contribution would be the winning strategy.