# TRAVELING WAVES OF A KINETIC TRANSPORT MODEL FOR THE KPP-FISHER EQUATION

Carlota M. Cuesta

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM
Facultad de Ciencias, Universidad Autónoma de Madrid
Crta. Colmenar Viejo Km 15, 28049, Madrid, Spain

Sabine Hittmeir

Vienna University of Technology
Institute for Analysis and Scientific Computing, Wiedner Hauptstr. 8-10
1040 Wien, Austria

Christian Schmeiser

University of Vienna
Faculty for Mathematics, Nordbergstraße 15
1090 Wien, Austria

Abstract. A reactive kinetic transport equation whose macroscopic limit is the KPP-Fisher equation is considered. In a scale, where collisions occur at a faster rate than reactions, existence of traveling waves close to those of the KPP-Fisher equation is shown. The method adapts a micro-macro decomposition in the spirit of the work of Caflisch and Nicolaenko for the Boltzmann equation. Stability of these waves is shown for perturbations in a weighted $L^2$-space, where the weight function is exponential and such that the (macroscopic) linearized operator in the weighted space is self-adjoint and negative definite. Similar approaches to stability of traveling waves are well-known for the KPP-Fisher equation.

## 1. Introduction

When the chemical reaction

$$A + B \leftrightarrow 2A$$

takes place in a setting, where the density of species $B$ can be assumed as constant and species $A$ is subject to one-dimensional diffusion, then the dynamics of the density $u(t,x)$ of species $A$ can be described (after non-dimensionalization) by the KPP-Fisher equation

$$\partial_t u = D\partial_x^2 u + u(\bar\rho - u)\,, \tag{1.1}$$

with the diffusion coefficient $D > 0$. This equation has two constant equilibrium states, $u \equiv 0$ and $u \equiv \bar\rho > 0$, the former linearly unstable and the latter linearly stable. Thus, an initial perturbation of $u \equiv 0$ grows to approach $u \equiv \bar\rho$. It is well-known that, in an unbounded domain, this growth may take the asymptotic form of a propagating wave front, i.e. as $t \to +\infty$ the solution approaches the form

2000 *Mathematics Subject Classification.* 35Q72, 35K57, 35B50.

*Key words and phrases.* kinetic transport equations, reaction diffusion equations, traveling waves.

The authors acknowledge support by the Austrian Science Fund under grant numbers P16174 and P18367. CMC has also been supported by a EPSRC Postdoctoral Research Fellowship while at the University of Nottingham.

$u(t,x) = u_{TW}(\xi)$ with the traveling wave variable $\xi = x - st$, the constant wave speed $s \in \mathbb{R}$, and $u_{TW}$ satisfying the ordinary differential equation

$$Du''_{TW} + su'_{TW} + u_{TW}(\bar\rho - u_{TW}) = 0\,. \tag{1.2}$$

We assume throughout that $s \ge 0$. This is no restriction, because (1.2) is invariant under the reflection $s \to -s$, $\xi \to -\xi$. The waves then propagate to the right and satisfy the far-field conditions

$$\lim_{\xi\to-\infty} u_{TW}(\xi) = \bar\rho\,, \qquad \lim_{\xi\to+\infty} u_{TW}(\xi) = 0\,. \tag{1.3}$$

Equation (1.1) has been introduced by Fisher [5] as a model in population genetics that describes the advance of individuals with a favorable gene. At the same time Kolmogorov, Petrovskii and Piskunov [9] investigated (1.1) with a more general nonlinearity. Some results concerning the traveling wave solutions (which have been studied extensively) will be reviewed below.

The subject of this work is a kinetic transport model for the same physical situation. The main modeling difference compared to a reaction-diffusion model is the replacement of the Brownian motion by a velocity jump process. The latter can be thought of being caused by collisions with a (non moving) background medium, which randomize the direction of movement. A kinetic equation for the phase space density $f(t,x,v)$ of particles of species $A$ can be written in the (dimensionless) form

$$\varepsilon^2\partial_t f + \varepsilon v\partial_x f = \mathcal{L}f + \varepsilon^2 Q(f)\,, \tag{1.4}$$

with time $t > 0$, position $x \in \mathbb{R}$ and velocity $v \in V \subset \mathbb{R}$. The left hand side of (1.4) describes the free streaming of particles, and the terms on the right hand side model collisions (described by the operator $\mathcal{L}$) and chemical reactions (described by the operator $Q$). The dimensionless parameter $\varepsilon$ is assumed to satisfy $0 < \varepsilon \ll 1$. Considering its occurrence on the right hand side of (1.4), this means that collisions are much more frequent than reactions. The powers of $\varepsilon$ on the left hand side can be achieved by appropriate scalings for time and position.

Collisions are described as instantaneous velocity jumps with an equilibrium distribution $M(v)$, satisfying the moment conditions

$$\int_V M\,dv = 1\,, \qquad \int_V vM\,dv = 0\,, \qquad \int_V v^2 M\,dv = D > 0\,, \qquad \int_V v^3 M\,dv = 0\,.$$

A typical example is the Maxwellian distribution $M(v) = (2\pi D)^{-1/2}e^{-v^2/(2D)}$, $V = \mathbb{R}$. The simplest collision model is the relaxation operator

$$\mathcal{L}f = \int_V [M(v)f(v') - M(v')f(v)]dv' = M\rho_f - f\,,$$

with the macroscopic density $\rho_f(t,x) = \int_V f(t,x,v)dv$. The collision process obviously conserves mass: $\int_V \mathcal{L}f\,dv = 0$. For the chemical reactions, it is assumed that they produce particles with the same equilibrium velocity distribution:

$$Q(f) = \int_V [\bar\rho M(v)f(v') - f(v)f(v')]dv' = \rho_f(M\bar\rho - f)\,.$$

We obtain the kinetic reaction model

$$\varepsilon^2\partial_t f + \varepsilon v\partial_x f = M\rho_f - f + \varepsilon^2\rho_f(M\bar\rho - f)\,. \tag{1.5}$$

A connection between (1.5) and (1.1) can be established by the macroscopic limit $\varepsilon \to 0$. Substitution of the Chapman-Enskog ansatz $f = M\rho_f + \varepsilon f^\perp$ into (1.5) and integration with respect to $v$ leads to the macroscopic equation

$$\partial_t\rho_f + \partial_x\int_V vf^\perp dv = \rho_f(\bar\rho - \rho_f)\,.$$

On the other hand, (1.5) implies

$$f^{\perp} = -vM\partial_x \rho_f + O(\varepsilon)\,.$$

Hence, in the formal limit $\varepsilon \to 0$, $\rho_f$ solves (1.1). This is an example of the derivation of reaction-diffusion equations from kinetic models. Formal asymptotics of this kind for much more general cases, in particular also systems, has been carried out by several authors (see, e.g., [1], [11]). However, a rigorous justification is only known for linear models [1].

It is our aim to study the existence and stability of traveling waves of (1.5). As a preliminary result, in Section 1.4, we prove global existence of solutions of the initial value problem for (1.5) for initial data bounded by a global equilibrium. Our approach for the analysis of traveling waves is based on the fact that, for $\varepsilon$ small, (1.5) can be approximated by (1.1). In Section 2 we present a constructive existence proof for traveling waves with speed $s \geq s_0 = 2\sqrt{D\,\bar\rho}$ of (1.5), which shows the asymptotic closeness of the kinetic profiles to the solutions of (1.2) with the same speed. We follow the approach of [4] (that is applied to traveling waves of kinetic BGK models for scalar conservation laws) by first constructing a formal asymptotic approximation, and then showing solvability of the problem for the correction term. For the latter we adapt the micro-macro decomposition introduced by Caflisch and Nicolaenko [2] for the Boltzmann equation. The major difficulty in the current problem is caused by the fact that, in contrast to [4] and [2], the macroscopic problem is not a conservation law. In Section 3 we show the asymptotic stability of kinetic profiles with $s > s_0$, under perturbations in suitable spaces. Traveling waves for the KPP-Fisher equation are stable under perturbations, which decay faster than (or at least as fast as) the waves. The analogous result is proven here. The required decay properties are built into an appropriately weighted $L^2$-space. This has the consequence that we can control the macroscopic terms in a similar way as for the KPP-Fisher equation. Concerning the control of the microscopic terms, we have only been successful under the additional assumption that the velocity space $V$ is bounded. In the remainder of this section we recall the stability results for traveling wave profiles of (1.1) and also show, how the stability of these profiles can be proven by using energy estimates. We also carry out the Chapman-Enskog argument for the approximation of kinetic traveling waves.

1.1. **Traveling waves for the KPP-Fisher equation.** Concerning existence of traveling waves of (1.1), the following result is well known.

**Theorem 1** ([9]). *For $s \geq s_0 := 2\sqrt{D\,\bar\rho}$ there exists a positive solution of (1.2), (1.3), which is unique, up to a shift in $\xi$, and strictly decreasing.*

*Proof.* One way of looking at the problem is by writing (1.2) as a planar system and analyzing the $(u_{TW}, u'_{TW})$ phase-plane. The critical points are clearly given by the zeroes $(0,0)$ and $(\bar\rho, 0)$ of the nonlinearity. Linearization shows that $(\bar\rho, 0)$ is a saddle point, with eigenvalues $(-s \pm \sqrt{s^2 + s_0^2})/(2D)$, and that there is a unique orbit coming out of it in the second quadrant.

The critical point $(0,0)$ has eigenvalues $(-s \pm \sqrt{s^2 - s_0^2})/(2D)$, thus it is a stable node for $s \geq s_0$ and a stable spiral for $s < s_0$. Hence, a positive solution to (1.2) satisfying (1.3) can only exist if $s \geq s_0$. Further, it is easy to see that the triangle

$$0 \leq u_{TW} \leq \bar\rho, \quad 0 \geq u'_{TW} \geq -\frac{s}{2D} u_{TW}\,, \tag{1.6}$$

is an invariant region, so that the unique orbit coming out of the saddle point enters the node, and it does so through the slow manifold when $s > s_0$. This gives existence of traveling waves (unique up to translation in $\xi$) for every $s \geq s_0$. □

The proof also provides the far-field behavior. On the one hand, we have

$$\bar\rho - u_{TW}(\xi) \sim c\, e^{\alpha_- \xi} \quad \text{as } \xi \to -\infty \quad \text{with} \quad \alpha_- = \frac{\sqrt{s^2+s_0^2}-s}{2D} > 0\,, \quad c>0\,. \tag{1.7}$$

On the other hand, for every $s > s_0$,

$$u_{TW}(\xi) \sim c_s\, e^{-\alpha_+ \xi} \quad \text{as } \xi \to +\infty\,, \quad \text{with} \quad \alpha_+ = \frac{s-\sqrt{s^2-s_0^2}}{2D} > 0\,, \quad c_s > 0\,, \tag{1.8}$$

and, for $s = s_0$,

$$u_{TW}(\xi) \sim c_0 \xi\, e^{-(s_0/2D)\,\xi} \quad \text{as } \xi \to +\infty\,, \quad c_0 > 0\,. \tag{1.9}$$

1.2. **Stability of traveling waves for the Fisher equation.** Throughout this section we let $u_{TW}$ be a traveling wave of (1.2) with speed $s > s_0$. We write (1.1) in the moving coordinates $t$ and $\xi = x - st$,

$$\partial_t \rho_f - s\partial_\xi \rho_f - D\partial_\xi^2 \rho_f - \rho_f(\bar\rho - \rho_f) = 0\,, \tag{1.10}$$

and look for solutions that are small perturbations of $\rho_{TW}$. Thus we assume $\rho_f = u_{TW} + \rho$ where $\rho \ll 1$, in a sense to be made precise later. The equation for the perturbation $\rho$ reads

$$\partial_t \rho - s\partial_\xi \rho - D\partial_\xi^2 \rho + \rho(2u_{TW} + \rho - \bar\rho) = 0\,. \tag{1.11}$$

It is well known that traveling waves of (1.1) are in general unstable to perturbations, c.f. Canosa [3]. In the classical approach to stability, one studies linear stability first by analyzing the spectrum of the linearized operator. In a $L^p$-setting with $p \geq 2$, the spectrum of the linearized operator about waves having $s > s_0$ extends to the right hand complex plane and always contains 0 as an eigenvalue with eigenfunction $\partial_\xi u_{TW}$ (this eigenfunction is the one generated by perturbations equivalent to small translations in the traveling wave). To overcome this problem one introduces norms with appropriate weights, that push the spectrum into the left hand plane and $\partial_\xi u_{TW}$ out of the space, thus creating a spectral gap. In the seminal work by Sattinger [10] such analysis is undertaken in $L^\infty$ with an exponential weight. We borrow this idea here, but, in our setting, it is more convenient to use $L^2$ estimates and we show next how this is done for (1.1). In the process we need to control $\|\rho\|_\infty$ for all times by an appropriate upper bound. For (1.1) this can be achieved by a comparison principle. Here, however, we use the continuous Sobolev embedding $H^1 \subset L^\infty$. The reason for choosing this approach is two-fold. First, for a kinetic model with complicated collision and reaction terms a comparison principle is difficult to prove, if available. And second, to our knowledge, the idea of applying integral estimates with a Sobolev embedding argument to prove stability of traveling waves of (1.1) is new. For the current kinetic model a comparison principle is easy to prove and stability can be achieved by this type of arguments too assuming a maximum principle for the macroscopic profile. This approach is outlined in the Appendix for completeness.

We define the weight function

$$W(\xi) = e^{\frac{s}{2D}\xi} \tag{1.12}$$

and introduce the Hilbert spaces $L^2_\xi = L^2(\mathbb{R})$, $H^1_\xi = H^1(\mathbb{R})$, $L^2_W$ and $H^1_W$ of functions of $\xi$ with the respective norms

$$\|\rho\|_\xi^2 = \int_{\mathbb{R}} \rho^2 d\xi\,, \quad \|\rho\|^2_{H^1_\xi} = \|\rho\|_\xi^2 + \|\partial_\xi \rho\|_\xi^2\,, \tag{1.13}$$

$$\|\rho\|_W = \|\rho W\|_\xi\,, \quad \|\rho\|^2_{H^1_W} = \|\rho\|_W^2 + \|\partial_\xi \rho\|_W^2\,. \tag{1.14}$$

Local existence of solutions of (1.11) in $H^1_\xi \cap H^1_W$ (which means the weight acts only as $\xi \to +\infty$) follows by a standard contraction argument. Hence, if we can show the decay of the solution in $H^1_\xi \cap H^1_W$ as time evolves, global existence follows by a continuation principle.

We assume that $\rho_f(0,\xi) \geq 0$, then $\rho_f = u_{TW} + \rho \geq 0$ holds as a consequence of the maximum principle. For definiteness, we assume that the traveling wave satisfies $u_{TW}(0) = 3\bar\rho/4$ (which makes it unique by monotonicity), implying

$$u_{TW}(\xi) \geq \frac{3\bar\rho}{4} \quad \text{for } \xi \leq 0\,. \tag{1.15}$$

Multiplication of (1.11) with $W$ gives

$$\partial_t(\rho W) - D\partial_\xi^2(\rho W) + (\kappa + 2u_{TW} + \rho)\,\rho W = 0\,, \tag{1.16}$$

with

$$\kappa := \frac{s^2}{4D} - \bar\rho > 0$$

by $s > s_0$. Testing (1.11) with $\rho$ and (1.16) with $\alpha\rho W$ (for some $\alpha > 0$) and adding the resulting equations leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) + D\left(\|\partial_\xi\rho\|_\xi^2 + \alpha\|\partial_\xi(\rho W)\|_\xi^2\right)\\ &+ \int_{\mathbb{R}}(2u_{TW} + \rho - \bar\rho)\rho^2\,d\xi + \alpha\int_{\mathbb{R}}(2u_{TW} + \rho + \kappa)(\rho W)^2\,d\xi = 0\,.\end{aligned} \tag{1.17}$$

The only problematic term is $-\bar\rho\int\rho^2\,d\xi$. In order to control it, we use the growth of the weight on $[0,+\infty)$ and the monotonicity of the wave on $(-\infty,0]$, i.e. (1.15). First, we write (1.17) in the more convenient form

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) + D\left(\|\partial_\xi\rho\|_\xi^2 + \alpha\|\partial_\xi(\rho W)\|_\xi^2\right) + \frac{1}{2}\int_{\mathbb{R}} u_{TW}(1+\alpha W^2)\rho^2\,d\xi\\ &+\alpha\int_{\mathbb{R}}(\kappa + u_{TW} + \rho)(\rho W)^2\,d\xi + \frac{\alpha}{2}\int_{\mathbb{R}} u_{TW}\,(\rho W)^2\,d\xi + \int_{\mathbb{R}}\left(\frac{3u_{TW}}{2} + \rho - \bar\rho\right)\rho^2\,d\xi\\ &= 0\,.\end{aligned}$$

All terms after the time derivative are obviously nonnegative, except the last one. Its contribution for $\xi < 0$ can be estimated using (1.15):

$$\int_{-\infty}^0 \rho^2\left(\frac{3u_{TW}}{2} - \bar\rho + \rho\right)d\xi \geq \left(\frac{\bar\rho}{8} - \|\rho\|_\infty\right)\int_{-\infty}^0 \rho^2 d\xi\,. \tag{1.18}$$

On the other hand, the contributions of the last two terms for $\xi > 0$ can be estimated by

$$\frac{\alpha}{2}\int_0^\infty (\rho W)^2 u_{TW}\,d\xi + \int_0^\infty \rho^2(3u_{TW}/2 - \bar\rho + \rho)d\xi \geq \int_0^\infty \rho^2(\alpha W^2 u_{TW}/2 - \bar\rho)d\xi\,.$$

Since $W^2$ increases faster than $u_{TW}$ decreases (see (1.8)), $\alpha$ can be chosen such that

$$\alpha\frac{W^2}{2}u_{TW} - \bar\rho \geq \frac{\bar\rho}{16} \quad \text{on} \quad [0,+\infty)\,. \tag{1.19}$$

If we succeed below in proving $\|\rho\|_\infty \leq \bar\rho/16$, then the inequality

$$\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) \leq -\frac{\bar\rho}{16}\|\rho\|_\xi^2 - \kappa\alpha\|\rho\|_W^2 \leq -\min\left\{\frac{\bar\rho}{16},\kappa\right\}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) \tag{1.20}$$

is satisfied, implying exponential decay of the perturbation $\rho$. For proving the required $L^\infty$-bound, we shall derive a $H^1$-bound and use Sobolev embedding. The third term in (1.18) is then needed to control terms resulting in the estimates on the $L^2$-norm of the derivatives of $r = \partial_\xi \rho$ and $rW$. In terms of $r$, the derivative with respect to $\xi$ of (1.11) and its product with $W$ read

$$\partial_t r - D\partial_\xi^2 r - s\partial_\xi r + r(2u_{TW} - \bar\rho + 2\rho) + 2\rho u'_{TW} = 0\,,$$
$$\partial_t (rW) - D\partial_\xi^2 (rW) + rW\,(\kappa + 2u_{TW} + 2\rho) + 2W\rho u'_{TW} = 0\,.$$

We proceed with this system in a similar way as for (1.11), (1.16), note however some small differences: In the parentheses, $2\rho$ replaces $\rho$, and there are additional terms containing $u'_{TW}$ in both equations. When testing with $r$ and, respectively, with $\alpha r W$, the identity $2r\rho = \partial_\xi(\rho^2)$ is used in these terms:

$$\frac{1}{2}\frac{d}{dt}\left(\|r\|_\xi^2 + \alpha\|r\|_W^2\right) + D\left(\|\partial_\xi r\|_\xi^2 + \alpha\|\partial_\xi(rW)\|_\xi^2\right) + \int_{\mathbb{R}} r^2(2u_{TW} - \bar\rho + 2\rho)d\xi$$
$$+\alpha\int_{\mathbb{R}}(rW)^2(\kappa + 2u_{TW} + 2\rho)d\xi = \int_{\mathbb{R}}\rho^2\partial_\xi((1+\alpha W^2)u'_{TW})d\xi\,. \tag{1.21}$$

For the right hand side we observe that

$$u''_{TW} = -\frac{s}{D}u'_{TW} - \frac{u_{TW}}{D}(\bar\rho - u_{TW}) \le \frac{s^2}{2D^2}u_{TW}$$

showing also that $\partial_\xi(W^2 u'_{TW}) \le 0$. This implies

$$\partial_\xi((1+\alpha W^2)u'_{TW}) \le \frac{s^2}{2D^2}u_{TW}\,.$$

Therefore the right hand side of (1.21) can be dominated by a multiple of the third term of (1.18). Similarly, the other problematic term in (1.21), $-\bar\rho\int_{\mathbb{R}} r^2 d\xi$, can be controlled by a multiple of the term $D\|\partial_\xi\rho\|_\xi^2$ in (1.18). Therefore, for a small enough positive constant $\beta$, the functional

$$J[\rho] := \|\rho\|_\xi^2 + \alpha\|\rho\|_W^2 + \beta\left(\|\partial_\xi\rho\|_\xi^2 + \alpha\|\partial_\xi\rho\|_W^2\right)\,,$$

is nonincreasing in time as long as

$$\|\rho\|_\infty \le \frac{\bar\rho}{16} \tag{1.22}$$

holds. Since, by Sobolev embedding, $\|\rho\|_\infty^2 \le cJ[\rho]$, (1.22) can be guaranteed for all time under the initial smallness assumption

$$J[\rho(t=0)] \le \frac{1}{c}\left(\frac{\bar\rho}{16}\right)^2\,.$$

As shown above, this implies (1.20). Obviously, when $\kappa = 0$ (or $s = s_0$) we cannot deduce exponential convergence by this procedure. In fact, the linearized operator in $L_\xi^2 \cap L_W^2$ has spectrum that extends up to the origin (see [10]). A more delicate treatment is needed here and without further discussion we refer the reader to Kirchgässner [8].

**Remark 2.** (i) *The weight in the norm implies that the initial perturbation decays faster than the travelling wave as $x \to \infty$, which is known to be necessary for stability. A decay of the perturbation is also required as $x \to -\infty$, which is a weakness of the $L^2$-approach.*

(ii) *Another weakness of the result is that the exponential decay rate $\lambda$ depends on the initial data through $\gamma$. This could be improved by $L^\infty$-decay of the perturbation, so possibly in the framework of the $H^1$-approach* [7] *mentioned above.*

(iii) *Obviously, when $s = s_0$, we cannot deduce exponential convergence by this procedure. In fact, the spectrum of the linearized operator in $L_\xi^2 \cap L_W^2$ extends to the origin (see* [10]*). A more delicate treatment is needed here, and without further discussion we refer the reader to Kirchgässner* [8].

1.3. **Formal approximation of kinetic traveling waves.** It is instructive to perform the formal limit $\varepsilon \to 0$ before proving existence of traveling waves. We look for traveling waves of (1.5), i.e. solutions of the form $f(t,x,v) = f_{TW}(\xi, v)$ with $\xi = x - st$ and $s > 0$, satisfying

$$\varepsilon(v - \varepsilon s)\partial_\xi f_{TW} = M\rho_{TW} - f_{TW} + \varepsilon^2 \rho_{TW}(M\bar\rho - f_{TW}), \quad \rho_{TW} := \rho_{f_{TW}}, \tag{1.23}$$

subject to the far-field conditions

$$f_{TW}(-\infty, v) = \bar\rho M(v) \quad \text{and} \quad f_{TW}(+\infty, v) = 0 \quad \text{for all } v \in V. \tag{1.24}$$

We make the ansatz

$$f_{TW}(\xi, v) = \rho_{TW}(\xi)M(v) + \varepsilon f_{TW}^\perp(\xi, v) \quad \text{with} \int_V f_{TW}^\perp\, dv = 0. \tag{1.25}$$

Substitution of (1.25) and integration in (1.23) lead to

$$-s\partial_\xi \rho_{TW} + \partial_\xi \int_V v f_{TW}^\perp dv = \rho_{TW}(\bar\rho - \rho_{TW}). \tag{1.26}$$

Substitution of (1.25) into (1.23) gives the asymptotic expansion of $f_{TW}^\perp$ as $\varepsilon \to 0$,

$$\begin{aligned} f_{TW}^\perp &= -vM\partial_\xi\rho_{TW} + \varepsilon[sM\partial_\xi \rho_{TW} - v\partial_\xi f_{TW}^\perp + M\rho_{TW}(\bar\rho - \rho_{TW})] + O(\varepsilon^2) \\ &= -vM\partial_\xi \rho_{TW} + \varepsilon(v^2 - D)M\partial_\xi^2 \rho_{TW} + O(\varepsilon^2), \end{aligned} \tag{1.27}$$

where in the last step we have used (1.26). Substitution of (1.27) into (1.26) shows that $\rho_{TW}$ formally solves (1.2) up to $O(\varepsilon^2)$-terms.

1.4. **Notation and preliminary results.** Next we introduce the underlying spaces of our analysis and establish the global existence of the Cauchy problem and a maximum principle as preliminary results.

We define the weighted inner product in the $v$-direction by

$$\langle f, g\rangle_v = \int_V \frac{fg}{M}\, dv$$

and denote the induced Hilbert space and norm by $(L^2_v, \|\cdot\|_v)$. With respect to $\langle\cdot,\cdot\rangle_v$, the linear collision operator $\mathcal{L}f = M\rho_f - f$ is symmetric and negative semidefinite, a consequence of

$$\langle \mathcal{L}f, g\rangle_v = -\langle \mathcal{L}f, \mathcal{L}g\rangle_v.$$

The standard norms and spaces of functions of $\xi$ are denoted by $(L^2_\xi, \|\cdot\|_\xi)$, $(H^k_\xi, \|\cdot\|_{H^k_\xi})$, and $(C^b_\xi, \|\cdot\|_\infty)$, and with weight (1.12) by $(L^2_W, \|\cdot\|_W)$ (see (1.14)). The Hilbert space $(L^2_\xi(L^2_v), \|\cdot\|_{\xi,v})$ is then naturally defined by the scalar product

$$\langle f, g\rangle_{\xi,v} = \int_{\mathbb{R}} \langle f, g\rangle_v \, d\xi.$$

For $k \in \mathbb{N}\cup\{0\}$, the space $H^k_\xi(L^2_v)$ of functions whose derivatives up to order $k$ with respect to $\xi$ are in $L^2_{\xi,v}$ is equipped with the norm

$$\|f\|_{H^k_\xi(L^2_v)} = \left(\|f\|^2_{\xi,v} + \cdots + \|\partial^k_\xi f\|^2_{\xi,v}\right)^{1/2}.$$

In a similar way $C^b_\xi(L^2_v)$ is defined by

$$\|f\|_{\infty,v} = \sup_{\xi\in\mathbb{R}} \|f\|_v.$$

Finally, we extend the definition of the norm with weight (1.12) to functions on $\mathbb{R}\times V$, leading to the space $(L^2_W(L^2_v), \|\cdot\|_{W,v})$ with norm

$$\|f\|_{W,v} = \|fW\|_{\xi,v}.$$

For later reference we note that the Cauchy-Schwarz inequality implies

$$\|\rho_f\|_\xi \le \|f\|_{\xi,v} \,. \tag{1.28}$$

A global existence and uniqueness result for the kinetic Cauchy problem is not hard to prove. We choose a simple setting, where the initial datum is bounded in terms of the equilibrium distribution.

**Theorem 3** (Global existence)**.** *Let $0 \le f_0(x,v) \le \hat\rho M(v)$ hold. Then the kinetic equation (1.5) subject to the initial condition $f(t=0) = f_0$ has a unique mild solution $f \in C([0,\infty);\ L^\infty(\mathbb{R}\times V))$, satisfying*

$$0 \le f(t,x,v) \le \max\{\bar\rho, \hat\rho\} M(v)\,, \quad \forall\,(t,x,v) \in [0,\infty)\times\mathbb{R}\times V\,. \tag{1.29}$$

*Proof.* The mild formulation of the initial value problem is given by

$$\begin{aligned} f(t,x,v) \;&=\; f_0(x - vt/\varepsilon, v) + M(v)\left(\frac{1}{\varepsilon^2} + \bar\rho\right)\int_0^t \rho_f(\tau, x - v\tau/\varepsilon)d\tau \\ &\quad - \int_0^t \left(\frac{1}{\varepsilon^2} + \rho_f(\tau, x - v\tau/\varepsilon)\right) f(\tau, x - v\tau/\varepsilon, v)d\tau\,. \end{aligned} \tag{1.30}$$

For $T > 0$, we introduce the Banach space

$$\mathcal{C}_T = \{f \in C([0,T];\ L^\infty(\mathbb{R}\times V)) :\ \|f\|_{\mathcal{C}_T} < \infty\}\,,$$
$$\|f\|_{\mathcal{C}_T} = \sup_{(t,x,v)\in[0,T]\times\mathbb{R}\times V} \frac{|f(t,x,v)|}{M(v)}\,.$$

Using the property $|\rho_f(t, x - vt/\varepsilon)| \le \|f\|_{\mathcal{C}_T}$ for all $(t,x,v) \in [0,T]\times\mathbb{R}\times V$, it is straightforward to uniquely solve (1.30) in $\mathcal{C}_T$ for small enough $T$ by Picard iteration. Global existence will follow from (1.29).

The nonnegativity of $f$ is an obvious consequence of the maximum principle for kinetic equations, after writing (1.5) in the form

$$\varepsilon^2\partial_t f + \varepsilon v\partial_x f + f(1 + \varepsilon^2\rho_f) = \rho_f M(1 + \varepsilon^2\bar\rho)\,,$$

and solving by a fixed point iteration, where $\rho_f$ is considered as given and nonnegative. The same argument applies to the function $h(t,x,v) = \max\{\bar\rho,\hat\rho\}M(v) - f(t,x,v)$, that satisfies

$$\varepsilon^2\partial_t h + \varepsilon v\partial_x h + h(1 + \varepsilon^2\rho_f) = \rho_h M + \varepsilon^2\rho_f M(\hat\rho - \bar\rho)_+\,, \qquad h(t=0) \ge 0\,,$$

proving $h \ge 0$ and, thus, (1.29). ☐

**Lemma 4.** *Let $f_1(t,x,v)$, $f_2(t,x,v) \ge 0$ be two solutions of (1.5). Let $f_2(0,x,v) \le \bar\rho M(v)$ and $f_1(0,x,v) \ge \gamma f_2(0,x,v)$, for all $x \in \mathbb{R}$ and $v \in V$, with $\gamma \le 1$. Then $f_1(t,x,v) \ge \gamma f_2(t,x,v)$, for all $t \ge 0$, $x \in \mathbb{R}$, $v \in V$.*

*Proof.* A simple computation shows that $g := f_1 - \gamma f_2$ satisfies

$$\varepsilon^2\partial_t g + \varepsilon v\partial_x g + (1 + \varepsilon^2\rho_1)g = \rho_g(M + \varepsilon^2(\bar\rho M - f_2)) + \varepsilon^2\rho_1 f_2(1-\gamma)\,.$$

Theorem 3 implies that $f_2 \le \bar\rho M$ for all times, such that the coefficient of $\rho_g$ is nonnegative. Since also the last term is nonnegative by the assumptions, the nonnegativity of $g$ for all times follows as in the proof of Theorem 3. ☐

## 2. Existence of traveling waves

We prove existence of traveling waves of (1.1) with a given $s \ge s_0$ for $\varepsilon \ll 1$. The proof follows the steps of that in [4], stated in the subsequent sections. Essentially, we make the expansion in Section 2.1 rigorous, but first produce a residual term whose zeroth order moment in $v$ vanishes.

2.1. **The asymptotic approximation.** We start by defining an asymptotic approximation of a traveling wave profile. In view of the computation of Section 1.3 we choose

$$f_{as}(\xi,v) = M(v)u_{TW}(\xi) + \varepsilon f^\perp[u_{TW}](\xi,v)\,,$$

where $u_{TW}$ is a traveling wave of the Fisher equation (i.e. satisfying (1.2), (1.3)), made unique by the requirement

$$u_{TW}(0) = \frac{\bar\rho}{2}\,. \tag{2.1}$$

Recalling the formal expansion (1.27), we set

$$f^\perp[u] = -vMu' + \varepsilon(v^2 - D)Mu''\,.$$

Integration shows that $\int_V f^\perp[u]dv = 0$, implying $\rho_{as} := \rho_{f_{as}} = u_{TW}$. Clearly, $f_{as}$ satisfies (1.24) and the equation (1.23) up to the residual

$$\begin{aligned}\varepsilon^3 h &= \varepsilon(v-\varepsilon s)\partial_\xi f_{as} - M\rho_{as} + f_{as} - \varepsilon^2\rho_{as}(M\bar\rho - f_{as})\\ &= \varepsilon^3\left(svMu_{TW}'' + (v-\varepsilon s)(v^2-D)Mu_{TW}''' + u_{TW}f^\perp[u_{TW}]\right).\end{aligned}$$

It is now not hard to prove that

$$\int_V h\,dv = 0, \qquad \text{and} \qquad \|h\|_{H^k_\xi(L^2_v)} \le C_k \quad \text{for any} \quad k\in\mathbb{N}\,, \tag{2.2}$$

with $\varepsilon$-independent constants $C_k$.

2.2. **The micro-macro decomposition and the correction term.** In terms of the correction $\varepsilon^2 g = f_{TW} - f_{as}$, the traveling wave equation reads

$$\varepsilon(v-\varepsilon s)\partial_\xi g = \mathcal{L}g + \varepsilon^2 Bg + \varepsilon^4 R[g] - \varepsilon h\,, \tag{2.3}$$

where

$$Bg = \rho_g(M\bar\rho - f_{as}) - \rho_{as}g\,, \qquad R[g] = -\rho_g g\,.$$

On the right hand side of (2.3), we have collected the linear collision operator, a linear term of $O(\varepsilon^2)$, a nonlinear term of $O(\varepsilon^4)$, and the residual. By the properties of $f_{as}$, a solution $g$ of (2.3) must satisfy the far-field conditions

$$g(\pm\infty, v) = 0 \qquad \text{for all } v\in V\,. \tag{2.4}$$

To prove the existence of such a $g$, we need some preparation. First, we observe that integration of (2.3) shows that necessarily

$$\partial_\xi \int_V (v-\varepsilon s)g\,dv = \varepsilon\rho_g(\bar\rho - 2\rho_{as}) - \varepsilon^3\rho_g^2\,. \tag{2.5}$$

We now decompose $g$ into a macroscopic term (with separated variables), containing the leading order terms, and a microscopic term of order $\varepsilon$:

$$g(\xi,v) = \Phi(v)z(\xi) + \varepsilon w(\xi,v)\,. \tag{2.6}$$

Here $\Phi$ is chosen such that $\mathcal{L}\Phi = -\varepsilon\tau(v-\varepsilon s)\Phi + O(\varepsilon^2)$ for some constant $\tau$, leading to

$$\Phi(v) = \left(1 + \varepsilon\frac{s}{D+\varepsilon^2 s^2}(v-\varepsilon s)\right)M(v)\,,$$

where the coefficient $s/(D+\varepsilon^2 s^2)$ guarantees that

$$\int_V (v-\varepsilon s)\Phi\,dv = 0\,, \tag{2.7}$$

and the decomposition of $g$ is unique by requiring

$$\int_V (v-\varepsilon s)^2 w\,dv = 0\,. \tag{2.8}$$

Integration also shows that $\rho_\Phi = 1 - \varepsilon^2\tau s$ and that

$$D_1 := \int_V (v-\varepsilon s)^2 \Phi dv = D\,\frac{D-\varepsilon^2 s^2}{D+\varepsilon^2 s^2} = D + O(\varepsilon^2)\,,$$

which is positive for $\varepsilon$ small enough. We also observe that, due to (2.7), (2.5) is equivalent to

$$\partial_\xi \int_V (v-\varepsilon s) w\, dv = \rho_g(\bar\rho - 2\rho_{as}) - \varepsilon^2\rho_g^2\,. \tag{2.9}$$

The problem we now solve is obtained by substituting (2.6) into (2.3), thus

$$(v-\varepsilon s)\Phi\, z' + \varepsilon(v-\varepsilon s)\partial_\xi w = \frac{1}{\varepsilon} z\,\mathcal{L}\Phi + \mathcal{L}w + \varepsilon Bg + \varepsilon^3 R(g) - h\,, \tag{2.10}$$

and, like $g$, its micro- and macro-components $z$ and $w$ have to satisfy the homogeneous far-field conditions

$$w(\pm\infty, v) \equiv 0\,, \qquad z(\pm\infty) = 0\,. \tag{2.11}$$

The next step consists of writing (2.10) as a system of two equations; one containing only derivatives of $z$ and the other containing only derivatives of $w$. This is achieved by applying the right macroscopic and microscopic projections. Applying

$$Pf := \int_V (v-\varepsilon s) f\, dv \tag{2.12}$$

to (2.10) we obtain, by (2.8),

$$D_1 z' + s\rho_\Phi z = P\mathcal{L}w + \varepsilon PBg + \varepsilon^3 PR(g) - Ph\,. \tag{2.13}$$

We differentiate (2.13) and use the moment relation (2.9). After multiplying the resulting equation by $D/D_1 = 1 + O(\varepsilon^2)$ and collecting the small linear and nonlinear terms on the right hand side we arrive at

$$Dz'' + sz' + z(\bar\rho - 2\rho_{as}) = \varepsilon B^z(z, z', w, \partial_\xi w) + \varepsilon^2 R^z(g, \partial_\xi g) - \tilde h\,, \tag{2.14}$$

where

$$\begin{aligned} B^z(z, z', w, \partial_\xi w) &= \frac{D}{D_1}\left[-\rho_w(\bar\rho - 2\rho_{as}) - s\rho_w' + \partial_\xi PBg\right] \\ &\quad + \frac{1}{\varepsilon}\left(1 - \frac{D}{D_1}\rho_\Phi\right)(sz' + z(\bar\rho - 2\rho_{as}))\,, \\ R^z(g, \partial_\xi g) &= \frac{D}{D_1}\left[\rho_g^2 + \varepsilon\partial_\xi PR(g)\right], \qquad \tilde h = -\frac{D}{D_1}\partial_\xi Ph\,. \end{aligned}$$

The right hand side of (2.14) is the linearization of the Fisher equation at $\rho_{as}$.

The microscopic projection

$$\Pi f := f - \frac{(v-\varepsilon s)\Phi}{D_1} Pf \tag{2.15}$$

has the properties $\Pi(v-\varepsilon s)\Phi = 0$ and $\Pi(v-\varepsilon s)w = (v-\varepsilon s)w$, by (2.8). Applying $\Pi$ to (2.10) we get the following equation for $w$:

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{L}w = \frac{(v-\varepsilon s)\Phi}{D_1}\int_V vw\, dv + \varepsilon\Lambda\, z + \varepsilon\Pi Bg + \varepsilon^3\Pi R(g) - \Pi h\,, \tag{2.16}$$

where

$$\Lambda := \frac{1}{\varepsilon^2}\Pi\mathcal{L}\Phi = s^2\frac{v^2 - D}{D^2 - \varepsilon^4 s^4}\, M = O(1)\,.$$

Since the symmetric operator $\mathcal{L}$ is only negative semidefinite, we introduce a new symmetric operator $\mathcal{M}$, which is strictly negative and coincides with $\mathcal{L}$ on the set of functions $w$ satisfying (2.8) (this idea is borrowed from [2]):

$$\mathcal{M}w := \mathcal{L}w - (v-\varepsilon s)^2 M \int_V (v-\varepsilon s)^2 w\, dv\,.$$

**Lemma 5.** *The operator $\mathcal{M}$ is symmetric and negative definite with respect to $\langle\cdot,\cdot\rangle_v$. There exists a constant $\sigma > 0$, such that*

$$-\langle \mathcal{M}w, w\rangle_v \ge \sigma \|w\|_v^2 \qquad \text{for all } w \in L_v^2 . \tag{2.17}$$

The proof is analogous to that in [4] and we do not repeat it here.

We now replace $\mathcal{L}$ in (2.16) by the operator $\mathcal{M}$:

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{M}w = \frac{(v-\varepsilon s)\Phi}{D_1}\int_V vw\,dv + \varepsilon\Lambda\, z + \varepsilon\Pi Bg + \varepsilon^3\Pi Rg - \Pi h. \tag{2.18}$$

The equivalence to the original problem is not obvious:

**Lemma 6.** *The function $g = \Phi z + \varepsilon w$ is a solution of (2.3), (2.4) if and only if $z$ and $w$ solve (2.14), (2.18) subject to (2.11).*

*Proof.* We follow the proofs in [2] and [4]. The problem (2.14), (2.18) (2.11) has been derived from (2.3), (2.4) using the properties (2.9), (2.8) of solutions of the latter. In particular (2.8) is not a necessary condition for existence. Hence we have to check that (2.8) also holds for solutions of (2.14), (2.18), (2.11), without requiring it as a side condition. Using

$$\int_V \Pi f\,dv = \int_V f\,dv, \qquad \int_V (v-\varepsilon s)\Pi f\,dv = 0\,,$$

integration of (2.18) implies

$$\begin{aligned}
\varepsilon\partial_\xi \int_V (v-\varepsilon s)w\,dv &= -(D+\varepsilon^2 s^2)\int_V (v-\varepsilon s)^2 w\,dv + \varepsilon(\rho_g(\bar\rho - 2\rho_{as}) - \varepsilon^2\rho_g^2)\,,\\
\varepsilon\partial_\xi \int_V (v-\varepsilon s)^2 w\,dv &= 2\varepsilon s D\int_V (v-\varepsilon s)^2 w\,dv\,.
\end{aligned}$$

The second equation is a linear ODE with constant coefficients for the unknown $\int_V (v-\varepsilon s)^2 w\,dv$. Since $w(\pm\infty, v) = 0$, the only possible solution is

$$\int_V (v-\varepsilon s)^2 w\,dv = 0\,.$$

Knowing this and returning to the first differential equation we also recover (2.9). □

We now eliminate the first term on the right hand side in (2.18) by substituting (2.13):

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{M}w = A(z,z') + \varepsilon Bg + \varepsilon^3 Rg - h\,, \tag{2.19}$$

where

$$A(z,z') = -\frac{(v-\varepsilon s)\Phi}{D_1}(D_1 z' + s\rho_\Phi z) + \varepsilon\Lambda\, z\,.$$

Thus we have arrived at our final differential problem (2.14), (2.19), subject to (2.11). In the following sections we show solvability via a fix-point argument.

2.3. **The Linear Problem.** We first analyze the leading order system of (2.14), (2.19), where the given inhomogeneity contains the higher order terms. In particular, we prove the solvability of

$$Dz'' + sz' + z(\bar\rho - 2\rho_{as}) = h_z\,, \qquad \text{with } h_z \in H_\xi^1\,, \tag{2.20}$$

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{M}w = A(z,z') + h_w\,, \qquad \text{with } h_w \in H_\xi^2(L_v^2)\,. \tag{2.21}$$

We shall look for solutions in the same spaces as the inhomogeneities. This replaces the homogeneous far-field conditions, and provides uniqueness for the solution of (2.21). This requirement allows, however, a one-parameter set of solutions of (2.20).

This reflects the arbitrary shift in the wave and uniqueness will be guaranteed by posing also an initial condition,

$$z(0) = z_0 \,, \quad z_0 \in \mathbb{R} \,. \tag{2.22}$$

For (2.20) we obtain

**Lemma 7.** *Let $h_z \in H^k_\xi$, $k \geq 0$. Then the problem (2.20), (2.22) with $s \geq s_0$ possesses a unique solution $z \in H^{k+2}_\xi$, satisfying (with $C > 0$ independent from $z_0$ and $h_z$)*

$$\|z\|_{H^{k+2}_\xi} \leq C(|z_0| + \|h_z\|_{H^k_\xi}) \,.$$

*Proof.* Since (2.20) is the linearization of (2.1) at its solution $\rho_{as}$, the derivative $\rho'_{as}$ is a solution of the homogeneous equation. The standard order reduction procedure then allows to rewrite (2.20) as the first order system

$$z' = \frac{\rho''_{as}}{\rho'_{as}} z + z_1 \,, \qquad z'_1 = -\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) z_1 + \frac{h_z}{D} \,. \tag{2.23}$$

Starting with the second equation, (2.1), $\rho'_{as} < 0$, and $0 < \rho_{as} < \bar\rho$ imply

$$-\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) = \frac{\rho_{as}(\bar\rho - \rho_{as})}{D\rho'_{as}} < 0 \,.$$

Since, by the asymptotic behavior of $\rho_{as}$, this coefficient converges to negative values as $\xi \to \pm\infty$, the stronger statement

$$-\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) \leq -\gamma < 0 \,,$$

holds. By standard ODE methods, a unique decaying solution $z_1$ of the second equation in (2.23) exists for decaying $h_z$ (using the 'boundary condition' $z_1(-\infty) = 0$). It can be estimated by testing the equation with $z_1$, giving

$$\|z_1\|_\xi \leq \frac{1}{\gamma D}\|h_z\|_\xi \,.$$

Turning to the first equation in (2.23), we observe that

$$\lim_{\xi\to\infty} \frac{\rho''_{as}(\xi)}{\rho'_{as}(\xi)} < 0 \,, \qquad \lim_{\xi\to-\infty} \frac{\rho''_{as}(\xi)}{\rho'_{as}(\xi)} > 0 \,.$$

This is the situation covered in Lemma 3.5 of [4], implying the existence of a unique solution satisfying

$$\|z\|_\xi \leq C'(|z_0| + \|z_1\|_\xi) \leq C'\left(|z_0| + \frac{1}{\gamma D}\|h_z\|_\xi\right) \,.$$

Testing (2.20) with $z$ and with $z''$ we obtain estimates for the first and second derivatives, implying $\|z\|_{H^2_\xi} \leq C(|z_0| + \|h_z\|_{L^2_\xi})$. Finally, the same procedure can be applied to differentiated versions of (2.20), completing the proof. □

We remark that the previous proof makes use of the positivity and strict monotonicity of $\rho_{as}$. The assumption $s \geq s_0$ is therefore crucial.

Now $A(z, z')$ can be considered as a given inhomogeneity in (2.21), and the following result from [4] can be used:

**Proposition 8.** *Let $\tilde h_w \in H^k_\xi(L^2_v)$, $k \geq 0$. Then there exists a unique solution $w \in H^k_\xi(L^2_v)$ of*

$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M}w = \tilde h_w \,,$$

*satisfying*

$$\|w\|_{H^k_\xi(L^2_v)} \leq \frac{1}{\sigma}\|\tilde h_w\|_{H^k_\xi(L^2_v)} \,,$$

*with $\sigma$ as in Lemma 5.*

*Sketch of the proof.* Uniqueness and the stability estimate are obtained by testing the equation with $w$ and the $k$-th derivative of the equation with $\partial_\xi^k w$. Existence can be proven in several ways, one of which is the approximation by a discrete velocity system with a finite number of discrete velocities. This reduces the problem to an ODE system. Care has to be taken in order not to destroy the definiteness of $\mathcal{M}$ by the approximation. □

The final result on the linear problem can now be easily proven.

**Lemma 9.** *Let $h_z \in H_\xi^k$ and $h_w \in H_\xi^l(L_v^2)$, then there exists a unique solution $(z,w) \in H_\xi^{k+2} \times H_\xi^m(L_v^2)$, $m = \min\{k+1, l\}$, of (2.20), (2.21), (2.22), satisfying*

$$\|z\|_{H_\xi^{k+2}(L_v^2)} \le C(|z_0| + \|h_z\|_{H_\xi^k})\,, \qquad \|w\|_{H_\xi^m(L_v^2)} \le C(|z_0| + \|h_z\|_{H_\xi^k} + \|h_w\|_{H_\xi^l(L_v^2)})\,.$$

*Proof.* The only thing left to note is the estimate

$$\|A(z,z')\|_{H_\xi^{k+1}(L_v^2)} \le \|z\|_{H_\xi^{k+2}}\,,$$

whose proof is straightforward by the definition of $A$. □

2.4. **The Nonlinear Problem.** In this section we prove existence and uniqueness of solutions of the nonlinear problem (2.19), (2.14), subject to $z(0) = z_0$, in the spaces $H_\xi^3$ and $H_\xi^2(L_v^2)$, respectively. After the preparations in the previous sections, the proof is a straightforward contraction argument. We need, however, estimates for the right hand sides of (2.19) and (2.14). In the following, $C$ denotes (possibly different) $\varepsilon$-independent constants.

**Lemma 10.** (i) *The linear terms $B$ and $B^z$ satisfy the estimate*

$$\|B(\Phi z + \varepsilon w)\|_{H_\xi^2(L_v^2)} + \|B^z(z, z', w, \partial_\xi w)\|_{H_\xi^1} \le C(\|z\|_{H_\xi^2} + \|w\|_{H_\xi^2(L_v^2)})\,.$$

(ii) *The nonlinearities $R$ and $R^z$ are quadratic: Let $g_1, g_2 \in H_\xi^2(L_v^2)$, then*

$$\|R(g_1) - R(g_2)\|_{H_\xi^2(L_v^2)} + \|R^z(g_1, \partial_\xi g_1) - R^z(g_2, \partial_\xi g_2)\|_{H_\xi^1}$$
$$\le C\left(\|g_1\|_{H_\xi^2(L_v^2)} + \|g_2\|_{H_\xi^2(L_v^2)}\right)\|g_1 - g_2\|_{H_\xi^2(L_v^2)}\,.$$

*Proof.* The proof is straightforward. All that is needed for (ii) is the one-dimensional Sobolev embedding $H_\xi^1 \subset C_\xi^b$ and (1.28). □

According to the spaces of the solutions and inhomogeneities of the linear problem we define the norm

$$\|(z,w)\| := \|z\|_{H_\xi^3} + \varepsilon\|w\|_{H_\xi^2(L_v^2)} \tag{2.24}$$

Clearly, $\|g\|_{H_\xi^2(L_v^2)}$ is bounded from above by $\|(z,w)\|$.

Before stating the existence result for traveling waves we note that in terms of the original unknown $f_{TW} = f_{as} + \varepsilon^2 g$, the condition $z(0) = z_0$ reads

$$\int_V (v - \varepsilon s)^2 (f_{TW}(0,v) - f_{as}(0,v))dv = \varepsilon^2 D_1 z_0\,. \tag{2.25}$$

**Theorem 11.** *Let the wave speed satisfy $s \ge s_0$. For every $z_0 \in \mathbb{R}$ and for $\varepsilon$ small enough, there exists a solution $f_{TW}$ of (1.23) satisfying (2.25), which is unique in a ball $\{f : \|f - f_{as}\| \le \delta\}$, where the radius $\delta$ can be chosen independently from $\varepsilon$. It satisfies*

$$\|f_{TW} - f_{as}\|_{H_\xi^2(L_v^2)} = O(\varepsilon^2)\,,$$

*or, more precisely,*

$$f_{TW} = f_{as} + \varepsilon^2\Phi z + \varepsilon^3 w = Mu_{TW} - \varepsilon v M u'_{TW} + \varepsilon^2(v^2 - D)Mu''_{TW} + \varepsilon^2\Phi z + \varepsilon^3 w\,, \tag{2.26}$$

*where $u_{TW}$ satisfies (1.2), (1.3) with (2.1), and $\|z\|_{H^3_\xi}$ and $\|w\|_{H^2_\xi(L^2_v)}$ are uniformly bounded as $\varepsilon \to 0$.*

*Proof.* Let $\varepsilon$ be small enough. Then as a consequence of Lemma 10 (i), the solvability results for the above linear problem (2.20), (2.21) can be extended to the full linear problem

$$Dz'' + sz' + z(\bar\rho - 2\rho) = \varepsilon B^z(z, z', w, \partial_\xi w) + h_z\,,$$
$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M}w = A(z, z') + \varepsilon B(z, w) + h_w\,,$$

with inhomogeneities $h_z, h_w$ and $z(0) = z_0$. Applying the solution operator to the nonlinear problem (2.14), (2.19), we obtain a fixed point problem $(z, w) = \mathcal{G}(z, w)$, where the fix point operator is bounded by

$$\|\mathcal{G}(z, w)\| \le C_0(1 + \varepsilon^2\|(z, w)\|^2)\,.$$

The constant $C_0$ bounds the initial condition and the residual terms, and the nonlinear terms are of order $\varepsilon^2$. We see that for $\varepsilon$ small enough, $\mathcal{G}$ maps both the ball with radius $2C_0$ and the ball with radius $1/(2\varepsilon^2 C_0)$ into themselves. Also, with the property of the nonlinearity, the fixed point operator $\mathcal{G}$ is a contraction on a ball with radius of order $O(\varepsilon^{-2})$.

We can conclude that for $\varepsilon$ small enough, the fixed point problem has a solution $(z, w)$ with $\|(z, w)\| \le 2C_0$, which is unique in a ball with an $O(\varepsilon^{-2})$-radius. Knowing this and returning to the fixed point problem, the boundedness of $\|w\|_{H^2_\xi(L^2_v)}$ follows. □

We remark that the contraction argument above could also be carried out in $H^k_\xi(L^2_v)$ for any $k \in \mathbb{N}$, by using Lemma 9, so the existence result also holds in $H^k_\xi(L^2_v)$ for $k \in \mathbb{N}$.

## 3. Dynamic stability of traveling waves

In this section we prove the local asymptotic stability of traveling waves with speed $s > s_0$. For this purpose it is necessary to make the assumption

**H1.** The set of velocities $V$ is bounded, and we let $v_{max} := \sup_{v\in V} |v|$.

As for the macroscopic equation in Section 1.2, we restrict our attention to nonnegative solutions. This can be done by taking nonnegative initial data, since Theorem 3 guarantees the nonnegativity of the solution.

In the traveling wave variable (1.5) becomes

$$\varepsilon^2\partial_t f + \varepsilon(v - \varepsilon s)\partial_\xi f = M\rho_f - f + \varepsilon^2\rho_f(M\bar\rho - f)\,. \tag{3.1}$$

The traveling wave $f_{TW}(\xi, v)$ constructed in Theorem 11 becomes a stationary solution. We choose $z_0$ in (2.22) such that the shift of $\rho_{TW}$ is fixed to

$$\rho_{TW}(0) = \frac{3}{4}\bar\rho\,. \tag{3.2}$$

The initial datum $G_0(v, \xi)$ of the perturbation

$$G(t, v, \xi) = f(t, v, \xi) - f_{TW}(v, \xi)\,, \qquad \rho(t, \xi) := \rho_G(t, \xi)\,,$$

is assumed to satisfy $G_0 + f_{TW} \ge 0$ guaranteeing $G(t, \cdot, \cdot) + f_{TW} \ge 0$ for all $t \ge 0$ and, in particular, $\rho + \rho_{TW} \ge 0$. Then $G$ satisfies

$$\varepsilon^2\partial_t G + \varepsilon(v - \varepsilon s)\partial_\xi G = M\rho - G + \varepsilon^2(M\rho\bar\rho - (\rho_{TW} + \rho)G - \rho\, f_{TW})\,. \tag{3.3}$$

Before proceeding with the energy estimates we apply a micro-macro decomposition to $G$ as follows

$$G = M\rho + \varepsilon g\,, \quad \text{i.e.} \int_V g\,dv = 0\,, \quad \text{implying} \quad \|G\|_v^2 = \rho^2 + \varepsilon^2\|g\|_v^2\,. \tag{3.4}$$

Using (3.4), the scalar product of (3.3) with $G$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|G\|_{\xi,v}^2 + \|g\|_{\xi,v}^2 + \int_{\mathbb{R}} (2\rho_{TW} + \rho - \bar{\rho})\rho^2 d\xi + \varepsilon^2 \int_{\mathbb{R}} (\rho_{TW} + \rho)\|g\|_v^2 d\xi \\ &= -\varepsilon \int_{\mathbb{R}} \rho \langle f_{TW}, g\rangle_v d\xi \le \varepsilon^2 C(\|\rho\|_\xi^2 + \|g\|_{\xi,v}^2)\,,\end{aligned} \tag{3.5}$$

with an $\varepsilon$-independent constant $C$, here we have used $f_{TW} = M\rho_{TW} + O(\varepsilon)$. As in the purely macroscopic case, the integrand of the third term of (3.5) is negative as $\xi \to +\infty$, and we shall control it by combining (3.5) with an estimate on $L^2_{W,v}$.

We rewrite (3.3) in terms of $GW$,

$$\begin{aligned}\partial_t(GW) + \frac{1}{\varepsilon}(v - \varepsilon s)\partial_\xi(GW) - \frac{1}{\varepsilon}\frac{s}{2D}(v - \varepsilon s)GW \\ = -\frac{1}{\varepsilon} gW + (M\bar{\rho}\rho - (\rho_{TW} + \rho)G - \rho f_{TW})W\,,\end{aligned}$$

and perform the scalar product with $GW$, which gives the estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|G\|_{W,v}^2 + \|g\|_{W,v}^2 + \int_{\mathbb{R}} (\kappa + 2\rho_{TW} + \rho)\,(\rho W)^2 d\xi + \varepsilon^2 \int_{\mathbb{R}} (\rho_{TW} + \rho)W^2\|g\|_v^2 d\xi \\ &\le -\frac{s^2}{4D}\|\rho\|_W^2 + \frac{s}{D}\int_{\mathbb{R}} \rho W^2 \int_V vg\,dv\,d\xi + \varepsilon\frac{s}{2D}\int_{\mathbb{R}} \langle (v - \varepsilon s)g, g\rangle_v^2 W^2 d\xi \\ &\quad -\varepsilon \int_{\mathbb{R}} \rho\langle f_{TW}, g\rangle_v W^2 d\xi \\ &\le \frac{\kappa}{2}\|\rho\|_W^2 + \frac{s^2}{s^2 + 2D\kappa}\|g\|_{W,v}^2 + \varepsilon\frac{s}{2D}v_{max}\|g\|_{W,v}^2 + \varepsilon^2 C(\|\rho\|_W^2 + \|g\|_{W,v}^2)\,.\end{aligned} \tag{3.6}$$

In the last inequality we have used (3.4), the Young inequality and $\left(\int_V vg\,dv\right)^2 \le D\|g\|_v^2$ and also that $\kappa > 0$ and **H1**.

We take $\alpha > 0$ such that (1.19) holds, with $\rho_{TW}$ replaced by $\rho_{as} = u_{TW}$. Using that there is a constant $\bar{C}$ for which

$$\rho_{TW}(\xi) \ge \rho_{as}(\xi) - \varepsilon^2 \bar{C} \quad \text{for } \xi \in \mathbb{R}$$

(by the representation (2.26) in Theorem 3) then

$$\alpha\frac{W^2(\xi)}{2}(\rho_{TW}(\xi) + \varepsilon^2\bar{C}) - \bar{\rho} \ge \frac{\bar{\rho}}{16} \qquad \text{for } \xi \ge 0\,. \tag{3.7}$$

We now combine (3.5) and (3.6). Here we apply the obvious simplification to (3.6) and let

$$\gamma := 1 - s^2/(s^2 + 2D\kappa) > 0\,.$$

We also rearrange the macroscopic terms conveniently to show how they are controlled below:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|G\|_{\xi,v}^2 + \alpha\|G\|_{W,v}^2) + (1 - \varepsilon^2 C)\|g\|_{\xi,v}^2 + \alpha\,(\gamma - \varepsilon C)\,\|g\|_{W,v}^2 \\ &\quad + \int_{\mathbb{R}} \left(\frac{3\rho_{TW}}{2} + \rho - \bar{\rho} - \varepsilon^2 C\right)\rho^2 d\xi + \frac{\alpha}{2}\int_{\mathbb{R}} \rho_{TW}(\rho W)^2 d\xi \\ &+ \frac{1}{2}\int_{\mathbb{R}} \rho_{TW}(1 + \alpha W^2)\rho^2 d\xi + \alpha\int_{\mathbb{R}} \left(\frac{\kappa}{2} - \varepsilon^2 C + (\rho_{TW} + \rho)\right)(\rho W)^2 d\xi \\ &\quad + \varepsilon^2 \int_{\mathbb{R}} (\rho_{TW} + \rho)\|g\|_v^2 (1 + \alpha W^2) d\xi \le 0\,.\end{aligned} \tag{3.8}$$

As in section 1.2, the sixth term will be needed to control terms coming from the estimates on the derivatives of $G$ and $GW$.

Using (1.15) we obtain

$$\int_{-\infty}^{0}\left(\frac{3\rho_{TW}}{2}-\bar\rho+\rho-\varepsilon^2C\right)\rho^2d\xi \ge \left(\frac{\bar\rho}{8}-\varepsilon^2C-\|\rho\|_\infty\right)\int_{-\infty}^{0}\rho^2d\xi\,. \tag{3.9}$$

Moreover, by (3.7),

$$\int_0^\infty\left(\frac{\alpha}{2}\rho_{TW}W^2-\bar\rho-\varepsilon^2C\right)\rho^2d\xi \ge \left(\frac{\bar\rho}{16}-\varepsilon^2C\right)\int_0^\infty\rho^2d\xi\;-\alpha\varepsilon^2\bar C\int_0^\infty(\rho W)^2d\xi\,. \tag{3.10}$$

We state the final estimate in the next lemma.

**Lemma 12.** *Let* **H1** *hold, let $\varepsilon>0$ be small enough, and let $f_{TW}$ be a traveling wave as constructed in Theorem 11. Let $G$ be a solution of (3.3) for initial data $G_0$ with $G_0+f_{TW}\ge 0$. Let $\alpha>0$ satisfy (3.7). Then there exists $C>0$ independent of $\varepsilon$ such that*

$$\begin{aligned}&\frac12\frac{d}{dt}(\|G\|^2_{\xi,v}+\alpha\|G\|^2_{W,v})+(1-\varepsilon^2C)\|g\|^2_{\xi,v}+\alpha\,(\gamma-\varepsilon C)\,\|g\|^2_{W,v}\\&+\left(\frac{\bar\rho}{8}-\varepsilon^2C-\|\rho\|_\infty\right)\int_{-\infty}^0\rho^2d\xi+\left(\frac{\bar\rho}{16}-\varepsilon^2C\right)\int_0^\infty\rho^2d\xi\\&+\frac12\int_{\mathbb R}\rho^2\rho_{TW}(1+\alpha W^2)d\xi+\alpha\int_{\mathbb R}\left(\frac{\kappa}{2}-\varepsilon^2C+(\rho_{TW}+\rho)\right)(\rho W)^2d\xi\\&+\varepsilon^2\int_{\mathbb R}(\rho_{TW}+\rho)\|g\|_v^2(1+\alpha W^2)d\xi\le 0\,.\end{aligned}$$

For controlling the nonlinearity we have to bound $\rho$ in $L^\infty(\mathbb R)$. By Sobolev embedding this can be done by controlling the $H^1(\mathbb R)$-norm. Thus we also derive estimates for the derivative in a similar procedure as above and denote

$$H=\partial_\xi G\,,\quad r=\partial_\xi\rho\,,\quad h=\partial_\xi g\,.$$

We start by differentiating (3.3):

$$\partial_tH+\frac1\varepsilon(v-\varepsilon s)\partial_\xi H=-\frac1\varepsilon h+M\bar\rho r-(\rho_{TW}+\rho)H-f_{TW}r-(\rho'_{TW}+r)G-\rho\partial_\xi f_{TW}\,, \tag{3.11}$$

yielding the estimate

$$\begin{aligned}&\frac12\frac{d}{dt}\|H\|^2_{\xi,v}+\|h\|^2_{\xi,v}+\int_{\mathbb R}(2(\rho_{TW}+\rho)-\bar\rho)r^2d\xi+\varepsilon^2\int_{\mathbb R}(\rho_{TW}+\rho)\|h\|_v^2d\xi\\&=\int_{\mathbb R}\rho''_{TW}\rho^2d\xi-\varepsilon^2\int_{\mathbb R}(\rho'_{TW}+r)\langle g,h\rangle_vd\xi-\varepsilon\int_{\mathbb R}\langle f_{TW},h\rangle_vrd\xi\\&\quad-\varepsilon\int_{\mathbb R}\langle\partial_\xi f_{TW},h\rangle_v\rho d\xi \qquad (3.12)\\&\le\frac{s^2}{2D^2}\int_{\mathbb R}\rho_{TW}\rho^2d\xi+\varepsilon^2C\left(\|\rho\|^2_{H^1_\xi}+\|g\|^2_{H^1_\xi(L^2_v)}\right)+\varepsilon^2\|g\|_{\infty,v}\left(\|r\|^2_\xi+\|h\|^2_{\xi,v}\right)\,.\end{aligned}$$

Here we have used the Cauchy-Schwarz inequality, $f_{TW}=M\rho_{TW}+O(\varepsilon)$, and the fact that $\rho_{TW}-\rho_{as}=O(\varepsilon^2)$, implying $D\rho''_{TW}=-s\rho'_{TW}-\rho_{TW}(\bar\rho-\rho_{TW})+O(\varepsilon^2)\le\frac{s^2}{2D}\rho_{TW}+O(\varepsilon^2)$, see (1.6). We now write (3.11) in terms of $HW$,

$$\begin{aligned}&\partial_t(HW)+\frac1\varepsilon(v-\varepsilon s)\partial_\xi(HW)-\frac1\varepsilon\frac{s}{2D}(v-\varepsilon s)HW+\frac1\varepsilon hW\\&=(M\bar\rho r-(\rho_{TW}+\rho)H-f_{TW}r-(\rho'_{TW}+r)G-\rho\partial_\xi f_{TW})W\,,\end{aligned}$$

and compute the scalar product with $HW$. Treating the terms on the right hand side similarly to (3.12) and the transport terms analogously to (3.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|H\|_{W,v}^2 + \left(\gamma - \varepsilon\frac{s}{2D}v_{max}\right)\|h\|_{W,v}^2 + \int_{\mathbb{R}}\left(\frac{\kappa}{2} + 2(\rho_{TW}+\rho)\right)(rW)^2 d\xi \\
&\qquad + \varepsilon^2\int_{\mathbb{R}}(\rho_{TW}+\rho)\|h\|_v^2 W^2 d\xi \\
&\le \varepsilon^2 C\left(\|\rho\|_{H_W^1}^2 + \|g\|_{H_W^1(L_v^2)}^2\right) + \varepsilon^2\|g\|_{\infty,v}\left(\|r\|_W^2 + \|h\|_{W,v}^2\right)\,,
\end{aligned}$$

where we have also used

$$-2\int \rho_{TW}'\rho\, rW^2 dx = \int(\rho_{TW}'' + \frac{s}{D}\rho_{TW}')\rho^2 W^2 d\xi \le \varepsilon^2 C\|\rho\|_W^2$$

since $D\rho_{TW}'' + s\rho_{TW}' = -u_{TW}(\bar\rho - u_{TW}) + O(\varepsilon^2) \le C\varepsilon^2$. We state the combined estimate for $H$ and $HW$ in a lemma.

**Lemma 13.** *Let the assumptions of Lemma 12 hold and let $\alpha > 0$. Then*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|H\|_{\xi,v}^2 + \alpha\|H\|_{W,v}^2\right) + \left(1 - \varepsilon^2(C + \|g\|_{\infty,v})\right)\|h\|_{\xi,v}^2 \\
&\quad + \alpha\left(\gamma - \varepsilon(C + \varepsilon\|g\|_{\infty,v})\right)\|h\|_{W,v}^2 + \varepsilon^2\int_{\mathbb{R}}(\rho_{TW}+\rho)\|h\|_v^2\left(1+\alpha W^2\right)d\xi \\
&\quad + 2\int_{\mathbb{R}}(\rho_{TW}+\rho)r^2\left(1+\alpha W^2\right)d\xi - \left(\bar\rho + \varepsilon^2(C + \|g\|_{\infty,v}\right)\|r\|_\xi^2 \\
&\quad + \alpha\left(\frac{\kappa}{2} - \varepsilon^2(C + \|g\|_{\infty,v})\right)\|r\|_W^2 \\
&\le \frac{s^2}{2D^2}\int_{\mathbb{R}}\rho_{TW}\rho^2 d\xi + \varepsilon^2 C\left(\|\rho\|_{\xi,v}^2 + \alpha\|\rho\|_{W,v}^2 + \|g\|_{\xi,v}^2 + \alpha\|g\|_{W,v}^2\right)\,.
\end{aligned}$$

We still need to control $-\bar\rho\|r\|_\xi^2$. In the purely macroscopic case we used the diffusion to control this term, but this is not directly available from the transport term in the kinetic equation. One can now proceed in two different ways. For the shortest way one can use the positivity of $\kappa$ instead. This would however result in requiring that the perturbation is initially bounded by a multiple of $\kappa$, which is very restrictive for $\kappa$ small. We rather perform a Chapman-Enskog approximation from which we can recover the diffusion term in the macroscopic part of the equation.

We split (3.3) into two equations that are, to leading order, an equation for $\rho$ and the other for $g$. Integrating (3.3) we obtain

$$\partial_t\rho - sr + (2\rho_{TW} + \rho - \bar\rho)\rho = -\int_V vh\, dv\,. \tag{3.13}$$

Applying $-\mathcal{L}$ to (3.3) gives the equation for $g$:

$$\varepsilon^2\partial_t g + vMr - \varepsilon M\int_V vh\, dv + \varepsilon(v - \varepsilon s)h + g = \varepsilon\rho\mathcal{L}f_{TW} - \varepsilon^2(\rho_{TW}+\rho)g\,.$$

We compute

$$-\int_V vh\, dv = D\partial_\xi r + \varepsilon^2\partial_t\int_V vh\, dv + \varepsilon\partial_\xi\mathcal{S}[g]\,,$$

where we denote

$$\mathcal{S}[g] := \int_V v(v - \varepsilon s)h\, dv + \varepsilon\left(\frac{\rho}{\varepsilon}\int_V vf_{TW}\, dv + (\rho_{TW}+\rho)\int_V vg\, dv\right)\,.$$

In the macroscopic equation we now recover the diffusion:

$$\partial_t\rho - D\partial_\xi r - sr + (2\rho_{TW} + \rho - \bar\rho)\rho = \varepsilon^2\partial_t\int_V vh\, dv + \varepsilon\partial_\xi\mathcal{S}[g]\,. \tag{3.14}$$

Testing this equation with $\rho$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\rho\|_\xi^2 + D\|r\|_\xi^2 + \int_{\mathbb{R}}(2\rho_{TW}+\rho-\bar\rho)\rho^2 d\xi = \varepsilon^2\frac{d}{dt}\int_{\mathbb{R}}\rho\int_V vh\,dvd\xi \\ &-\varepsilon^2\int_{\mathbb{R}}\left(sr-(2\rho_{TW}+\rho-\bar\rho)\rho-\int_V vh\,dv\right)\int_V vh\,dv\,d\xi - \varepsilon\int_{\mathbb{R}}\mathcal{S}[g]rd\xi\,,\end{aligned} \tag{3.15}$$

where we have substituted the expression for $\partial_t\rho$ given by according to (3.13) and integrated by parts to obtain the first term on the right hand side. Applying the Young inequality we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\rho\|_\xi^2 - \varepsilon^2\int_{\mathbb{R}}\rho\int_V vh\,dv\,d\xi\right) + D\|r\|_\xi^2 + \int_{\mathbb{R}}(2\rho_{TW}+\rho-\bar\rho)\rho^2 d\xi \\ &\le \frac{D}{4}\|r\|_\xi^2 + \varepsilon^2 C\left(\|g\|^2_{H^1_\xi(L^2_v)} + \|\rho\|^2_{H^1_\xi}\right) \\ &\quad +\varepsilon\int_{\mathbb{R}}(\rho_{TW}+\rho)\left(\rho^2+r^2+\varepsilon^2 C(\|g\|_v^2+\|h\|_v^2)\right)d\xi\,.\end{aligned}$$

As before we rewrite (3.14) in terms of $\rho W$:

$$\partial_t(\rho W) - D\partial_\xi^2(\rho W) + (\kappa + 2\rho_{TW}+\rho)\rho W = \varepsilon^2\partial_t\int_V vhW\,dv + \varepsilon\partial_\xi\mathcal{S}[g]W\,.$$

And the scalar product with $\rho W$ gives

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\rho\|_W^2 - \varepsilon^2\int_{\mathbb{R}}\rho W^2\int_V vh\,dv\,d\xi\right) + \frac{3D}{4}\|\partial_\xi(\rho W)\|_\xi^2 \\ &\quad + \int_{\mathbb{R}}(\kappa+2\rho_{TW}+\rho)(\rho W)^2 d\xi \\ &\le \varepsilon C(\|h\|^2_{W,v}+\|\rho\|_W^2) + \varepsilon^2 C\left(\|g\|^2_{H^1_W(L^2_v)} + \|\rho\|^2_{H^1_W}\right) \\ &\quad +\varepsilon\int_{\mathbb{R}}(\rho_{TW}+\rho)\left(\rho^2+r^2+\varepsilon^2 C(\|g\|_v^2+\|h\|_v^2)\right)W^2 d\xi\,.\end{aligned} \tag{3.16}$$

We combine (3.16) and (3.16), treating the macroscopic part as in (3.9), (3.10), and obtain the following Lemma.

**Lemma 14.** *Let the assumptions of Lemma 12 hold, let $\alpha$ satisfy (3.7), and let $\varepsilon$ be small. Then*

$$\begin{aligned}&\frac{d}{dt}I + \left(\frac{3D}{4}-\varepsilon^2 C\right)(\|r\|_\xi^2 + \alpha\|\partial_\xi(\rho W)\|_\xi^2) + \left(\frac{\bar\rho}{8}-\varepsilon C - \|\rho\|_\infty\right)\int_{-\infty}^0\rho^2 d\xi \\ &\quad + \left(\frac{\bar\rho}{16}-\varepsilon C\right)\int_0^\infty \rho^2 d\xi + \alpha(\kappa-\varepsilon C)\|\rho\|_W^2 \\ &\le \varepsilon C\|h\|^2_{W,v} + \varepsilon^2 C\left(\|g\|^2_{H^1_\xi(L^2_v)} \quad + \alpha\|g\|^2_{H^1_\xi(L^2_v)}\right) \\ &\quad +\varepsilon\int_{\mathbb{R}}(\rho_{TW}+\rho)\left(r^2+\varepsilon^2 C\left(\|g\|_v^2+\|h\|_v^2\right)\right)(1+\alpha W^2)d\xi\,,\end{aligned}$$

*where*

$$I = \frac{1}{2}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2 - 2\varepsilon^2\int_{\mathbb{R}}\rho(1+\alpha W^2)\int_V vhdv\,d\xi\right)\,.$$

We are now ready to prove the main result of this section.

**Theorem 15.** *Let* **H1** *hold, let $f_{TW}$ be the traveling wave from Theorem 11 with speed $s > s_0$ made unique by (3.2), and let $\varepsilon$ be small. Let $f_0(v,\xi)$ satisfy*

$$0 \le f_0 \le \hat\rho M\,, \qquad \text{and} \qquad \|f_0 - f_{TW}\|_{H^1_\xi(L^2_v)} + \|f_0 - f_{TW}\|_{H^1_W(L^2_v)} \le \delta\,,$$

*for a $\delta > 0$ small enough, but independently of $\varepsilon$, and $\hat\rho > 0$.*

*Then the solution of (3.1) with initial datum $f_0$ satisfies*

$$\begin{aligned}&\|f-f_{TW}\|^2_{H^1_\xi(L^2_v)}(t)+\|f-f_{TW}\|^2_{H^1_W(L^2_v)}(t)\\&\quad\le Ce^{-\lambda t}(\|f_0-f_{TW}\|^2_{H^1_\xi(L^2_v)}+\|f_0-f_{TW}\|^2_{H^1_W(L^2_v)}),\end{aligned}$$

*with an exponential decay rate $\lambda>0$.*

**Proof.** Applying a standard contraction argument, one can show the well posedness of (3.3) in $H^1_\xi(L^2_v)\cap H^1_W(L^2_v)$ for initial data $G_0\in H^1_\xi(L^2_v)\cap H^1_W(L^2_v)$. Hence it only remains to derive the *a priori* estimate.

We construct a Lyapunov functional by combining the above estimates. We introduce

$$J(t)=I(t)+\frac{1}{2}\left(\|G\|^2_{\xi,v}+\alpha\|G\|^2_{W,v}+\beta(\|H\|^2_{\xi,v}+\alpha\|H\|^2_{W,v})\right),$$

where $\beta>0$ is determined below and $\alpha>0$ satisfies (3.7). The functional $J$ is bounded from above and below by

$$\alpha_*\left(\|G\|^2_{H^1_\xi(L^2_v)}+\|G\|^2_{H^1_W(L^2_v)}\right)\le J\le\alpha^*\left(\|G\|^2_{H^1_\xi(L^2_v)}+\|G\|^2_{H^1_W(L^2_v)}\right),$$

where $\alpha_*,\alpha^*>0$ are independent of $\kappa$ and $\varepsilon$, if the same is true for $\beta$. For $\varepsilon$ small enough, the estimate for $J$ reads

$$\begin{aligned}&\frac{d}{dt}J+2\left(\frac{\bar\rho}{16}-\|\rho\|_\infty\right)\int_{-\infty}^0\rho^2d\xi+\frac{\bar\rho}{16}\int_0^\infty\rho^2d\xi+\frac{\alpha\kappa}{4}\|\rho\|^2_W\\&+\left(\frac12-\beta\frac{s^2}{2D^2}\right)\int_{\mathbb R}\rho_{TW}\rho^2(1+\alpha W^2)d\xi+\left(\frac{D}{2}-\beta\bar\rho-\varepsilon^2\beta\|g\|_{\infty,v}\right)\|r\|^2_\xi\\&+\alpha\beta\left(\frac{\kappa}{4}-\varepsilon^2\|g\|_{\infty,v}\right)\|r\|^2_W+\frac12\|g\|^2_{\xi,v}+\alpha\frac{\gamma}{2}\|g\|^2_{W,v}\\&+\left(\frac{\beta}{2}-\varepsilon^2\|g\|_{\infty,v}\right)\|h\|^2_{\xi,v}+\alpha\beta\left(\frac{\gamma}{2}-\varepsilon^2\|g\|_{\infty,v}\right)\|h\|^2_{W,v}\le0.\end{aligned}$$

We choose $\beta=\min\{\frac{D}{4\bar\rho},\frac{D^2}{2s^2}\}$. By the Sobolev embedding

$$\|G\|_{\infty,v}\le\|G\|_{H^1_\xi(L^2_v)}\le\sqrt{\frac{J}{\alpha_*}},$$

such that in particular $\|\rho\|_\infty$ and $\varepsilon\|g\|_{\infty,v}$ are bounded by $\sqrt{J/\alpha_*}$. We now denote $L=\sqrt{J(0)/\alpha_*}$ and let the initial data be small enough such that

$$L<\frac{\bar\rho}{16},$$

which in particular implies $\varepsilon L<\min\{\kappa/4,\beta/2,\gamma/2,D/(4\beta)\}$ for $\varepsilon$ small enough. Then all coefficients above are positive initially and therefore $J$ is decreasing at $t=0$. Since in turn $J$ controls the coefficients, the functional $J$ decreases for all times. This in particular means that there exists a constant $\lambda>0$ such that

$$\frac{d}{dt}J\le-\lambda J,$$

and we get the exponential decay of $J$, which is equivalent to $\|G\|^2_{H^1_\xi(L^2_v)}+\|G\|^2_{H^1_W(L^2_v)}$.
□

## Appendix. The comparison principle approach to stability

Assuming additionally that the macroscopic density of the traveling wave profile remains between the far-field values 0 and $\bar\rho$ (which we know only up to error terms of order $O(\varepsilon^2)$), we can derive a comparison argument also for the kinetic profile, based on ideas of Golse [6].

**Lemma 16.** *Let $f_{TW}$ be a traveling wave solution of (1.23) constructed in Theorem 11 and let us assume additionally that $0 \le \rho_{TW} \le \bar\rho$. The $f_{TW}$ satisfies the maximum principle*

$$0 \le f_{TW}(\xi, v) \le M(v)\bar\rho\,, \quad \textit{for all } (\xi, v) \in \mathbb{R}\times V\,.$$

*Proof.* We rearrange terms in (1.23) and write it as

$$\varepsilon(v-\varepsilon s)\partial_\xi f_{TW} + (1+\varepsilon^2\rho_{TW})f_{TW} = (1+\varepsilon^2\bar\rho)M\rho_{TW} \ge 0\,.$$

We also observe that if we set $\bar f = M\bar\rho - f_{TW}$, then $\bar f$ satisfies

$$\varepsilon(v-\varepsilon s)\partial_\xi \bar f + (1+\varepsilon^2\rho_{TW})\bar f = M(\bar\rho-\rho_{TW}) \ge 0\,.$$

Thus, we only need to prove that for a given $g$, continuous in $\xi$, such that $g \to M\rho_\pm$ as $x \to \pm\infty$ at an exponential rate, for constants $\rho_\pm \ge 0$, then the inequality

$$\varepsilon(v-\varepsilon s)\partial_\xi g + (1+\varepsilon^2\rho_{TW})g \ge 0 \tag{A.1}$$

implies $g \ge 0$, where $\rho_{TW}$ is the macroscopic profile of a given traveling wave $f_{TW}$. The key to prove this is to rewrite (A.1) as

$$\varepsilon(v-\varepsilon s)e^{\left(-\xi-\varepsilon^2\int_{\xi_0}^{\xi}\rho_{TW}(y)\,dy\right)/(\varepsilon(v-\varepsilon s))}\partial_\xi\left(e^{\left(\xi+\varepsilon^2\int_{\xi_0}^{\xi}\rho_{TW}(y)\,dy\right)/(\varepsilon(v-\varepsilon s))}g\right) \ge 0$$

for a $\xi_0 \in \mathbb{R}$. Then the function

$$\xi \to e^{\left(\xi+\varepsilon^2\int_{\xi_0}^{\xi}\rho_{TW}(y)\,dy\right)/(\varepsilon(v-\varepsilon s))}g$$

is nondecreasing when $v-\varepsilon s>0$ and nonincreasing when $v-\varepsilon s<0$. Now, taking any sequence $\{\xi_n\}_n$ such that $\xi_n \to -\infty$ as $n\to\infty$, when $v-\varepsilon s>0$, implies that for all $\xi > \xi_n$,

$$e^{\left(\xi+\varepsilon^2\int_{\xi_0}^{\xi}\rho_{TW}(y)\,dy\right)/(\varepsilon(v-\varepsilon s))}g(\xi,v) \ge e^{\left(\xi_n+\varepsilon^2\int_{\xi_0}^{\xi_n}\rho_{TW}(y)\,dy\right)/(\varepsilon(v-\varepsilon s))}g(\xi_n,v) \to 0$$

as $n\to\infty$. Taking sequences $\xi_n\to\infty$ when $v-\varepsilon s<0$, gives that $g(\xi,v)\ge 0$ for all $\xi<\xi_n$, and the result follows. ☐

We indicate how a comparison principle can be used to control $\|\rho\|_\infty$, where $\rho$ is the perturbation about a traveling wave solution, in the integral estimates for both (1.1) and (1.5). This avoids getting estimates on the derivatives of the perturbation, simplifying the analysis for this particular model.

**Lemma 17.** *Let $f_{TW}$ be a traveling wave solution of (1.23) as in Lemma 16 and let $G$ satisfy (3.3). We assume for the initial data $G_0$ that*

$$0 \le f_{TW}(\xi,v) + G_0(\xi,v) \qquad \text{and} \qquad 0 \le f_{TW}(\xi,v) + c\,G_0(\xi,v)$$

*for a $c\in\mathbb{R}$. Then*

$$0 \le f_{TW}(\xi,v) + c\,G(t,\xi,v) \quad \text{for all } t \ge 0.$$

Proof. Let $h = f_{TW} + c\,G$, then

$$\varepsilon^2\partial_t h + (v - \varepsilon s)\partial_\xi h = M\rho_h - h + \varepsilon^2(M\bar{\rho}\rho_h - f_{TW}\rho_h - cG(\rho_{TW} + \rho)).$$

The terms can be rearranged such that

$$\begin{aligned}&\varepsilon^2\partial_t h + (v - \varepsilon s)\partial_\xi h + (1 + \varepsilon^2(\rho_{TW} + \rho))h \\ &= M\rho_h + \varepsilon^2(\rho_h(M\bar{\rho} - f_{TW}) + f_{TW}(\rho_{TW} + \rho)) \geq 0 \quad \text{if} \quad \rho_h \geq 0\,,\end{aligned}$$

where we have used Lemma 16 and Theorem 3 to obtain $f_{TW} + G \geq 0$ for all $t \geq 0$. Solving this equation by a fixed-point iteration gives the desired result $0 \leq h = f_{TW} + c\,G$ for all $t \geq 0$. □.

In particular for $c = 2$ we obtain

$$0 \leq \frac{\rho_{TW}}{2} + \rho, \qquad \text{if} \quad 0 \leq \frac{f_{TW}}{2} + G_0. \tag{A.2}$$

The corresponding comparison argument also holds for the purely macroscopic case.

**Lemma 18.** *Let $\rho_{TW}$ be a traveling wave solution of the KPP-Fisher equation (1.2) and let $\rho$ satisfy (1.11). We assume for the initial data $\rho_0$ that it is bounded and satisfies*

$$0 \leq \rho_{TW}(\xi) + \rho_0(\xi) \qquad \text{and} \qquad 0 \leq \rho_{TW}(\xi) + c\,\rho_0(\xi)$$

*for a $c \in \mathbb{R}$. Then*

$$0 \leq \rho_{TW}(\xi) + c\,\rho(t,\xi) \quad \text{for all } t \geq 0.$$

Proof. Let, as above, $\rho_h = \rho_{TW} + c\,\rho$, then

$$\partial_t\rho_h - D\partial_x^2\rho_h + a(x,t)\rho_h \geq 0,$$

where $a(x,t) = 2\rho_{TW} + \rho - \bar{\rho}$. Here we have used $\rho_{TW} + \rho \geq 0$. The function $a$ is bounded from below, and by standard arguments (see e.g. [9]) we can deduce $\rho_h \geq 0$. □

Let us now briefly discuss the integral estimates that give asymptotic stability. We start with the estimates for (1.1) and proceed as in Section 1.2. We derive the estimate in $L^2 \cap L^2_W$, without extracting the third term in (1.18), which reads

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) + D\left(\|\partial_\xi\rho\|_\xi^2 + \alpha\|\partial_\xi(\rho W)\|_\xi^2\right) \\ &+\alpha\int_{\mathbb{R}}(\kappa + \rho_{TW} + \rho)(\rho W)^2\,d\xi + \alpha\int_{\mathbb{R}}\rho_{TW}\,(\rho W)^2\,d\xi + \int_{\mathbb{R}}(2\rho_{TW} + \rho - \bar{\rho})\rho^2\,d\xi = 0\,.\end{aligned}$$

Assuming $\rho_{TW} + 2\rho_0 \geq 0$, Lemma 18 and (1.15) imply

$$2\rho_{TW} + \rho - \bar{\rho} \geq \frac{\bar{\rho}}{8} \qquad \text{on } (-\infty, 0]\,, \tag{A.3}$$

One then chooses $\alpha > 0$ such that $\alpha W^2\rho_{TW} - \bar{\rho} > 0$ on $[0, +\infty)$, and asymptotic stability can be deduced without estimating the derivatives.

We now argue for the kinetic equation (1.5). We follow the steps of Section 3 to conclude with the following (combined) estimate for a perturbation in $L^2_{\xi,v} \cap L^2_{W,v}$, again, without extracting the sixth term in (3.8),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|G\|_{\xi,v}^2 + \alpha\|G\|_{W,v}^2) + (1 - \varepsilon^2 C)\|g\|_{\xi,v}^2 + \alpha\,(\gamma - \varepsilon C)\,\|g\|_{W,v}^2 \\ &+ \int_{\mathbb{R}}\left(2\rho_\phi + \rho - \bar{\rho} - \varepsilon^2 C\right)\rho^2 d\xi + \frac{\alpha}{2}\int_{\mathbb{R}}\rho_\phi(\rho W)^2 d\xi \\ &+ \alpha\int_{\mathbb{R}}\Big(\frac{\kappa}{2} - \varepsilon^2 C + (\rho_{TW} + \rho)\Big)\,(\rho W)^2 d\xi \leq 0\,.\end{aligned}$$

Assuming $f_{TW} + 2G_0 \geq 0$ and again $0 \leq \rho_{TW} \leq \bar{\rho}$, we obtain (A.3) also for the macroscopic densities of the kinetic profile and perturbation. Now asymptotic stability follows from this estimate only, by requiring that $\varepsilon$ is small enough and by choosing $\alpha > 0$ such that $\alpha W^2 u_{TW} - \bar{\rho} > \beta\bar{\rho}$ on $[0, +\infty)$ for some $0 < \beta < 1$. It is remarkable that no further estimates are needed in this case.

## References

[1] M. Bisi and L. Desvillettes. From reactive Boltzmann equations to reaction-diffusion systems. *J. Stat. Phys.*, 125 (1):249–280, 2006.

[2] R. E. Caflisch and B. Nicolaenko. Shock profile solutions of the Boltzmann equation. *Comm. Math. Phys.*, 86:161–194, 1982.

[3] J. Canosa. On a nonlinear diffusion equation describing population growth. *IBM J. Res. Develop.*, 17:307–313, 1973.

[4] C. M. Cuesta and C. Schmeiser. Weak shocks for a one-dimensional BGK kinetic model for conservation laws. *SIAM J. Math. Anal.*, 38:637–656, 2006.

[5] R. A. Fisher. The wave of advance of advantageous genes, *Ann. Eugenics*, 7:353–369, 1937.

[6] F. Golse. Shock profiles for the Perthame-Tadmor kinetic model, *Comm. Partial Differential Equations*, 23:1857–1874, 1998.

[7] S. Hittmeir. *Nonlinear Waves in Kinetic Transport and Nonlocal Diffusion Equations,* PhD thesis, University of Vienna, 2009.

[8] K. Kirchgässner. On the nonlinear dynamics of travelling fronts. *J. Differ. Equations*, 96 (2):256–278, 1992.

[9] A. Kolmogorov, I. Petrovskii and N. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique. *Bull. Univ. Etat Moscou*, pages 1–25, 1937.

[10] D. H. Sattinger. On the stability of waves of nonlinear parabolic systems. *Advances in Math.*, 22:312–355, 1976.

[11] R. Spigler and D. H. Zanette. Asymptotic analysis and reaction-diffusion approximation for BGK kinetic models of chemical processes in multispecies gas mixtures. *Z. Angew. Math. Phys.*, 44 (5):812–827, 1993.

*E-mail address*: `carlota.cuesta@icmat.es`
*E-mail address*: `sabine.hittmeir@tuwien.ac.at`
*E-mail address*: `christian.schmeiser@univie.ac.at`