On nonlinear conservation laws with a nonlocal diffusion term

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Abstract

Scalar one-dimensional conservation laws with a nonlocal diffusion term corresponding to a Riesz-Feller differential operator are considered. Solvability results for the Cauchy problem in L^∞ are adapted from the case of a fractional derivative with homogeneous symbol. The main interest of this work is the investigation of smooth shock profiles. In case of a genuinely nonlinear smooth flux function we prove the existence of such travelling waves, which are monotone and satisfy the standard entropy condition. Moreover, the dynamic nonlinear stability of the travelling waves under small perturbations is proven, similarly to the case of the standard diffusive regularization, by constructing a Lyapunov functional.

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1. Introduction

We consider one-dimensional conservation laws for a density $u(t,x)$, $t>0$, $x\in\mathbb{R}$, of the form

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u\,, \tag{1}$$

where $\mathcal{D}^\alpha$ is the non-local operator

$$(\mathcal{D}^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^x \frac{u'(y)}{(x-y)^\alpha}dy\,, \tag{2}$$

with $0<\alpha<1$. The flux function $f(u)$ is smooth and satisfies $f(0)=0$.

We shall analyse the local and global solvability of the Cauchy problem for (1), as well as the existence and stability of travelling wave solutions. In particular, we shall show that smooth travelling wave solutions exist, which are asymptotically stable. These waves are shock profiles satisfying the standard entropy conditions like those derived from the standard parabolic regularization with $\mathcal{D}^\alpha$ replaced by ∂_x.

Since $\mathcal{D}^\alpha u$ can be written as the convolution of the derivative u' with $\Gamma(1-\alpha)^{-1}\theta(x)x^{-\alpha}$ (with the Heaviside function θ), $\mathcal{D}^\alpha$ is a pseudo-differential operator with symbol

$$\frac{ik\sqrt{2\pi}}{\Gamma(1-\alpha)}\mathcal{F}\left(\frac{\theta(x)}{x^\alpha}\right)(k) = ik\left(a_\alpha - ib_\alpha\, sgn(k)\right)|k|^{\alpha-1} = \left(b_\alpha + ia_\alpha\, sgn(k)\right)|k|^\alpha\,,$$

i.e. $\mathcal{F}(\mathcal{D}^\alpha u)(k) = (b_\alpha + ia_\alpha\, sgn(k))\,|k|^\alpha\widehat{u}(k)$. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}\varphi(k) = \widehat{\varphi}(k) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx}\varphi(x)dx\,,$$

and

$$a_\alpha = \sin\left(\frac{\alpha\pi}{2}\right) > 0\,, \qquad b_\alpha = \cos\left(\frac{\alpha\pi}{2}\right) > 0\,,$$

(see [2] for the details of the computation). Obviously, the operator on the right hand side of (1) also is a pseudo-differential operator with symbol

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha) = -\left(a_\alpha - ib_\alpha\, sgn(k)\right)|k|^{\alpha+1} . \tag{3}$$

Due to the negativity of its real part, it is dissipative.

Remark 1. *For $s \in \mathbb{R}$, we use the Sobolev space*

$$H^s := \{u :\, \|u\|_{H^s} < \infty\}, \qquad \|u\|_{H^s} := \|(1+|k|)^s \widehat{u}\|_{L^2(\mathbb{R})},$$

and the corresponding homogeneous norm

$$\|u\|_{\dot{H}^s} := \||k|^s \widehat{u}\|_{L^2(\mathbb{R})} .$$

The fact $\|\mathcal{D}^\alpha u\|_{\dot{H}^s} = \sqrt{a_\alpha^2 + b_\alpha^2}\, \|u\|_{\dot{H}^{s+\alpha}}$ justifies to interpret $\mathcal{D}^\alpha$ as a differentiation operator of order α. It is bounded as a map from H^s to $H^{s-\alpha}$.

Denoting by C_b^m, $m \geq 0$, the set of functions, whose derivatives up to order m are continuous and bounded on $\mathbb{R}$, $\mathcal{D}^\alpha u : C_b^1 \to C_b$ is bounded. This can be easily seen by splitting the domain of integration in (2) into $(-\infty, x-\delta]$ and $[x-\delta, x]$ for some positive $\delta > 0$. Then integration by parts in the first integral shows the boundedness of $\mathcal{D}^\alpha u$.

The operator $\partial_x \mathcal{D}^{1/3}$ occurs in applications. It has been derived as the physically correct viscosity term in two layer shallow water flows by performing formal asymptotic expansions associated to the triple-deck regularization used in fluid mechanics (see, e.g., [18]). Moreover $\mathcal{D}^{1/3}$ appears in the work of Fowler [12] in an equation for dune formation:

$$\partial_t u + \partial_x u^2 = \partial_x^2 u - \partial_x \mathcal{D}^{1/3} u . \tag{4}$$

Here the fractional derivative appears with the negative sign, but this instability is regularized by the second order derivative. Alibaud et al. showed the well-posedness of (4) in L^2 as well as the violation of the maximum principle, which is intuitive in the context of the application due to underlying erosions [1]. Travelling wave solutions of (4) have been analysed by Alvarez-Samaniego and Azerad in [2].

Fractal conservation laws of the form

$$\partial_t u + \partial_x f(u) = D^{\alpha+1} u , \tag{5}$$

where $D^{\alpha+1}$ is the pseudo-differential operator with symbol $-|k|^{\alpha+1}$ (meaning $D^{\alpha+1}u = \mathcal{F}^{-1}(-|k|^{\alpha+1}\hat{u})$) have been investigated in several works, see e.g. Biler et al. [5] and Droniou et al. [10].

This work is organized as follows. In the remainder of this section we present an existence result for the Cauchy problem in L^∞. The crucial property here is the nonnegativity of the semigroup generated by $\partial_x \mathcal{D}^\alpha$, which is a consequence of its interpretation as a Riesz-Feller derivative [11, 13]. This allows to prove a maximum principle for solutions of (1) as in [10].

Section 2 is devoted to the analysis of travelling wave solutions connecting different far-field values. Such wave profiles are typically smooth. Working with the original representation (2) of $\mathcal{D}^\alpha$, we obtain a nonlinear Volterra integral equation as the travelling wave version of (1). Assuming (even a bit less than) convexity of the flux function and that the solutions of the associated linear Volterra integral equation form a one-dimensional subspace of $H^2(\mathbb{R}_-)$, we can show the existence and uniqueness of monotone solutions satisfying the entropy condition for classical shock waves of the inviscid conservation law underlying (1). This essentially requires to extend the well known results for the existence of viscous shock profiles, which solve (local) ordinary differential equations.

Biler et al. [5] showed that no travelling wave solutions of (5) can exist for $\alpha \in (-1, 0]$. For the case $\alpha \in (0, 1)$ also no existence result is available.

To show the asymptotic stability of the travelling waves, we use the antiderivative method typically applied in the case of the classical viscous regularisation and derive a Lyapunov functional. This allows to deduce the decay of initially small perturbations.

In the appendix we consider linear Volterra integral equations and prove the assumption on the dimension of the solution space with respect to subspaces of $H^2(\mathbb{R}_-)$.

The Cauchy Problem

In the following, we verify the applicability of the work of Droniou et al. [10] on the Cauchy problem of (5) in L^∞ to

$$\partial_t u + \partial_x f(u) = \partial_x \mathcal{D}^\alpha u, \quad u(0,x) = u_0(x). \tag{6}$$

Applying the Fourier transform to the linear evolution equation $\partial_t u = \partial_x \mathcal{D}^\alpha u$, we see that the semigroup generated by the fractional derivative is formally given by the convolution with the kernel

$$K(t,x) = \mathcal{F}^{-1} e^{-\Lambda(k)t}(x), \quad \text{where } \Lambda(k) = (a_\alpha - i b_\alpha sgn(k))|k|^{\alpha+1}. \tag{7}$$

To analyse the well-posedness, we use the mild formulation of (6),

$$u(t,x) = K(t,.) * u_0(x) - \int_0^t K(t-\tau,.) * \partial_x f(u(\tau,.))(x) d\tau. \tag{8}$$

As a main ingredient in [10], Droniou et al. used the non-negativity of the kernel associated to the semigroup generated by $D^{\alpha+1}$. To make use of their methods in the analysis of the Cauchy problem (6), we need to investigate the properties of the kernel K associated to the operator $\partial_x \mathcal{D}^\alpha$.

Lemma 1. *For $0 < \alpha < 1$, the kernel K given by (7) is non-negative:*

$$K(t,x) \geq 0, \qquad \text{for all } \; t > 0, x \in \mathbb{R}.$$

Additionally, the kernel K satisfies the properties:

(i) For all $t > 0$ and $x \in \mathbb{R}$, $K(t,x) = \frac{1}{t^{1/(1+\alpha)}} K\left(1, \frac{x}{t^{1/(1+\alpha)}}\right)$.

(ii) For all $t > 0$, $\|K(t,.)\|_{L^1(\mathbb{R})} = 1$.

(iii) $K(t,x)$ is C^∞ on $(0,\infty) \times \mathbb{R}$ and for all $m \geq 0$ there exists a B_m such that

$$\forall (t,x) \in (0,\infty) \times \mathbb{R}, \qquad |\partial_x^m K(t,x)| \leq \frac{1}{t^{(1+m)/(1+\alpha)}} \frac{B_m}{(1 + t^{-2/(1+\alpha)}|x|^2)}. \tag{9}$$

(iv) There exists a C_0 such that for all $t > 0$: $\|\partial_x K(t,.)\|_{L^1(\mathbb{R})} = \frac{C_0}{t^{1/(1+\alpha)}}$.

Proof. We already mentioned that the operator $\partial_x \mathcal{D}^\alpha$ is a Riesz-Feller differential operator, see also Gorenflo and Mainardi [13]. Due to Feller [11], the symbol of $\partial_x \mathcal{D}^\alpha$ is the characteristic exponent of a random variable with Lévy stable distribution. Hence the kernel K is the scaled probability density function of a Lévy stable distribution and is non-negative.

The additional properties of the kernel K are verified in the same manner as in [10]: (i) follows from the change of variable $\eta = t^{1/(1+\alpha)}k$ under the integral sign. Since the kernel K is non-negative, we deduce $\|K(1,.)\|_{L^1(\mathbb{R})} = \int K(1,x)dx = \mathcal{F}(K(1,.))(0) = 1$, which together with (i) implies (ii). To show (iii), we write $\partial_x^m K(1,x) = \int (ik)^m e^{ikx} e^{-\Lambda(k)t} dk$. Since $\alpha > 0$, we can integrate by parts twice and obtain $\partial_x^m K(1,x) = O(1/x^2)$. Together with the boundedness of $\partial_x^m K(1,x)$, we get the estimate for $t = 1$ and deduce the estimate for arbitrary $t > 0$ from (i). Finally, (iv) follows from (i) and (iii). $\square$

Hence the kernel associated to $\partial_x \mathcal{D}^\alpha$ satisfies the same properties as the one for $D^{\alpha+1}$ required in the work of Droniou et al. [10]. Thus their analysis carries over to our problem and we obtain the analogous result:

Theorem 1. *If $u_0 \in L^\infty$, then there exists a unique solution $u \in L^\infty((0,\infty) \times \mathbb{R})$ of (6) satisfying the mild formulation (8) almost everywhere. In particular*

$$\|u(t,.)\|_\infty \leq \|u_0\|_\infty, \qquad \text{for } t > 0.$$

Moreover, the solution has the following properties:

1. $u \in C^\infty((0,\infty)\times\mathbb{R})$ *and* $u \in C_b^\infty((t_0,\infty)\times\mathbb{R})$ *for all* $t_0 > 0$.
2. u *satisfies equation* (1) *in the classical sense.*
3. $u(t) \to u_0$, *as* $t \to 0$, *in* $L^\infty(\mathbb{R})$ *weak-*$*$ *and in* $L^p_{loc}(\mathbb{R})$ *for all* $p \in [1,\infty)$.

To motivate the well-posedness, we estimate the terms in (8) for $t > 0$, with the help of the properties of the kernel K, as follows: $|K(t,.) * u_0(x)| \le \|u_0\|_\infty$ and

$$\left|\int_0^t \partial_x K(t-s,.) * f(u(s,.))ds\right| \le C\|f(u)\|_{L^\infty((0,t)\times\mathbb{R})} t^{1-\frac{1}{1+\alpha}} \,.$$

Due to the Lipschitz continuity of f, we get a contraction for small times t_0 on $L^\infty((0,t_0)\times\mathbb{R})$ and therefore the well-posedness.

To show the global existence as well as the maximum principle, Droniou et al. [10] constructed an approximate solution by a splitting method and used a compactness argument to pass to the limit.

We shall also mention that an alternative L^2-based existence theory of (1) can be obtained by standard approaches such as contraction arguments and Lyapunov functionals. Here the main ingredient is the a priori decay of the L^2-norm. Testing (1) with u and assuming vanishing far-field values of u, the flux term vanishes

$$\int_\mathbb{R} u\partial_x f(u)dx = \int_\mathbb{R} uf'(u)\partial_x u dx = \int_\mathbb{R} \partial_x G(u)dx = 0, \quad G(u) = \int_0^u vf'(v)dv,$$

since G is smooth and $G(0) = 0$. We obtain the L^2-estimate:

$$\frac{1}{2}\frac{d}{dt}\int_\mathbb{R} u^2 dx = -a_\alpha \int_\mathbb{R} |k|^{1+\alpha}|\hat{u}|^2 dk \le 0\,.$$

Here we have used Plancherel's theorem together with $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$, implying

$$\int_\mathbb{R} \mathrm{sgn} k|k|^j |\hat{u}(k,t)|^2 dk = 0\,.$$

This relation shows that in an L^2-framework the operator $\partial_x\mathcal{D}^\alpha$ behaves similarly to $D^{\alpha+1}$. Due to the decay of the L^2-norm of the solution to (1), one would hope for well-posedness of the Cauchy problem with initial data in L^2 allowing us to deduce the global existence. Using a contraction argument similar to the one by Dix for the classical viscous Burgers equation, we can show the well-posedness in L^2 for the quadratic flux $f(u) = u^2$ in the case $\alpha > 1/2$. This critical value was already mentioned by Biler, Funaki and Woyczynski [5] for (5). For the general flux and $\alpha \in (0,1)$ we have to require higher regularity of the initial data: $u_0 \in H^1$. To deduce global existence of solutions in H^1, a Lyapunov functional can be derived under an additional smallness assumption on $\|u_0\|_{H^1}$. These results follow from the proofs we carry out in Section 2.2. Since obviously the assumptions on the initial data are much more restrictive as in the L^∞-based existence result, we do not go into more details here.

2. Travelling wave solutions

2.1. Existence of travelling wave solutions

We introduce the travelling wave variable $\xi = x - st$ with the wave speed s and look for solutions $u(x,t) = u(\xi)$ of (1), which are connecting the different far-field values u_- and u_+. A straightforward calculation shows that if u depends on x and t only through the travelling wave variable ξ, then so does $\mathcal{D}^\alpha u$, and we arrive at

$$-su' + f(u)' = (\mathcal{D}^\alpha u)'\,, \qquad u(-\infty) = u_-\,, \quad u(\infty) = u_+\,,$$

where the prime denotes differentiation with respect to ξ. Integration gives the travelling wave equation

$$h(u) := -s(u-u_-) + f(u) - f(u_-) = \mathcal{D}^\alpha u = d_\alpha \int_0^\infty \frac{u'(\xi-y)}{y^\alpha} dy \,, \tag{10}$$

with $d_\alpha = 1/\Gamma(1-\alpha)$. If the derivative u' decays to zero fast enough as $\xi \to \pm\infty$, then we obtain, at least formally, the Rankine-Hugoniot conditions, which correspond to shock solutions of the inviscid conservation law and relate the far-field values and the wave speed via

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \,. \tag{11}$$

If the flux function $f(u)$ is convex between the far-field values u_- and u_+, then the left hand side $h(u)$ of (10) is negative between its zeroes u_- and u_+. If $u(\xi)$ is monotone, the right hand side in (10) has the same sign as u'. Therefore if a monotone solution exists, it has to be nonincreasing, leading to the standard entropy condition

$$u_- > u_+ \,,$$

derived by replacing $\mathcal{D}^\alpha u$ by u'. Under this assumption, the existence of a smooth monotone travelling wave will be proved. The precise assumptions on the flux function will be formulated in terms of $h(u)$: We require

$$\begin{aligned} & h \in C^\infty([u_+, u_-]) \,, \quad h(u_+) = h(u_-) = 0 \,, \quad h < 0 \text{ in } (u_+, u_-) \,, \\ & \exists\, u_m \in (u_+, u_-) \text{ such that } \quad h' < 0 \text{ in } (u_+, u_m) \,, \quad h' > 0 \text{ in } (u_m, u_-] \,. \end{aligned} \tag{12}$$

Note that this is a little less than asking for convexity of f, and it allows for the slightly weakened form $f'(u_+) \le s < f'(u_-)$ of the Lax entropy condition.

The integral operator

$$\mathcal{D}^\alpha u(\xi) = d_\alpha \int_{-\infty}^{\xi} \frac{u'(y)}{(\xi-y)^\alpha} dy$$

in the travelling wave problem

$$h(u) = \mathcal{D}^\alpha u \,, \qquad u(-\infty) = u_- \,, \quad u(\infty) = u_+ \,, \tag{13}$$

is of the Abel type. It is well known that it can be inverted by multiplying (13) with $(z-\xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z. This leads to

$$u(\xi) - u_- = \mathcal{D}^{-\alpha}(h(u))(\xi) := d_{1-\alpha} \int_{-\infty}^{\xi} \frac{h(u(y))}{(\xi-y)^{1-\alpha}} dy \,. \tag{14}$$

Equations (13) and (14) are equivalent if $u \in C_b^1(\mathbb{R})$ and $u' \in L^1(\mathbb{R}_-)$, hence in particular if $u \in C_b^1(\mathbb{R})$ is monotone. We will use both formulations to deduce the existence result. An important property of both integral equations is their translation invariance, which will be used several times below.

The equation (14) is a nonlinear Volterra integral equation with a locally integrable kernel, where a well developed theory exists for problems on bounded intervals. Therefore we shall start our investigations by proving a 'local' existence result around $\xi = -\infty$. The subsequent monotonicity and boundedness results will lead to global existence for $\xi \in \mathbb{R}$.

The local existence result is based on linearisation at $\xi = -\infty$ (or, equivalently, at $u = u_-$). This can be done for either (13) or (14) with the same result. As could be expected for ordinary differential equations, the linearisations

$$h'(u_-)v = \mathcal{D}^\alpha v \,, \qquad v = h'(u_-)\mathcal{D}^{-\alpha} v \,, \tag{15}$$

have solutions of the form $v(\xi) = be^{\lambda\xi}$, $b \in \mathbb{R}$, where a straightforward computation gives $\lambda = h'(u_-)^{1/\alpha}$, see also [6]. We will need that these are the only non-trivial solutions of (15) in the space $H^2(-\infty, \xi_0]$ for some $\xi_0 \le 0$. In particular, we assume that

$$\mathcal{N}\big(id - h'(u_-)\mathcal{D}^{-\alpha}\big) = \text{span}\{\exp(\lambda\xi)\} \qquad \text{with} \qquad \lambda = h'(u_-)^{1/\alpha}\,, \tag{16}$$

which is reasonable due to our analysis in the appendix Appendix A. The main result of this section is the following.

Theorem 2. *Let* (12) *and* (16) *hold. Then there exists a decreasing solution $u \in C_b^1(\mathbb{R})$ of the travelling wave problem* (13)*. It is unique (up to a shift) among all $u \in u_- + H^2((-\infty,0)) \cap C_b^1(\mathbb{R})$.*

The following local existence result shows that the nonlinear problem has, up to translations, only two nontrivial solutions, which can be approximated by $u_- \pm e^{\lambda\xi}$ for large negative ξ. The choice 1 of the modulus of the coefficient of the exponential is irrelevant due to the translation invariance of the solution.

Lemma 2. *(Local existence) Let* (16) *hold. Then, for every small enough $\varepsilon > 0$, the equation* (13) *has solutions $u_{up}, u_{down} \in u_- + H^2(I_\varepsilon)$, $I_\varepsilon = (-\infty, \xi_\varepsilon]$, $\xi_\varepsilon = \log\varepsilon/\lambda$, satisfying*

$$u_{up}(\xi_\varepsilon) = u_- + \varepsilon\,, \qquad u_{down}(\xi_\varepsilon) = u_- - \varepsilon\,. \tag{17}$$

These are unique among all functions u satisfying $\|u - u_-\|_{H^2(I_\varepsilon)} \le \delta$, with δ small enough, but independently from ε. They satisfy (with an ε-independent constant C)

$$\|u_{up} - u_- - e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \le C\varepsilon^2\,, \qquad \|u_{down} - u_- + e^{\lambda\xi}\|_{H^2(I_\varepsilon)} \le C\varepsilon^2\,.$$

Proof. The proof will only be given for existence and uniqueness of u_{down}, which will be of greater interest below, but the proof for u_{up} is analogous.

We start by writing (13) and the initial condition (17) in terms of the perturbation $\bar{u}(\xi) = u_{down}(\xi) - u_- + e^{\lambda\xi}$:

$$(\mathcal{D}^\alpha - h'(u_-))\bar{u} = h(u_- - e^{\lambda\xi} + \bar{u}) + h'(u_-)(e^{\lambda\xi} - \bar{u})\,, \quad \bar{u}(\xi_\varepsilon) = 0\,. \tag{18}$$

The idea is to write this as a fixed point problem considering the right hand side as given. Since we shall use the Fourier transform for constructing a particular solution, we need a smooth enough extension to $\xi \in \mathbb{R}$, although we are only interested in $\xi < \xi_\varepsilon$. For $f \in H^2(I_\varepsilon)$, let the extension $\mathcal{E}(f) \in H^2(\mathbb{R})$ satisfy

$$\mathcal{E}(f)\,\Big|_{I_\varepsilon} = f\,, \quad \|\mathcal{E}(f)\|_{H^2(\mathbb{R})} \le \gamma \|f\|_{H^2(I_\varepsilon)}\,.$$

The bounded solution of the equation

$$(\mathcal{D}^\alpha - h'(u_-))u_{part} = \mathcal{E}(f)\,,$$

and of its derivatives with respect to ξ can be written as

$$u_{part}^{(m)} = \mathcal{F}^{-1}\left[\left(b_\alpha|k|^\alpha - h'(u_-) + ia_\alpha \text{sgn}(k)|k|^\alpha\right)^{-1} \mathcal{F}\mathcal{E}(f)^{(m)}\right]\,, \quad m = 0,1,2\,.$$

The coefficient can easily be seen to be bounded uniformly in k, leading to the estimate

$$\|u_{part}\|_{H^2(I_\varepsilon)} \le \|u_{part}\|_{H^2(\mathbb{R})} \le C\|\mathcal{E}(f)\|_{H^2(\mathbb{R})} \le C\gamma\|f\|_{H^2(I_\varepsilon)}\,.$$

By the assumption (16), $U[f](\xi) = u_{part}(\xi) - u_{part}(\xi_\varepsilon)e^{\lambda(\xi-\xi_\varepsilon)}$ is the unique solution of

$$(\mathcal{D}^\alpha - h'(u_-))U = f \quad \text{in } I_\varepsilon\,, \qquad U(\xi_\varepsilon) = 0\,.$$

This allows to write (18) as a fixed point problem:

$$\bar{u} = U\left[h(u_- - e^{\lambda\xi} + \bar{u}) + h'(u_-)(e^{\lambda\xi} - \bar{u})\right]\,.$$

The right hand side of (18) can be written as

$$\frac{h''(\tilde u)}{2}\left(e^{\lambda\xi}-\bar u\right)^2 = \frac{h''(\tilde u)}{2}\left(\varepsilon^2 e^{2\lambda(\xi-\xi_\varepsilon)} - 2\varepsilon e^{\lambda(\xi-\xi_\varepsilon)}\bar u + \bar u^2\right).$$

Using the continuous imbedding of $H^2(I_\varepsilon)$ in $C_b(I_\varepsilon)$, it can easily be shown that

$$\begin{aligned}&\left\| h(u_- - e^{\lambda\xi} + \bar u) + h'(u_-)(e^{\lambda\xi} - \bar u)\right\|_{H^2(I_\varepsilon)} \\ &\quad \le L(\|\bar u\|_{H^2(I_\varepsilon)})\left(\varepsilon^2 + \varepsilon\|\bar u\|_{H^2(I_\varepsilon)} + \|\bar u\|^2_{H^2(I_\varepsilon)}\right),\end{aligned}$$

where L is a positive nondecreasing function. Now it is easily seen that the fixed point map is a contraction in (independently of ε) small enough balls and that it maps a ball with an $O(\varepsilon^2)$ radius into itself. □

Lemma 3. *(Local monotonicity) Let the assumptions of Lemma 2 hold. Then, in I_ε,*

$$u_{up} > u_-\,, \quad u'_{up} > 0\,, \qquad u_{down} < u_-\,, \quad u'_{down} < 0\,.$$

Proof. Again we restrict our attention to u_{down} and skip the analogous proof for u_{up}. As a consequence of Lemma 2 and of Sobolev imbedding

$$|u_{down}(\xi) - u_- + e^{\lambda\xi}| \le C\varepsilon^2\,, \quad \xi \le \xi_\varepsilon\,.$$

Thus, there exists ξ^* satisfying

$$u_{down}(\xi^*) = u_- - 2C\varepsilon^2\,, \qquad \xi_{C\varepsilon^2} \le \xi^* \le \xi_{3C\varepsilon^2}\,.$$

Since $u_{down}(\xi) < u_-$ for $\xi \ge \xi^*$, we may restrict our attention in the following to $\xi \le \xi^*$. Thus, we eliminated a subinterval of length $d_1 \ge \xi_\varepsilon - \xi_{3C\varepsilon^2}$. Now we set $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 2C\varepsilon_1^2$, and, by a shift in ξ, replace ξ^* by ξ_{ε_2}. This means that the shifted solution becomes the unique u_{down} from Lemma 2, where ε_1 has been replaced by ε_2. Of course, the argument can be iterated to produce a sequence $\{\varepsilon_n\}$, determined by $\varepsilon_{n+1} = 2C\varepsilon_n^2$, and in each step a subinterval of length $d_n \ge \xi_{\varepsilon_n} - \xi_{3C\varepsilon_n^2}$ can be eliminated, where $u_{down} < u_-$ holds. It is easily seen that, for $\varepsilon_1 = \varepsilon$ small enough, $\sum_{n=1}^\infty d_n = \infty$ completing the proof of $u_{down} < u_-$ in I_ε.

The proof of the second property of u_{down} is completely analogous noting that, again by Sobolev imbedding,

$$|u'_{down}(\xi) + \lambda e^{\lambda\xi}| \le C\varepsilon^2 \quad \text{for } \xi \le \xi_\varepsilon\,.$$

□

Remark 2. *Together with $u_{up} - u_-, u_{down} - u_- \in L^2(I_\varepsilon)$, the result of the lemma implies*

$$\lim_{\xi\to-\infty} u_{up}(\xi) = \lim_{\xi\to-\infty} u_{down}(\xi) = u_-\,.$$

Together the two solutions constitute the 'unstable manifold' of the point u_-.

The Lemmata 2 and 3 show the existence of a solution u of (13), which satisfies $u \in C_b^1$ and is monotone. Thus u is also a solution of equation (14).

Lemma 4. *(Continuation principle) Let $u \in C_b^1((-\infty,\xi_0])$ be a (continuation of a) solution of (14) as constructed in Lemma 2. Then there exists a $\delta > 0$, such that it can be extended uniquely to $C_b^1((-\infty,\xi_0+\delta))$.*

Proof. Defining

$$f(\xi) = u_- + d_{1-\alpha}\int_{-\infty}^{\xi_0} \frac{h(u(y))}{(\xi-y)^{1-\alpha}}dy\,,$$

which can be considered as given and smooth by the assumptions, (14) can be written as

$$u(\xi) = f(\xi) + d_{1-\alpha} \int_{\xi_0}^{\xi} \frac{h(u(y))}{(\xi - y)^{1-\alpha}} dy \, .$$

Local existence of a smooth solution for ξ close to ξ_0 is a standard result for Volterra integral equations, see e.g. Linz [16]. □

It is now obvious that, as for ordinary differential equations, boundedness will be enough for global existence.

Lemma 5. *(Global uniqueness) Let* $u \in u_- + H^2((-\infty, \xi_0))$ *be a solution of* (14)*. Then, up to a shift in* ξ*, it is the continuation of* u_{up} *or of* u_{down}*, or* $u \equiv u_-$*.*

Proof. For every $\delta > 0$ there exists a $\xi^* \le \xi_0$, such that $\|u - u_-\|_{H^2((-\infty,\xi^*))} < \delta$, and therefore, by Sobolev imbedding, also $|u(\xi^*) - u_-| < \delta$. Choosing δ small enough, there are only the options $u(\xi^*) = u_-$ (implying $u \equiv u_-$) or $u(\xi^*) \neq u_-$ whence, by local uniqueness, u is up to a shift either equal to u_{up} or to u_{down}, depending on the sign of $u(\xi^*) - u_-$. □

This result already implies the uniqueness of the travelling wave, if it exists.

Lemma 6. *(Global monotonicity) Let* $u \in C_b^1(-\infty, \xi_0]$ *be (a continuation of) the solution* u_{down} *of* (14) *as constructed in Lemma 2. Then* u *is nonincreasing.*

Proof. We recall the properties of h given in (12). We shall use both formulations (13) and (14). First we prove that the derivative of u remains negative as long as $u \ge u_m$. Assume to the contrary that

$$u(\xi_*) \ge u_m \, , \quad u'(\xi_*) = 0 \, , \quad u' < 0 \text{ in } (-\infty, \xi_*) \, .$$

Then we obtain from the derivative of (14), evaluated at $\xi = \xi_*$, the contradiction

$$0 = u'(\xi_*) = d_{1-\alpha} \int_{-\infty}^{\xi_*} \frac{h'(u(y))u'(y)}{(\xi_* - y)^{1-\alpha}} dy < 0 \, .$$

Now we show that u cannot become increasing for $u < u_m$. Again, assume the contrary

$$u(\xi_*) < u_m \, , \quad u' > 0 \text{ in } (\xi_*, \xi_* + \delta) \, , \quad u' \le 0 \text{ in } (-\infty, \xi_*] \, ,$$

where we assume additionally that δ is small enough such that $u(\xi_* + \delta) < u_m$. This implies

$$\begin{aligned} \int_{-\infty}^{\xi_*+\delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy &= \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy + \int_{\xi_*}^{\xi_*+\delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy \\ &> \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy \, . \end{aligned}$$

But on the other hand we know

$$\begin{aligned} 0 &> h(u(\xi_* + \delta)) - h(u(\xi_*)) \\ &= d_\alpha \int_{-\infty}^{\xi_*+\delta} \frac{u'(y)}{(\xi_* + \delta - y)^\alpha} dy - d_\alpha \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy \, > \, 0 \, , \end{aligned}$$

leading again to a contradiction. Therefore u' cannot get positive. □

Lemma 7. *(Boundedness) Let* $u \in C_b^1(-\infty, \xi_0]$ *be (a continuation of) the solution* u_{down} *of* (14) *as constructed in Lemma 2. Then* $u_+ < u < u_-$*.*

Proof. Suppose the solution would reach the value u_+ in finite time, i.e. there exists a ξ_*, such that $u(\xi_*) = u_+$. Since u is nonincreasing and, by Lemma 3, strictly decreasing at least close to $\xi = -\infty$, we obtain the contradiction

$$0 = h(u_+) = d_\alpha \int_{-\infty}^{\xi_*} \frac{u'(y)}{(\xi_* - y)^\alpha} dy < 0 .$$

□

The proof of Theorem 2 is completed by proving $\lim_{\xi\to\infty} u(\xi) = u_+$. Assuming to the contrary $\lim_{\xi\to\infty} u(\xi) > u_+$, would imply $\lim_{\xi\to\infty} h(u(\xi)) < 0$. Then, however, $-\mathcal{D}^{-\alpha} h(u) = u_- - u$ would increase above all bounds, which is impossible by Lemma 7.

2.2. Asymptotic stability of travelling waves for convex fluxes

We change to the moving coordinate $\xi = x - st$ in (1),

$$\partial_t u + \partial_\xi (f(u) - su) = \partial_\xi \mathcal{D}^\alpha u , \tag{19}$$

and look for solutions of (19), which are small perturbations of travelling wave solutions and in particular share the same far-field values. Let $u_0(\xi)$ be an initial datum and $\phi(\xi)$ a travelling wave solution as constructed in the previous section, with the shift chosen such that

$$\int_{\mathbb{R}} (u_0(\xi) - \phi(\xi)) d\xi = 0 . \tag{20}$$

Due to the conservation property of the equation (19) we see that (formally)

$$\int_{\mathbb{R}} (u(t,\xi) - \phi(\xi)) d\xi = 0 , \qquad \text{for all } t \geq 0 .$$

The flux function will be assumed to be convex between the far-field values of the travelling wave, i.e.

$$f''(\phi(\xi)) \geq 0 , \qquad \text{for all } \xi \in \mathbb{R} .$$

The perturbation $U = u - \phi$ satisfies the equation

$$\partial_t U + \partial_\xi ((f'(\phi) - s)U) + \frac{1}{2} \partial_\xi \left(f''(\phi + \vartheta U) U^2 \right) = \partial_\xi \mathcal{D}^\alpha U , \tag{21}$$

for some $\vartheta \in (0,1)$. The aim is to show local stability of travelling waves, i.e. the decay of U for small initial perturbations $U_0 = u_0 - \phi$. Testing (21) with U, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}} f''(\phi) \phi' U^2 d\xi - \frac{1}{2} \int_{\mathbb{R}} f''(\phi + \vartheta U) U^2 \partial_\xi U \, d\xi \\ = -a_\alpha \|U\|_{\dot{H}^{(1+\alpha)/2}}^2 , \end{aligned} \tag{22}$$

where several integrations by parts have been carried out. Recalling $\phi' \leq 0$, we see that the second term has the unfavourable sign. As one would do for the conservation law with the classical viscous regularisation, we introduce the primitive of the perturbation:

$$W(t,\xi) = \int_{-\infty}^{\xi} U(t,\eta) d\eta , \qquad W_0(\xi) = \int_{-\infty}^{\xi} U_0(\eta) d\eta .$$

Integration of (21) gives the equation for W,

$$\partial_t W + (f'(\phi) - s) \partial_\xi W + \frac{1}{2} f''(\phi + \vartheta U)(\partial_\xi W)^2 = \partial_\xi \mathcal{D}^\alpha W , \tag{23}$$

which we test with W to obtain

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} f''(\phi)\phi' W^2 d\xi + \frac{1}{2}\int_{\mathbb{R}} f''(\phi+\vartheta U)(\partial_\xi W)^2 W\, d\xi$$
$$= -a_\alpha \|W\|^2_{\dot{H}^{(1+\alpha)/2}}\,. \tag{24}$$

This equation has the crucial property that the quadratic terms have the favour-able sign. From the cubic term (arising from the nonlinearity) we pull out the L^∞-norm of W (and of U if f'' is not constant), which we shall control by Sobolev imbedding.

Well-posedness of the perturbation equation

Before deriving decay estimates, we have to guarantee the well-posedness of the Cauchy problem for (23),

$$\partial_t W + (f'(\phi)-s)\partial_\xi W + \frac{1}{2}f''(\phi+\vartheta U)(\partial_\xi W)^2 = \partial_\xi \mathcal{D}^\alpha W\,, \quad W(0,x) = W_0(x). \tag{25}$$

Therefore we use a contraction argument. Assuming $f(u) = u^2$ and $\alpha > 1/2$ allows to estimate the nonlinearity in the fashion of Dix [7] implying the well-posedness in H^1. For the general flux and $\alpha \in (0,1)$ we have to require more regularity of the initial data, $W_0 \in H^2$.

We recall the definition (7) of the kernel K associated to the linear evolution equation and rewrite (25) in its mild formulation

$$W(t,x) = K(t,.) * W_0(x)$$
$$- \int_0^t K(t-\tau,.) * \left((f'(\phi)-s)U(\tau,.) + \frac{f''(\phi+\vartheta U))}{2}(U(\tau,.))^2\right)(x)d\tau. \tag{26}$$

Before proceeding with the contraction arguments, we note that for any $W_0 \in H^s$ we have $K(t,.)* W_0 \to W_0$ as $t \to 0$ in H^s. In particular, the integral

$$\|K(t,.) * W_0 - W_0\|^2_{H^s} = \int (1+|k|)^{2s} |e^{-\Lambda(k)t} - 1|^2 |\widehat{W}_0(k)|^2 dk$$

is bounded by $4\|W_0\|^2_{H^s}$ and we can apply the Dominated Convergence Theorem to pass to the limit under the integral sign. Moreover $\|K(t,.) * W_0\|_{H^s} \le \|W_0\|_{H^s}$.

Proposition 1. *Let $f(u) = u^2$ and $\alpha > \frac{1}{2}$. Then for any $W_0 \in H^1$ there exists a $T > 0$ such that* (25) *has a unique solution $W \in H^1$ for $t \in [0,T)$.*

Proof. Denoting the right hand side of (26) with $\mathcal{G}W$ the mild formulation gives a fixed point problem $W = \mathcal{G}W$. We note that $f'' = 2$ and briefly explain how to carry out the contraction argument. Let $T > 0$ and denote $\|W\|^*_{H^s} = \sup_{t\in[0,t_0]} \|W\|_{H^s}$. Applying Plancherel's Theorem we can bound the H^1 norm of $\mathcal{G}W$ by

$$\begin{aligned}
\|\mathcal{G}W\|^*_{H^1} &\le \|W_0\|_{H^1} + \int_0^T \left\|(1+|k|)e^{-\Lambda(k)(t-\tau)}\mathcal{F}((2\phi - s)U + U^2)(\tau,k)\right\|_{L^2} d\tau \\
&\le \|W_0\|_{H^1} + C\int_0^T \sup_{k\in\mathbb{R}} \left|(1+|k|)e^{-\Lambda(k)(t-\tau)}\right| \|U(\tau,.)\|_{L^2} d\tau \\
&\qquad + \int_0^T \left\|(1+|k|)e^{-\Lambda(k)(t-\tau)}\right\|_{L^2} \sup_{k\in\mathbb{R}} |(U(\tau,.)^2)^\frown| d\tau
\end{aligned}$$

Using Cauchy-Schwarz inequality it is easy to see that $\|(gh)^\frown\|_\infty \le \|g\|_{L^2}\|h\|_{L^2}$, hence $\sup_{k\in\mathbb{R}} |(U(\tau,.)^2)^\frown| \le {\|U\|^*_{L^2}}^2$. We then bound

$$\begin{aligned}
\sup_{k\in\mathbb{R}} \left|(1+|k|)e^{-\Lambda(k)(T-\tau)}\right| &\le 1 + \frac{\left\|ye^{-a_\alpha |y|^{\alpha+1}}\right\|_\infty}{(T-\tau)^{\frac{1}{1+\alpha}}} \\
&\le C\left(1 + (T-\tau)^{-\frac{1}{1+\alpha}}\right), \qquad (27) \\
\|(1+|k|)e^{-\Lambda(k)(T-\tau)}\|_{L^2} &\le C\left((T-\tau)^{-\frac{1}{2(1+\alpha)}} + (T-\tau)^{-\frac{3}{2(1+\alpha)}}\right),
\end{aligned}$$

where we have performed the substitution $k \mapsto k(t-\tau)^{\frac{1}{\alpha+1}}$ in the integrand. For $\alpha > 1/2$, the terms on the right hand side are integrable from 0 to T and the operator $\mathcal{G}$ is a contraction for small times T: There exists a constant $C_0 > 0$, such that

$$\|\mathcal{G}W\|^*_{H^1} \le C_0 \left(1 + (T + T^{1-\frac{1}{1+\alpha}})\|W\|^*_{H^1} + (T^{1-\frac{1}{2(1+\alpha)}} + T^{1-\frac{3}{2(\alpha+1)}})\|W\|^{*\,2}_{H^1}\right),$$

Then for T small enough, $\mathcal{G}$ maps the ball $B_{2C_0}(T) = \{W \in C([0,T], H^1) : \|W\|^*_{H^1} \le 2C_0\}$ into itself. With Banach's fixed point argument we can conclude the existence of a solution $W \in B_{2C_0}(T)$ of (26), which is therefore the solution of (25) on $[0,T)$. The uniqueness result is only local in B_{2C_0}. Hence let us now assume $W, V \in C([0,T], H^1)$ are two solutions of (26) and let $M = \max\{\|W\|^*_{H^1}, \|V\|^*_{H^1}\}$. Then $W - V$ solves a fixed point equation, where for a small enough $T_0 > 0$ the fixed point operator is again a contraction on $B_{2M}(T_0)$. Therefore $W = V$ on $[0, T_0]$. Repetition of this argument provides uniqueness on the whole time interval of existence. □

Proposition 2. *Let $W_0 \in H^2$. Then there exists a $T > 0$ such that the Cauchy problem* (25) *has a unique solution $W \in H^2$ for $t \in [0,T)$.*

Proof. We again consider the fix point operator $\mathcal{G}W$ associated to the right hand side of (26), where now f'' is not constant. This requires to pull out the L^∞-norm of U and therefore, by Sobolev-Imbedding, we shall control W in H^2. We estimate the nonlinearity as follows:

$$\begin{aligned}
&\left\| K(T-\tau,.) * f''(\phi + \vartheta U)U^2(\tau,.)\right\|_{H^2} \\
&\quad = \left\|(1+|k|)\widehat{K}\ \ (1+|k|)\mathcal{F}(f''(\phi+\vartheta U)U^2)\right\|_{L^2} \\
&\quad \le C\left(1 + (T-\tau)^{-\frac{1}{1+\alpha}}\right)\|f''(\phi+\vartheta U)U^2\|_{H^1} \\
&\quad \le L(\|U\|_{H^1})\|U\|^2_{H^1}\left(1 + (T-\tau)^{-\frac{1}{1+\alpha}}\right),
\end{aligned}$$

where we have used (27) and Sobolev Imbedding. L is a positive non-decreasing function. The linear terms are estimated in a similar fashion as above, such that for a $C_0 > 0$

$$\|\mathcal{G}W\|^*_{H^2} \le C_0 \left(1 + (T + T^{1-\frac{1}{1+\alpha}})\,(1 + L(\|W\|^*_{H^2})\|W\|^*_{H^2})\,\|W\|^*_{H^2}\right).$$

The proof can be concluded in a similar way as before. □

Global existence will be the consequence of the existence of a Lyapunov functional, which also allows to deduce the asymptotic stability of travelling waves. The Lyapunov functional is also easier to derive in the case of the Burgers flux. Mainly for pedagogical reasons we first derive the result in this simplified situation and then proceed with the stability for the general convex flux function.

Stability of travelling waves for the quadratic flux

Assuming $f(u) = u^2$ and $\alpha > 1/2$, the Cauchy problem for (23) is well-posed in H^1. Since $f'' = 2$, the nonlinear term in (22) vanishes. Therefore to derive the global existence as well as asymptotic stability it suffices to construct a Lyapunov-functional controlling the H^1-norm of W.

Theorem 3. *Let $f(u) = u^2$ and $\alpha > 1/2$. Let ϕ be a travelling wave solution as in Theorem 2, and let $u_0(\xi)$ be an initial datum for* (19)*, such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))d\eta$ satisfies $W_0 \in H^1$. If $\|W_0\|_{H^1}$ is small enough, the Cauchy problem for equation* (19) *with initial datum u_0 has a unique global solution converging to the travelling wave in the sense that*

$$\lim_{t\to\infty}\int_t^\infty \|u(\tau,\cdot) - \phi\|_{L^2} d\tau = 0\,.$$

Remark 3. *Note that the condition* (20)*, which can be translated to $W_0(\pm\infty) = 0$, is incorporated in the condition $W_0 \in H^1$.*

Proof. Equations (22) and (24) imply the estimates

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 \le -a_\alpha \|U\|_{\dot H^{(1+\alpha)/2}}^2 \,, \tag{28}$$

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \le -a_\alpha \|W\|_{\dot H^{(1+\alpha)/2}}^2 \,, \tag{29}$$

with $C_0 = \|\phi'\|_{L^\infty}$. We shall construct a Lyapunov functional by a linear combination of these estimates. For $\gamma > 0$, we denote $\gamma_* = \min\{1,\gamma\}$ and $\gamma^* = \max\{1,\gamma\}$. Then

$$J(t) = \frac{1}{2}\left(\|W\|_{L^2}^2 + \gamma\|U\|_{L^2}^2\right)$$

is bounded from above and below by

$$\frac{\gamma_*}{2}\|W\|_{H^1}^2 \le J \le \frac{\gamma^*}{2}\|W\|_{H^1}^2 \,. \tag{30}$$

The combined estimate reads

$$\frac{dJ}{dt} - \left(\gamma C_0 + \|W\|_{L^\infty}\right)\|W\|_{\dot H^1}^2 + a_\alpha\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma\|W\|_{\dot H^{(3+\alpha)/2}}^2\right) \le 0 \,.$$

The idea is to control the second term by the third, which seems plausible, since the interpolation inequality

$$\|W\|_{\dot H^1}^2 \le \|W\|_{\dot H^{(1+\alpha)/2}}^2 + \|W\|_{\dot H^{(3+\alpha)/2}}^2 \,, \tag{31}$$

holds as a consequence of $k^2 \le |k|^{1+\alpha} + |k|^{3+\alpha}$, $k \in \mathbb{R}$. The same inequality with k replaced by $k(a_\alpha/(2C_0))^{1/(1+\alpha)}$ implies

$$\gamma C_0\|W\|_{\dot H^1}^2 \le \frac{a_\alpha}{2}\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma\|W\|_{\dot H^{(3+\alpha)/2}}^2\right) \,,$$

with $\gamma = (a_\alpha/(2C_0))^{2/(1+\alpha)}$. For the term arising from the nonlinearity we use the consequence $\|W\|_{\dot H^1}^2 \le \frac{1}{\gamma_*}(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma\|W\|_{\dot H^{(3+\alpha)/2}}^2)$ of (31), which leads to

$$\frac{dJ}{dt} + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}\|W\|_{L^\infty}\right)\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma\|W\|_{\dot H^{(3+\alpha)/2}}^2\right) \le 0 \,.$$

By Sobolev imbedding and (30) we have

$$\|W\|_{L^\infty} \le \|W\|_{H^1} \le \sqrt{\frac{2}{\gamma_*}J} \,.$$

We now let the initial data be small enough such that $J(0) < (\gamma_*)^3 a_\alpha^2/8$. This immediately implies the existence of a $\lambda > 0$, such that

$$\frac{dJ}{dt} \le -\lambda\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma\|W\|_{\dot H^{(3+\alpha)/2}}^2\right) \le -\lambda\gamma_*\|U\|_{L^2}^2 \,, \qquad \text{for all } t > 0 \,.$$

Integration with respect to time concludes the proof. □

Stability for a general convex flux function

In contrary to the quadratic flux, now the nonlinearity in estimate (22) does not vanish:

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 - L(\|U\|_{L^\infty})\|U\|_{L^\infty}\|U\|_{\dot H^1}^2 \le -a_\alpha\|U\|_{\dot H^{(1+\alpha)/2}}^2 \,, \tag{32}$$

with a positive nondecreasing function L and, similarly to above, $C_0 = \|f''(\phi)\phi'\|_{L^\infty}/2$. The estimate for W reads

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2} - L(\|U\|_{L^\infty})\|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \le -a_\alpha\|W\|_{\dot H^{(1+\alpha)/2}}^2 \,, \tag{33}$$

We see that now we have to control U and W in $H^1 \subset L^\infty$, and therefore also need to derive an estimate for $\partial_\xi U$. As we have mentioned above, the Cauchy problem for (23) is well-posed in H^2. Hence the decay of W in H^2 is needed to repeat the local existence as well as to control the nonlinearities. We differentiate (21) and test it with $\partial_\xi U$. After several integrations by parts, we can estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_\xi U\|_{L^2}^2 - C_1\|U\|_{H^1}^2 - L(\|U\|_{L^\infty})\left(\|U\|_{L^\infty}\|\partial_\xi U\|_{L^2}^2 + \|\partial_\xi U\|_{L^3}^3\right)& \\ \leq -a_\alpha\|\partial_\xi U\|_{\dot H^{(1+\alpha)/2}}^2\,,& \qquad (34)\end{aligned}$$

where C_1 depends on the travelling wave and its derivatives up to order 2. We now apply a generalisation of the celebrated Gagliardo-Nirenberg inequalities (see e.g. [14]) to Sobolev spaces with fractional order, which was proven by Amann [4] (Proposition 4.1):

$$\|\partial_\xi U\|_{L^3}^3 \quad\leq\quad C\|\partial_\xi U\|_{H^{\frac{\alpha+1}{4}}}^2\|\partial_\xi U\|_{L^2} \leq C\|U\|_{H^1}\|U\|_{H^{\frac{5+\alpha}{4}}}^2 \qquad (35)$$

We are now ready to prove a stability result similar to Theorem 3 for the general convex flux function:

Theorem 4. *Let* (12) *hold and let ϕ be a travelling wave solution as in Theorem 2. Let u_0 be an initial datum for* (19) *such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))d\eta$ satisfies $W_0 \in H^2$. If $\|W_0\|_{H^2}$ is small enough, then the Cauchy problem for equation* (19) *with initial datum u_0 has a unique global solution converging to the travelling wave in the sense that*

$$\lim_{t\to\infty}\int_t^\infty \|u(\tau,\cdot) - \phi\|_{H^1}d\tau = 0\,.$$

Proof. We proceed similarly to above and define

$$J(t) = \frac{1}{2}(\|W\|_{L^2}^2 + \gamma_1\|U\|_{L^2}^2 + \gamma_2\|\partial_\xi U\|_{L^2}^2)\,,$$

with positive constants $\gamma_1, \gamma_2 > 0$. We denote $\gamma_* = \min\{1, \gamma_1, \gamma_2\}$ and $\gamma^* = \max\{1, \gamma_1, \gamma_2\}$. Then, as a functional of W, J is equivalent to the square of the H^2-norm. Combining (33), (32) and (34) together with (35) gives the complete estimate

$$\begin{aligned}&\frac{d}{dt}J + a_\alpha\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1\|W\|_{\dot H^{(3+\alpha)/2}}^2 + \gamma_2\|W\|_{\dot H^{(5+\alpha)/2}}^2\right)\\ &-\gamma_1 C_0\|U\|_{L^2}^2 - \gamma_2 C_1\|U\|_{H^1}^2 - L(\|W\|_{H^2})\|W\|_{H^2}\|U\|_{H^{(5+\alpha)/4}}^2 \leq 0\,.\end{aligned}$$

Similarly to above we now choose $\gamma_1, \gamma_2 > 0$ such that

$$\begin{aligned}&\gamma_1 C_0\|U\|_{L^2}^2 + \gamma_2 C_1\|U\|_{H^1}^2\\ &\quad\leq \frac{a_\alpha}{2}\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1\|W\|_{\dot H^{(3+\alpha)/2}}^2 + \gamma_2\|W\|_{\dot H^{(5+\alpha)/2}}^2\right),\end{aligned}$$

and get the final estimate

$$\begin{aligned}\frac{d}{dt}J + \left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}L(\|W\|_{H^2})\|W\|_{H^2}\right)\left(\|W\|_{\dot H^{(1+\alpha)/2}}^2 + \gamma_1\|W\|_{\dot H^{(3+\alpha)/2}}^2\right)&\\ +\gamma_2\left(\frac{a_\alpha}{2} - \frac{1}{\gamma_*}L(\|W\|_{H^2})\|W\|_{H^2}\right)\|W\|_{\dot H^{(5+\alpha)/2}}^2 \;\leq\; 0\,.&\end{aligned}$$

Letting again the initial data be such that $J(0)$ is small enough, we can deduce that J is nonincreasing for all times and moreover

$$\int_0^\infty \|U(t,\cdot)\|_{H^1}^2 dt < \infty\,.$$

$\square$

Appendix A. Linear Integral Equation

In this appendix we analyse the assumption (16) in more detail. We will show that all continuous and bounded solutions on $\mathbb{R}_-$ of the linear equation

$$v(\xi) = C_0 \int_{-\infty}^{\xi} \frac{v(y)}{(\xi - y)^{1-\alpha}} dy, \qquad v(-\infty) = 0, \qquad C_0 = h'(u_-)/\Gamma(\alpha), \tag{A.1}$$

are given by the one-parameter family $\{be^{\lambda\xi} : b \in \mathbb{R}\}$ with $\lambda = h'(u_-)^{1/\alpha}$. A proof for the space $C_b(\mathbb{R}_-)$ cannot be carried out directly, since the kernel is only locally integrable. Therefore we first derive the uniqueness result in the space of continuous functions with exponential decay as $\xi \to -\infty$. We also present a less direct, but more general approach, which gives a similar result for the underlying space $L^\infty(\mathbb{R}_-)$. In addition we show that no continuous solutions with polynomial decay can exist.

We start by analysing solutions of (A.1) in $C_b(-\infty, \xi_0]$ for a $\xi_0 < 0$. Since it is easier to work with integral operators acting on a finite domain, we perform the transformation

$$w(\eta) = u(\xi), \qquad \text{where} \quad \eta = -\frac{1}{\xi} \in [0, \eta_0], \quad \text{for an} \quad \eta_0 > 0,$$

leading to the following equation for w

$$w(\eta) = C_0\, \eta^{1-\alpha} \int_0^{\eta} \frac{w(s)}{(\eta - s)^{1-\alpha} s^{1+\alpha}} ds, \qquad w(0) = 0. \tag{A.2}$$

To prove that the only non-trivial solutions with exponential decay are $w(\eta) = be^{-\frac{\lambda}{\eta}}$, we adapt the approach of Wolfersdorf for another integral equation (see the Appendix in [20]):

Lemma 8. *All solutions of* (A.1) *within the space*

$$C_w(\mathbb{R}_-) = \{f \in C_b(\mathbb{R}_-) : \; f(\xi) = e^{\mu\xi} g(\xi) \;\text{ for a }\; 0 < \mu < \lambda, \;\text{ where } g \in C_b(\mathbb{R}_-)\}$$

are given by the one-parameter family $\{be^{\lambda\xi} : b \in \mathbb{R}\}$ *with* $\lambda = h'(u_-)^{1/\alpha}$.

Proof. Let $w(\eta) = e^{-\frac{\mu}{\eta}} z(\eta)$ be a solution of (A.2), where $0 < \mu < \lambda$. For $z \in C_b[0, \eta_0]$ we assume w.l.o.g. $z(0) = 0$ (otherwise we can shift some decay of the exponential function onto z). We shall show that $z = be^{-\frac{\lambda-\mu}{\eta}}$. Therefore we introduce

$$\phi(\eta) = z(\eta) - C_1 e^{-\frac{\lambda-\mu}{\eta}} \int_0^{\eta_0} z(s) ds, \qquad 1 = C_1 \int_0^{\eta_0} e^{-\frac{\lambda-\mu}{s}} ds$$

and note that $\phi(0) = 0$. Its primitive $\Phi(\xi) = \int_0^{\eta} \phi(s) ds$ satisfies $\Phi(0) = \Phi(\eta_0) = 0$. Due to Rolle's Theorem there exists an $\eta_1 > 0$ such that $\Phi'(\eta_1) = \phi(\eta_1) = 0$. If $\phi \equiv 0$, the proof is finished. Let now $\phi \neq 0$. W.l.o.g. we assume that $\eta_1 > 0$ is the smallest value with $\phi(\eta_1) = 0$ and that $\phi(\eta) \geq 0$ in $[0, \eta_1]$ with $\phi(\eta) > 0$ in (η_2, η_1) for an $\eta_2 \in [0, \eta_1)$. Then we obtain

$$\begin{aligned} z(\eta_1) &= C_0\, \eta_1^{1-\alpha} \int_0^{\eta_1} \frac{e^{\mu\left(\frac{1}{\eta_1} - \frac{1}{s}\right)} z(s)}{(\eta_1 - s)^{1-\alpha} s^{1+\alpha}} ds \\ &> \underbrace{C_0\, \eta_1^{1-\alpha} \int_0^{\eta_1} \frac{e^{\lambda\left(\frac{1}{\eta_1} - \frac{1}{s}\right)}}{(\eta_1 - s)^{1-\alpha} s^{1+\alpha}} ds}_{=1} \; C_1\, e^{-\frac{\lambda-\mu}{\eta_1}} \int_0^{\eta_0} z(s) ds \;=\; z(\eta_1), \end{aligned}$$

leading again to a contradiction, and thus $\phi \equiv 0$. ∎

We shall also mention a more general approach, which was introduced for integral equations of Fredholm type. A similar result to Lemma 8 with the underlying space being $L^\infty(\mathbb{R}_-)$, can also be deduced from results on the *Wiener-Hopf equation*, which has the standard form

$$W(\xi) - \int_0^\infty K(\xi - y)W(y)dy = 0, \qquad \xi \geq 0. \tag{A.3}$$

Wiener and Hopf related the Fredholm property of the associated operator in (A.3) to conditions on its symbol [19]. Krein extended the Wiener-Hopf method to equations with L^1-integrable kernels [15]. We only state the part of his result which we will use in the following:

Let $K \in L^1(\mathbb{R})$. If the symbol $a(z) := 1 - \sqrt{2\pi}\mathcal{F}(K)(z)$ is elliptic, i.e. $\inf_{z\in\mathbb{R}} |a(z)| > 0$, and the winding number of the curve $\{a_\mu(z) : z \in (-\infty, \infty)\}$ around the origin is a non-positive number r. Then equation (A.3) *has exactly $|r|$ linearly independent solutions in any of the Lebesgue spaces $L^p(\mathbb{R}_+)$, $1 \leq p \leq \infty$.*

Since the kernel in (A.1) is only locally integrable we introduce as above exponential weights, which will allow to apply this result.

For a generalization of the Wiener-Hopf method to other spaces than the Lebesgue ones, we refer to the work of Duduchava [9], in which also the Theorem of Krein is given more detailed.

Lemma 9. *All solutions of* (A.1) *within the space*

$$L^\infty_w(\mathbb{R}_-) = \{f \in L^\infty(\mathbb{R}_-) : \; f(\xi) = e^{\mu\xi}g(\xi) \text{ for a } \; 0 < \mu < \lambda \text{ and } g \in L^\infty(\mathbb{R}_-)\}$$

are given by the one-parameter family $\{be^{\lambda\xi} : b \in \mathbb{R}\}$ with $\lambda = h'(u_-)^{1/\alpha}$.

Proof. Consider solutions v of (A.1) of the form $v(\xi) = e^{\mu\xi}w(\xi)$ for some $0 < \mu < \lambda$ and $w \in L^\infty(\mathbb{R}_-)$. Setting $W(\xi) = w(-\xi)$ and $K(\xi) = e^{-\mu\xi}\theta(\xi)\xi^{\alpha-1}$, equation (A.1) becomes a Wiener Hopf equation in the form (A.3). The kernel K is integrable, since $\mu > 0$. Thus, to apply the result of Krein, it remains to investigate the properties of the symbol

$$\begin{aligned} a_\mu(z) &= 1 - \frac{h'(u_-)\sqrt{2\pi}}{\Gamma(\alpha)}\mathcal{F}\left(\frac{\theta(\xi)}{\xi^{1-\alpha}}\right)(z - i\mu) \;=\; 1 - h'(u_-)(\mu + iz)^{-\alpha} \\ &= 1 - h'(u_-)(\mu^2 + z^2)^{-\alpha/2}(\cos(\alpha\,\varphi_{\mu,z}) - i\sin(\alpha\,\varphi_{\mu,z}))\,, \end{aligned}$$

where $\varphi_{\mu,z} = \arctan\frac{z}{\mu}$ and $\frac{\sqrt{2\pi}}{\Gamma(\alpha)}\mathcal{F}\big(\frac{\theta(\xi)}{\xi^{1-\alpha}}\big)(z) = (iz)^{-\alpha}$ for $z \in \mathbb{C}$. To check the ellipticity of the symbol, rewrite $|a_\mu(z)|^2$ as follows

$$|a_\mu(z)|^2 = \big(1 - h'(u_-)(\mu^2 + z^2)^{-\alpha/2}\big)^2 + 2h'(u_-)(\mu^2 + z^2)^{-\alpha/2}\big(1 - \cos(\alpha\,\varphi_{\mu,z})\big),$$

which attains its minimum with respect to z at $z = 0$ and does not vanish if $0 < \mu < \lambda$. Thus the symbol a_μ is elliptic and forms a closed curve $\{a_\mu(z) : z \in (-\infty, \infty)\}$, since $a_\mu(\pm\infty) = 1$. Thus the winding number of the closed curve is a well-defined integer, which remains to be computed. We note that $Re(a_\mu)$ is an even function and $Re(a_\mu(0)) < 0$ for $0 < \mu < \lambda$. Moreover $Im(a_\mu)$ is an odd function and $Im(a_\mu(z)) = 0$ only if $z = 0$ or $z = \pm\infty$. Hence the parametrization of the closed curve runs once around the origin in the counter clockwise sense. Thus the winding number is -1 and the result of Krein implies the statement. □

Finally, we show that no bounded solutions with polynomial decay can exist.

Lemma 10. *(i) If $v \in C_b(\mathbb{R}_-)$ is a solution of* (A.1)*, then v cannot change the sign.*
(ii) Equation (A.1) has no solution $v \in C_b(\mathbb{R}_-)$ with polynomial decay as $\xi \to -\infty$.

Proof. Again it easier to consider equation (A.2) instead. Solutions cannot change sign due to the nonlocality: If a smooth solution w is positive (negative) on $(0, \eta_*)$ for some $\eta_* > 0$, then the solution remains positive (negative). In contrast, if $w = 0$ on $[0, \eta_*)$, then $w(\eta)$ is a solution of equation (A.2) where the integration starts at η_* instead of $s = 0$. Therefore, we avoid the singularity of the kernel at $s = 0$ and are left with the integrable singularity at $s = \eta$. Given the initial value $w(\eta_*) = 0$, we conclude from standard theory that there exists only the trivial solution.

We prove statement (ii) by contradiction. Suppose that there exists a solution with polynomial decay $w(\eta) = \eta^\beta z(\eta)$ for some $\beta > 0$ and $z \in C_b(-\infty, \eta_0]$ which satisfies w.l.o.g. $z(\eta) \geq z_* > 0$. Then

$$z(\eta) \geq z_* \, \frac{h'(u_-)}{\Gamma(\alpha)} \, \eta^{1-\alpha-\beta} \int_0^\eta \frac{1}{(\eta - s)^{1-\alpha} s^{1+\alpha-\beta}} ds = \frac{h'(u_-)}{\Gamma(\alpha)} \, z_* \, B(\alpha, \beta - \alpha) \, \eta^{-\alpha},$$

where B denotes the Beta function. We see that for any β the right hand side grows unbounded as $\eta \to 0$, which contradicts our assumption $z \in C_b(-\infty, \eta_0]$. □

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