Travelling waves for a non-local Korteweg-de Vries-Burgers equation

F. Achleitner[*], C. M. Cuesta[†], S. Hittmeir[*]

Abstract

We study travelling wave solutions of a Korteweg-de Vries-Burgers equation with a non-local diffusion term. This model equation arises in the analysis of a shallow water flow by performing formal asymptotic expansions associated to the triple-deck regularisation (which is an extension of classical boundary layer theory). The resulting non-local operator is of fractional type with order between 1 and 2. Travelling wave solutions are typically analysed in relation to shock formation in the full shallow water problem. We show rigorously the existence of these waves. In absence of the dispersive term, the existence of travelling waves and their monotonicity was established previously by two of the authors. In contrast, travelling waves of the non-local KdV-Burgers equation are not in general monotone, as is the case for the corresponding classical (or local) KdV-Burgers equation. This requires a more complicated existence proof compared to the previous work. Moreover, the travelling wave problem for the classical KdV-Burgers equation is usually analysed via a phase-plane analysis, which is not applicable here due to the presence of the non-local diffusion operator. Instead, we apply fractional calculus results available in the literature and a Lyapunov functional. In addition we discuss the monotonicity of the waves in terms of a control parameter and prove their dynamic stability in case they are monotone.

Keywords. non-local evolution equation, fractional derivative, travelling waves
Math.Subj.Class. 47J35, 26A33, 35C07

1 Introduction

In this paper we study existence and stability of travelling waves of the following one-dimensional evolution equation:

$$\partial_t u + \partial_x u^2 = \partial_x \mathcal{D}^\alpha u + \tau \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0 \tag{1.1}$$

with $\tau > 0$ and $\mathcal{D}^\alpha$ denotes the non-local operator

$$\mathcal{D}^\alpha u(x) = d_\alpha \int_{-\infty}^{x} \frac{u'(y)}{(x-y)^\alpha} dy\,, \quad \text{with} \quad 0 < \alpha < 1\,, \quad d_\alpha = \frac{1}{\Gamma(1-\alpha)} > 0\,, \tag{1.2}$$

where Γ denotes the Gamma function.

[*]Vienna University of Technology, Institute for Analysis and Scientific Computing, Wiedner Hauptstrasse 8-10, 1040 Wien, Austria. E-mail: franz.achleitner@tuwien.ac.at, sabine.hittmeir@tuwien.ac.at

[†]University of the Basque Country (UPV/EHU), Faculty of Science and Technology, Department of Mathematics, Aptdo. 644, 48080 Bilbao, Spain. E-mail: carlotamaria.cuesta@ehu.es

Equation (1.1) with $\alpha = 1/3$ and either a quadratic flux, as above, or a cubic one, has been derived from one (quadratic flux) and two (cubic flux) layer shallow water flows, respectively, by performing formal asymptotic expansions associated to the triple-deck (boundary layer) theory used in fluid mechanics (see, e.g. [12] and [19]). In [19] numerical simulations indicate the existence of travelling waves that resemble the inner structure in a very particular limit of small amplitude shock waves for the original shallow water problem. In this manuscript we aim to study rigorously the existence and stability of these type of solutions for the quadratic flux.

In [1] travelling waves for (1.1) with $\tau = 0$ were analysed. In this case travelling waves are monotone, as it is the case for the classical (or local) Burgers equation. The existence prove relies on this fact. However, travelling waves are in general non-monotone if τ is larger that certain value $\tau_0 > 0$ in the (local) KdV-Burgers equation, see e.g. [3] (this can be inferred by linearisation of the critical points of the resulting travelling wave equation, an ODE in the local case). Numerical computations performed in [19] and in[12] suggest that we may expect a similar oscillatory behaviour of the travelling waves of (1.1). This has an immediate implication that the present existence proof (with $\tau > 0$) differs significantly from the existence proof in [1] as we shall see below. On the other hand, and in contrast to the classical KdV-Burgers equation, the presence of the non-local operator in (1.1) does not allow to approach the problem using phase-plane analysis of the travelling wave equation, since this becomes a (non-linear) integro-differential equation.

Let us first recall some basic properties of the fractional differential operator $\mathcal{D}^\alpha u$. Since it can be written as the convolution of u' with $\theta(x)x^{-\alpha}/\Gamma(1-\alpha)$ (where θ is the Heaviside function), $\mathcal{D}^\alpha$ is a pseudo-differential operator with symbol

$$\frac{ik\sqrt{2\pi}}{\Gamma(1-\alpha)}\mathcal{F}\left(\frac{\theta(x)}{x^\alpha}\right)(k) = (b_\alpha + ia_\alpha \operatorname{sgn}(k))\,|k|^\alpha\,, \tag{1.3}$$

i.e. $\mathcal{F}(\mathcal{D}^\alpha u)(k) = (b_\alpha + ia_\alpha\, sgn(k))\,|k|^\alpha \hat{u}(k)$ where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}\varphi(k) = \hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx}\varphi(x)dx\,,$$

and the coefficients a_α and b_α are given by

$$a_\alpha = \sin\left(\frac{\alpha\pi}{2}\right) > 0\,, \qquad b_\alpha = \cos\left(\frac{\alpha\pi}{2}\right) > 0\,, \tag{1.4}$$

(we refer to [2] for the details of the computation to obtain (1.3)). The operator on the right-hand side of (1.1) then is a pseudo-differential operator with symbol

$$\mathcal{F}(\partial_x\mathcal{D}^\alpha) = -\,(a_\alpha - ib_\alpha\, sgn(k))\,|k|^{\alpha+1}\,, \tag{1.5}$$

which is dissipative in the sense that the real part of (1.5) is negative.

For $s \in \mathbb{R}$ we shall adopt the following notation for the Sobolev of square integrable functions,

$$H^s := \{u :\, \|u\|_{H^s} < \infty\}\,, \qquad \|u\|_{H^s} := \|(1+|k|^2)^{s/2}\hat{u}\|_{L^2(\mathbb{R})}\,,$$

and the corresponding homogeneous norm

$$\|u\|_{\dot{H}^s} := \||k|^s\hat{u}\|_{L^2(\mathbb{R})}\,.$$

Using that $(a_\alpha^2 + b_\alpha^2) = 1$ it is easy to see that $\|\mathcal{D}^\alpha u\|_{\dot{H}^s} = \|u\|_{\dot{H}^{s+\alpha}}$, and this suggests that one can interpret $\mathcal{D}^\alpha$ as a differentiation operator of order α. We also observe that $\mathcal{D}^\alpha$ is a bounded linear operator from H^s to $H^{s-\alpha}$.

We shall also let denote C_b^m with $m \geq 0$, the set of functions, whose derivatives up to order m are continuous and bounded. Then one can also infer that $\mathcal{D}^\alpha u$ is a bounded linear operator from $C_b^1(\mathbb{R})$ to $C_b(\mathbb{R})$. As explained in [1], this can be easily seen by splitting the domain of integration in (1.2) into $(-\infty, x-\delta]$ and $[x-\delta, x]$ for some positive $\delta > 0$. Then integration by parts in the first integral shows the boundedness of $\mathcal{D}^\alpha u$.

It is also known that $\mathcal{D}^\alpha$ can be inverted by multiplying it with $(z-\xi)^{-(1-\alpha)}$ and integrating with respect to ξ from $-\infty$ to z. Applying this to (1.2) we obtain:

$$\mathcal{I}^\alpha \mathcal{D}^\alpha (u(x)) = u(x) - \lim_{x \to -\infty} u(x) \,, \tag{1.6}$$

with the integral operator

$$\mathcal{I}^\alpha u(x) = d_{1-\alpha} \int_{-\infty}^{x} \frac{u(y)}{(x-y)^{1-\alpha}} dy \qquad u \in C_b^1(\mathbb{R}) \,. \tag{1.7}$$

We shall use this inversion of $\mathcal{D}^\alpha$ in Section 2.

In some instances we shall also need to split the integral operator (1.2) as follows

$$(\mathcal{D}^\alpha u)(x) = d_\alpha \int_{-\infty}^{x_0} \frac{u'(y)}{(x-y)^\alpha} dy + d_\alpha \int_{x_0}^{x} \frac{u'(y)}{(x-y)^\alpha} dy \,, \quad \text{for some} \quad x_0 < x \,, \tag{1.8}$$

and treat the first term as a known function, whereas the second one can be viewed as a left-sided Caputo derivative, see e.g. [11], and that we denote by $\mathcal{D}_{x_0}^\alpha$, indicating that the integration is from a finite value x_0, i.e. $u \in C_b^1([x_0, \infty))$ and $\alpha \in (0, 1]$

$$\mathcal{D}_{x_0}^\alpha u(x) = \mathcal{I}_{x_0}^{1-\alpha} u'(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} \frac{u'(y)}{(x-y)^\alpha} \, dy \,. \tag{1.9}$$

Notice that the first term in the right-hand side of (1.8), which is a function of x, is not equal to $(\mathcal{D}^\alpha u)(x_0)$, which is a number for fixed x_0.

2 Existence of Travelling Wave Solutions

We introduce the travelling wave variable $\xi = x - ct$ with wave speed c and look for solutions $u(x,t) = \phi(\xi)$ of (1.1) which connect two different far-field real values ϕ_- and ϕ_+. A straightforward calculation shows that if ϕ depends on x and t only through the travelling wave variable, then so does $\mathcal{D}^\alpha \phi$, and so the travelling wave problem becomes

$$-c\phi' + (\phi^2)' = (\mathcal{D}^\alpha \phi)' + \tau \phi''' \,, \tag{2.1}$$

subject to

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_- \,, \quad \lim_{\xi \to \infty} \phi(\xi) = \phi_+ \,. \tag{2.2}$$

Here $'$ denotes differentiation with respect to ξ. We can then integrate (2.1) with respect to ξ and use (2.2) to arrive at the following travelling wave equation:

$$h(\phi) = \mathcal{D}^\alpha \phi + \tau \phi'' \,, \quad \text{where} \quad h(\phi) := -c(\phi - \phi_-) + \phi^2 - \phi_-^2 \,. \tag{2.3}$$

If ϕ' decays to zero fast enough as $\xi \to \pm\infty$, then we obtain the Rankine-Hugoniot condition

$$c = \phi_+ + \phi_- \tag{2.4}$$

that we assume throughout. Since $h(\phi)$ is convex, the left hand side of (2.3) is negative between its only zeroes $\phi = \phi_-$ and $\phi = \phi_+$. In what follows we shall show the existence of solutions of (2.3) provided the entropy condition

$$\phi_- > \phi_+ \,, \tag{2.5}$$

is satisfied. We shall not make further assumptions on the far-field values (regarding the sign, for example), but just note (2.4) and (2.5) imply that

$$h'(\phi_-) = \phi_- - \phi_+ > 0 \quad \text{and} \quad h'(\phi_+) = \phi_+ - \phi_- < 0\,. \tag{2.6}$$

We observe that (2.5) is a necessary condition for existence of the travelling wave if $\alpha = 1$. Their existence for $\tau = 0$ and $\alpha \in (0,1)$, where this condition is crucial, is shown in [1].

As in [1], we shall start our analysis by proving a 'local' existence result on $(-\infty, \tilde{\xi}]$ with $\tilde{\xi} < 0$ and $|\xi|$ sufficiently large. Global existence will then follow by a continuation argument and global boundedness of solutions. The lack of monotonicity for $\tau > 0$ requires additional investigations in order to show that a travelling wave solution tends to ϕ_+ as $\xi \to \infty$. In order prove this we use that the functional $H(\phi) - H(\phi_-)$, where

$$H(\phi) = \int_0^\phi h(y)dy = -c\frac{\phi^2}{2} + \frac{\phi^3}{3} + A\phi\,, \quad \text{with} \quad A = c\phi_- - \phi_-^2\,, \tag{2.7}$$

is increasing with respect to ξ. This step allows to show that if a travelling wave tends to a constant value as ξ tends to ∞ then that constant must be ϕ_+. Then we show that indeed the solutions of (2.3) satisfying $\phi(-\infty) = \phi_-$ tend to a constant as ξ tends to ∞.

The local existence result is based on linearisation about $\xi = -\infty$ (or, equivalently, $\phi = \phi_-$). As it could be expected for ordinary differential equations, the linearisation about $\phi \equiv \phi_-$,

$$h'(\phi_-)v = \mathcal{D}^\alpha v + \tau v''\,, \tag{2.8}$$

has solutions of the form $v(\xi) = be^{\lambda\xi}$, $b \in \mathbb{R}$, where $\lambda > 0$ is a root of

$$P(z) = \tau z^2 + z^\alpha - h'(\phi_-)\,. \tag{2.9}$$

We observe that there is a unique positive real root of (2.9). Indeed, this follows from the fact that $P(z) \to \infty$ as $z \to \infty$ and

$$P(0) = -h'(\phi_-) < 0\,, \quad P'(z) = 2\tau z + \alpha z^{\alpha-1} \geq 0 \quad \text{for} \quad z \geq 0\,.$$

In Lemma B.1 of Appendix B we show, using Rouche's theorem, that (2.9) has exactly three roots, one positive real one and two complex conjugates with negative real part.

We assume for the moment that the only solutions of (2.8) that decay to 0 as $\xi \to -\infty$ are of the form $be^{\lambda\xi}$ for some constant b and λ being the real root of (2.9). We have not fully succeeded in proving this, however in Appendix A we do it in suitable weighted spaces (see Theorem A.2).

Henceforth, we assume that

$$\mathcal{N}\big(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id}\big) = \mathrm{span}\{e^{\lambda\xi}\} \quad \text{in} \quad H^4(\mathbb{R}) \tag{2.10}$$

where Id denotes the identity operator.

The main result of this section is the following:

Theorem 2.1 *Let (2.5) and (2.10) hold. Then, there exists a solution $\phi \in C_b^3(\mathbb{R})$ of (2.1)-(2.2) that is unique (up to a shift in ξ) among all $\phi \in \phi_- + H^4((-\infty,0)) \cap C_b^3(\mathbb{R})$.*

We prove Theorem 2.1 in several steps that we write as lemmas. The first one below is a 'local' existence result that says that the nonlinear problem has, up to translations, only two nontrivial solutions, which can be approximated by $\phi_- \pm e^{\lambda\xi}$ for large negative ξ (observe that the shift in ξ gives a positive constant multiplying the exponential and that we have taken equal to 1 without loss of generality).

Lemma 2.1 *[Local existence] Let the assumptions of Theorem 2.1 hold. Then, for every small enough $\varepsilon > 0$, (2.3) has solutions ϕ_{up}, $\phi_{down} \in \phi_- + H^4(I_\varepsilon)$, where $I_\varepsilon = (-\infty, \xi_\varepsilon]$ and $\xi_\varepsilon = \log\varepsilon/\lambda$, such that*

$$\phi_{up}(\xi_\varepsilon) = \phi_- + \varepsilon\,, \qquad \phi_{down}(\xi_\varepsilon) = \phi_- - \varepsilon\,. \tag{2.11}$$

Moreover, these are unique among all functions ϕ satisfying $\|\phi - \phi_-\|_{H^4(I_\varepsilon)} \le \delta$, with δ small enough, but independent of ε. They satisfy, with an ε-independent constant C,

$$\|\phi_{up} - \phi_- - e^{\lambda\xi}\|_{H^4(I_\varepsilon)} \le C\varepsilon^2\,, \qquad \|\phi_{down} - \phi_- + e^{\lambda\xi}\|_{H^4(I_\varepsilon)} \le C\varepsilon^2\,.$$

Proof. We follow the proof of [1]. We only prove existence and uniqueness for ϕ_{down}, the proof of for ϕ_{up} is analogous and we do not do it here.

We start by writing (2.3) and the initial condition (2.11) in terms of the perturbation $\Phi(\xi) = \phi_{down}(\xi) - \phi_- + e^{\lambda\xi}$:

$$(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id})\Phi = h(\phi_- - e^{\lambda\xi} + \Phi) + h'(\phi_-)(e^{\lambda\xi} - \Phi)\,, \quad \Phi(\xi_\varepsilon) = 0\,. \tag{2.12}$$

We then define a fixed-point map by considering the right-hand side of (2.12) as given.

In order to use Fourier methods, we need a smooth enough extension of functions to $\xi \in \mathbb{R}$. Then, in general, for a $f \in H^4(I_\varepsilon)$ we let $\mathcal{E}(f) \in H^4(\mathbb{R})$ denote a smooth extension of f that satisfies

$$\mathcal{E}(f)\Big|_{I_\varepsilon} = f\,, \quad \|\mathcal{E}(f)\|_{H^4(\mathbb{R})} \le \gamma\|f\|_{H^4(I_\varepsilon)}\,.$$

And denote by Φ a bounded solution of

$$(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id})\Phi = \mathcal{E}(f) \quad \text{in} \quad \mathbb{R}\,,$$

then Φ and its derivatives with respect to ξ can be written as

$$\frac{d^m\Phi}{d\xi^m} = \mathcal{F}^{-1}\left[\left(-\tau k^2 + b_\alpha|k|^\alpha - h'(\phi_-) + ia_\alpha\mathrm{sgn}(k)|k|^\alpha\right)^{-1}\mathcal{F}\left(\frac{d^m\mathcal{E}(f)}{d\xi^m}\right)\right]\,, \quad m = 0,1,2,3,4\,. \tag{2.13}$$

The Fourier symbol in (2.13) is uniformly bounded in k and this implies that there exist constants C_1, $C_2 > 0$ such that

$$\|\Phi|_{I_\varepsilon}\|_{H^4(I_\varepsilon)} \le \|\Phi\|_{H^4(\mathbb{R})} \le C_1\|\mathcal{E}(f)\|_{H^4(\mathbb{R})} \le C_2\|f\|_{H^4(I_\varepsilon)}\,.$$

By the assumption (2.10), the unique solution of

$$(\tau\partial_\xi^2 + \mathcal{D}^\alpha - h'(\phi_-)\mathrm{Id})U = f \quad \text{in } I_\varepsilon\,, \quad U(\xi_\varepsilon) = 0\,,$$

is $U[f](\xi) = \Phi(\xi) - \Phi(\xi_\varepsilon)e^{\lambda(\xi-\xi_\varepsilon)}$. This allows to write (2.12) as the fixed-point problem

$$\bar{\phi}(\xi) = U\left[h(\phi_- - e^{\lambda\xi} + \bar{\phi}(\xi)) + h'(\phi_-)(e^{\lambda\xi} - \bar{\phi}(\xi))\right].$$

The continuous embedding of $H^4(I_\varepsilon)$ in $C_b^3(I_\varepsilon)$ gives

$$\left\|h(\phi_- - e^{\lambda\xi} + \bar{\phi}) + h'(\phi_-)(e^{\lambda\xi} - \bar{\phi})\right\|_{H^4(I_\varepsilon)} \leq L\left(\varepsilon^2 + \varepsilon\|\bar{\phi}\|_{H^4(I_\varepsilon)} + \|\bar{\phi}\|^2_{H^4(I_\varepsilon)}\right),$$

where L is a positive non-decreasing function. It is now easily seen that the fixed point map is a contraction in small enough balls (independent of ε) and that maps a ball with radius of $O(\varepsilon^2)$ into itself (see [1] for such similar details). ∎

Lemma 2.2 *[Continuation principle] Let $\phi \in C_b^3((-\infty,\xi_0])$ be a solution of (2.3) as constructed in Lemma 2.1. Then there exists a $\delta > 0$, such that ϕ can be extended uniquely to $C_b^3((-\infty,\xi_0+\delta))$.*

Proof. The idea is to write the integro-differential equation as a system of Caputo-differential equations. We use the definition of the Caputo derivative and the inversion formula for it(1.9). Since $\phi \in C^1([\xi_0,\infty))$ and $\alpha \in (0,1]$ then this allows to write down derivatives of entire order by using that $\mathcal{D}^\alpha_{\xi_0}\mathcal{I}^\alpha_{\xi_0} \equiv \mathrm{Id}$ (cf. [11]). Indeed, we can write

$$\phi'(\xi) = \mathcal{D}^\alpha_{\xi_0}\mathcal{D}^{1-\alpha}_{\xi_0}\phi(\xi) = \mathcal{D}^{1-\alpha}_{\xi_0}\mathcal{D}^\alpha_{\xi_0}\phi(\xi),$$

hence, also

$$\phi''(x) = \mathcal{D}^{1-\alpha}_{\xi_0}\mathcal{D}^\alpha_{\xi_0}\mathcal{D}^{1-\alpha}_{\xi_0}\mathcal{D}^\alpha_{\xi_0}\phi(\xi).$$

We can now express (2.3) as a system:

$$\mathcal{D}^\alpha_{\xi_0}\phi = \psi, \quad \mathcal{D}^{1-\alpha}_{\xi_0}\psi = \theta, \quad \mathcal{D}^\alpha_{\xi_0}\theta = \chi, \tag{2.14}$$

$$\tau\mathcal{D}^{1-\alpha}_{\xi_0}\chi = h(\phi) - \psi - \int_{-\infty}^{\xi_0}\frac{\phi'(y)}{(\xi-y)^\alpha}dy. \tag{2.15}$$

The system is locally Lipschitz continuous in $C_b^3(\xi_0,\xi_0+\delta)$. Local existence then follows by using a Picard-Lindelöf type of argument, taking as initial conditions the values of ϕ, $\mathcal{D}^\alpha_{\xi_0}\phi$, ϕ' and $\mathcal{D}^\alpha_{\xi_0}\phi'$ at $\xi = \xi_0$. The well-posedness of linear integro-differential systems of this form is given by Jafari and Daftardar-Gejji [9], so we do not give further details. ∎

It is now clear that boundedness of the solutions will guarantee global existence by applying repeatedly Lemma 2.2 as long as ϕ' remains integrable. First we show that a solution of (2.3) as constructed in lemmas 2.1 and 2.2 is uniformly bounded.

Lemma 2.3 (Uniform boundedness) *Let $\phi \in C_b^3((-\infty,\xi_0])$ be a solution of (2.3) as constructed in Lemma 2.1. Then the solution is bounded for $\xi \in (-\infty,\xi_0)$ by*

$$\bar{\phi} < \phi(\xi) < \phi_-, \quad \textit{where} \quad \bar{\phi} = \frac{3\phi_+ - \phi_-}{2} < \phi_+ \tag{2.16}$$

is the second root of

$$\frac{H(\phi) - H(\phi_-)}{\phi - \phi_-} = 0.$$

Proof. We first derive an energy type of estimate for (2.3). This is done, as in the local case, by multiplying the equation by ϕ' and integrating with respect to ξ:

$$H\left(\phi(\xi)\right) - H\left(\phi_-\right) = \frac{\tau}{2}\left(\phi'(\xi)\right)^2 + \int_{-\infty}^{\xi} \phi'(y)\, \mathcal{D}^\alpha \phi(y) dy\,. \tag{2.17}$$

The first term on the right-hand side of (2.17) is clearly non-negative.

Let us show that the second term is also non-negative.

We first observe that

$$\int_{-\infty}^{\xi} \phi'(y)\, \mathcal{D}^\alpha \phi(y) dy = \frac{1}{2}\int_{-\infty}^{\xi} \phi'(y) \int_{-\infty}^{\xi} \frac{\phi'(x)}{|x-y|^\alpha}\, dx\, dy \tag{2.18}$$

this is shown by noticing that

$$\int_{-\infty}^{\xi} \phi'(y) \int_{y}^{\xi} \frac{\phi'(x)}{(x-y)^\alpha}\, dx\, dy = \int_{-\infty}^{\xi} \phi'(x) \int_{-\infty}^{x} \frac{\phi'(y)}{(x-y)^\alpha}\, dy\, dx\,.$$

Then, we can consider an extension $\phi'_E \in L^2(\mathbb{R})$ of ϕ' to $\mathbb{R}$ so that $\phi'_E(y) = 0$ for $y > \xi$. Then, by applying Theorem 9.8[15] to (2.18) with this extension we obtain that

$$\int_{-\infty}^{\xi} \phi'(y)\, \mathcal{D}^\alpha \phi(y) dy = \frac{1}{2}\int_{\mathbb{R}} \phi'_E(x) \int_{\mathbb{R}} \frac{\phi'_E(y)}{|x-y|^\alpha}\, dy\, dx \geq 0\,. \tag{2.19}$$

Let us now prove the upper bound. Suppose that there exists a $\bar{\xi} < \infty$ such that $\phi(\bar{\xi}) = \phi_-$, then from (2.17) one gets that $\int_{-\infty}^{\bar{\xi}} \phi'(y)\, \mathcal{D}^\alpha \phi(y) dy = 0$, and (2.19) implies that $\phi'(\xi) = 0$ for all $\xi \in (-\infty, \bar{\xi}]$ (see [15]). Assume now that $\lim_{\xi\to\infty} \phi(\xi) = \phi_-$, then $\int_{-\infty}^{\infty} \phi'(y)\, \mathcal{D}^\alpha \phi(y) dy = 0$. But, we can write (2.18) with $\xi = \infty$ and without using an extension of ϕ';

$$\int_{-\infty}^{\xi} \phi'(y)\, \mathcal{D}^\alpha \phi(y) dy = \frac{1}{2}\int_{\mathbb{R}} \phi'(x) \int_{\mathbb{R}} \frac{\phi'(y)}{|x-y|^\alpha}\, dy\, dx = 0\,,$$

thus also $\phi'(\xi) = 0$ for all $\xi \in \mathbb{R}$. Then a non constant solution is always below ϕ_-.

In order to get the lower bound, we use that the right hand side of (2.17) is non-negative, thus

$$H\left(\phi\right) - H\left(\phi_-\right) = -\frac{c}{2}(\phi^2 - (\phi_-)^2) + \frac{1}{3}(\phi^3 - (\phi_-)^3) + A(\phi - \phi_-) \geq 0\,.$$

Since we have just shown that $\phi - \phi_- < 0$ in $(-\infty, \xi_0]$, we obtain the condition

$$\frac{H(\phi) - H(\phi_-)}{\phi - \phi_-} = -\frac{c}{2}(\phi + \phi_-) + \frac{1}{3}(\phi^2 + \phi\phi_- + (\phi_-)^2) + A < 0$$

and this implies (2.16). ■

Lemma 2.4 *(Global uniqueness) Let $\phi \in \phi_- + H^4((-\infty, \xi_0))$ be a solution of (2.3). Then, up to a shift in ξ, ϕ is the continuation of either ϕ_{up} or ϕ_{down}, otherwise $\phi \equiv \phi_-$.*

Proof. For every $\delta > 0$ there exists a $\xi^* \leq \xi_0$, such that $\|\phi - \phi_-\|_{H^4((-\infty,\xi^*))} < \delta$ and, therefore, by Sobolev embedding, also $|\phi(\xi^*) - \phi_-| < \delta$. Choosing δ small enough, there are only two possibilities, either $\phi(\xi^*) = \phi_-$ (implying that $\phi \equiv \phi_-$) or $\phi(\xi^*) \neq \phi_-$. Whence, by local uniqueness, ϕ is, up to a shift, either equal to ϕ_{up} or ϕ_{down}, depending on the sign of $\phi(\xi^*) - \phi_-$. ■

It remains to analyse the far-field behaviour.

Lemma 2.5 *Let $\phi \in \phi_- + H^4((-\infty, \xi_0))$ be a continuation of ϕ_{down} as in Lemma 2.4. Suppose that*

$$\lim_{\xi\to\infty} \phi = \phi_0 \in \mathbb{R}\,, \tag{2.20}$$

then $\phi_0 = \phi_+$.

Proof. We argue by contradiction. Assume that (2.20) holds with $\phi_0 \neq \phi_+$, then $h(\phi(\xi)) \to h(\phi_0) \neq 0$. Suppose first that for $\xi > \xi_0$, $h(\phi(\xi)) > C_+ > 0$, then applying the integral operator (1.7) to $h(\phi)$ we get

$$\begin{aligned} d_{1-\alpha}^{-1}\mathcal{I}^\alpha h(\phi(\xi)) &> \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + C_+ \int_{\xi_0}^{\xi} \frac{dy}{(\xi-y)^{1-\alpha}} \\ &= \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + \frac{C_+}{\alpha}(\xi-\xi_0)^\alpha \to \infty \quad \text{as} \quad \xi\to\infty\,. \end{aligned}$$

This and (2.3) imply that $\mathcal{I}^\alpha\phi'' \to \infty$ as $\xi \to \infty$. Similarly, if for all $\xi > \xi_0$ we have $h(\phi(\xi)) < C_- < 0$, we obtain

$$\begin{aligned} d_{1-\alpha}^{-1}\mathcal{I}^\alpha h(\phi) &< \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + C_- \int_{\xi_0}^{\xi} \frac{dy}{(\xi-y)^{1-\alpha}} \\ &= \int_{-\infty}^{\xi_0} \frac{h(\phi(y))}{(\xi-y)^{1-\alpha}}dy + C_-(\xi-\xi_0)^\alpha \to -\infty \quad \text{as} \quad \xi\to\infty \end{aligned}$$

as before, this implies that $\mathcal{I}^\alpha\phi'' \to -\infty$ as $\xi \to \infty$. In both cases and using (2.3) we obtain that $|\mathcal{I}^\alpha\phi''|$ is unbounded as well. Let us see that this contradicts (2.20).

Since $\phi \in C_b^3(\mathbb{R})$ by Lemma 2.2 and (2.20) holds, we can take for any $\varepsilon > 0$ and ξ large enough, $\xi^* = \xi - \delta$ for a fixed $\delta > 0$ such that $|\phi''(\xi)| < \varepsilon$ for all $\xi > \xi^*$. Then

$$\left| \int_{\xi^*}^{\xi} \frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy \right| < \varepsilon(\xi-\xi^*)^\alpha = \varepsilon\delta^\alpha\,. \tag{2.21}$$

Now if we write

$$d_{1-\alpha}^{-1}\mathcal{I}^\alpha\phi'' = \int_{-\infty}^{\xi} \frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy = \int_{-\infty}^{\xi^*} \frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy + \int_{\xi^*}^{\xi} \frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy\,, \tag{2.22}$$

(2.21) implies that the second term of (2.22) converges. We integrate by parts the first term:

$$\begin{aligned} \int_{-\infty}^{\xi^*} \frac{\phi''(y)}{(\xi-y)^{1-\alpha}}dy &= \frac{\phi'(\xi^*)}{(\xi-\xi^*)^{1-\alpha}} - \lim_{y\to-\infty} \frac{\phi'(y)}{(\xi-y)^{1-\alpha}} + (1-\alpha)\int_{-\infty}^{\xi^*} \frac{\phi'(y)}{(\xi-y)^{2-\alpha}}dy \\ &= \frac{\phi'(\xi^*)}{\delta^{1-\alpha}} + (1-\alpha)\int_{-\infty}^{\xi^*} \frac{\phi'(y)}{(\xi-y)^{2-\alpha}}dy\,. \end{aligned}$$

The absolute value of the second term on the right hand side is also bounded by $C/\delta^{1-\alpha}$. Since δ was a fixed number, this contradicts the unboundedness of $\mathcal{I}^\alpha\phi''$. ■

Next we show that a solution as constructed in Lemma 2.1 approaches a constant value as $\xi \to \infty$. Once this is proved we can conclude the proof of Theorem 2.1 since this then implies that $\lim_{\xi\to\infty}\phi = \phi_+$ by Lemma 2.5.

Lemma 2.6 *Let ϕ be a solution of (2.3) as in Lemma 2.4. Then there exist a constant $\phi_0 \in \mathbb{R}$ such that $\lim_{\xi\to\infty}\phi(\xi) = \phi_0$.*

Proof. The solution ϕ can be extended to any interval of the form $(-\infty, \xi_0]$ by repeating the continuation result of Lemma 2.2 as necessary, since (2.16) is satisfied. Now, knowing that the smooth wave profile exists, we split the non-local differential operator and rewrite the travelling wave equation in the following form

$$\tau\phi'' + \mathcal{D}^{\alpha}_{\xi_0}\phi + \phi = q(\phi, \xi) \tag{2.23}$$

for $\xi \geq \xi_0$, where

$$q(\phi, \xi) = -d_\alpha \int_{-\infty}^{\xi_0} \frac{\phi'(y)}{(\xi - y)^\alpha} dy + h(\phi(\xi)) + \phi(\xi)\,.$$

We can know write down the solution to (2.23) implicitly. In order to do that one applies Laplace transform methods as in e.g. [8] to obtain a 'variations of constants' representation of the solution with initial conditions at $\xi = \xi_0$. One gets

$$\phi(\xi) = \phi(\xi_0)\, v(\xi) - \phi'(\xi_0)\, v'(\xi) - \int_{\xi_0}^{\xi} q(\phi(\xi - s), \xi - s) v'(s) ds$$

where the function v and its derivatives are uniformly bounded and satisfy (we give more details in Appendix C, see (C.12)-(C.14)):

$$\lim_{\xi\to\infty} (\xi - \xi_0)^\alpha v(\xi) = \frac{d_\alpha}{\tau}\,, \qquad \lim_{\xi\to\infty} (\xi - \xi_0)^{\alpha+1} v'(\xi) = \frac{d_{\alpha-1}}{\tau}\,.$$

Now using that ϕ is uniformly bounded in $\mathbb{R}$, we conclude that $q(\phi, \xi)$ is also uniformly bounded and it is easy to see the integrability of the term with the inhomogeneity q as well as the decay of ϕ towards a constant. ■

We end the section with the proof of the main theorem:

Proof of Theorem 2.1. The proof follows by applying the previous lemmas. First, Lemma 2.1 (local existence), then Lemma 2.2 (continuation principle) and then lemmas 2.3 and 2.4 imply the global existence and uniqueness up to translation in ξ of solutions of (2.3) satisfying $\phi(-\infty) = \phi_-$. Finally, Lemma 2.6 implies that such solution satisfies $\lim_{\xi\to\infty}|\phi(\xi)| < \infty$ and from Lemma 2.5 we conclude that in fact $\lim_{\xi\to\infty}\phi(\xi) = \phi_+$. ■

3 Analysis of the monotonicity of travelling waves

In this section we discuss the role of the parameter τ in the monotonicity of the travelling waves. To start with, we remark that one can show 'local' monotonicity for all $\tau > 0$ in the interval I_ε for ε small enough:

Lemma 3.1 (Local monotonicity) *Let the assumptions of Lemma 2.1 hold. Then, for ε small enough,*

$$\phi_{up} > \phi_-\,, \quad \phi'_{up} > 0\,, \qquad \phi_{down} < \phi_-\,, \quad \phi'_{down} < 0\,, \quad in \quad I_\varepsilon\,.$$

Proof. The proof follows as in [1]. ■

Now, if $\tau = 0$ we know from [1] that travelling waves are monotone decreasing. Moreover, if $\tau \neq 0$ and $\alpha = 1$, thus in the classical KdV-Burgers case, it is easy to see that the waves are monotone if τ is smaller than some critical value (see [3]). In fact, travelling waves are heteroclinic connections of the corresponding ODE system. The critical points represent the far-field values ϕ_- and ϕ_+, linearisation about these points shows that the one associated to ϕ_- is a saddle point and the one associated to ϕ_+ is an attractor. It is important to notice that the attractor has the eigenvalues

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{1 + 4\tau h'(\phi_+)}}{2\tau}$$

and that $h'(\phi_+) < 0$ (see (2.6)). It then becomes clear that heteroclinic connections give monotone travelling waves when $\tau \leq -1/(4h'(\phi_+))$.

We expect a similar behaviour for (2.3), although the decay of ϕ towards ϕ_+ is not exponential, as we have seen in the proof of Lemma 2.6.

Let us now prove that if τ is small enough then the solution of (2.2)-(2.3) that is a extension of ϕ_{down} is close to the solution with $\tau = 0$ (as constructed in [1]) on a large interval, thus implying monotonicity for small values of τ on such intervals. Before we give the result let us introduce the appropriate notation. Let us denote by ϕ_τ a travelling wave solution for a given τ and ϕ_0 a travelling wave of the problem with $\tau = 0$. Then:

Theorem 3.1 (Monotonicity) *If τ is small enough, then there exist an interval $I_\tau = (-\infty, \xi_\tau]$ with $\xi_\tau = O(\tau^{-\frac{1}{2-\alpha}})$ as $\tau \to 0$, and a value $\xi = \xi_\tau^0 < \xi_\tau$ such that $\phi_\tau(\xi_\tau^0) = \phi_0(\xi_\tau^0)$, moreover, $|\phi_\tau(\xi) - \phi_0(\xi)| \leq \tau C$ and $|\phi_\tau'(\xi) - \phi_0'(\xi)| \leq \tau^{1/(2-\alpha)} C$ for all $\xi \in I_\tau$. Thus for τ small enough ϕ_τ is also monotone decreasing in I_τ.*

We prove this theorem in several lemmas. First we fix the shift in ξ:

Lemma 3.2 *For a given small τ there exists a $\xi_\tau^0 < \log\tau/(h'(\phi_-))^{1/\alpha}$ small and a travelling wave solution ϕ_τ such that, if ϕ_0 is the travelling wave solution of the problem with $\tau = 0$ such that $\phi_0(\log\tau/(h'(\phi_-))^{1/\alpha}) = \phi_- - \tau$, then ϕ_τ is monotone decreasing in $(-\infty, \xi_\tau^0]$ and*

$$\phi_\tau(\xi_\tau^0) = \phi_0(\xi_\tau^0)\,, \quad |\phi_\tau'(\xi) - \phi_0'(\xi)|\,,\ |\phi_\tau''(\xi) - \phi_0''(\xi)| \leq \tau C \quad \textit{for} \quad \xi \in (-\infty, \xi_\tau^0] \tag{3.1}$$

with some order one constant $C > 0$.

Proof. We want to compare travelling wave solutions for a small $\tau > 0$ with solutions of the problem with $\tau = 0$. The later ones are monotone and are constructed 'locally' near $-\infty$ as in Lemma 2.1 in [1]. In particular, for a given small enough ε then $\phi_0(\xi_\varepsilon^0) = \phi_- - \varepsilon$ where $\xi_\varepsilon^0 = \log\varepsilon/(h'(\phi_-))^{1/\alpha}$. On the other hand, if λ_τ denotes the real root of (2.9) then $\phi_\tau(\log\varepsilon)/\lambda_\tau) = \phi_- - \varepsilon$. The asymptotic behaviour of λ_τ as $\tau \to 0$ (see (B.3) in Appendix B) and (2.6) imply that if τ is small enough then $\xi_\varepsilon^0 < \log\varepsilon/\lambda_\tau$, hence, by local monotonicity, $\phi_- - \varepsilon < \phi_\tau(\xi_\varepsilon^0) < \phi_-$ i.e. $\phi_\tau(\xi_\varepsilon^0) - \phi_0(\xi_\varepsilon^0) < \varepsilon$. Again by monotonicity, we can find a value $\xi_\varepsilon < \xi_\varepsilon^0$, by shifting ϕ_τ in ξ if necessary, such that $\phi_\tau(\xi_\varepsilon) = \phi_0(\xi_\varepsilon)$. Finally, by the construction of these waves, they are close to ϕ_- by an exponential difference in $H^3(-\infty, \xi_\tau^0)$, it holds that $|\phi_\tau'(\xi) - \phi_0'(\xi)|$, $|\phi_\tau''(\xi) - \phi_0''(\xi)| \leq \varepsilon C$ with an order 1 constant $C > 0$. Finally, we can do this same construction by taking $\varepsilon = \tau$. ■

Let $\psi_\tau := \phi_\tau - \phi_0$, then ψ_τ satisfies the following equation

$$\tau\psi_\tau'' + \mathcal{D}^\alpha_{\xi_\tau^0}\psi_\tau = \mathcal{R}(\phi_0, \phi_\tau, \xi) - \tau\phi_0''(\xi) \tag{3.2}$$

where

$$\mathcal{R}(\phi_0,\phi_\tau,\xi) = [-c + (\phi_\tau(\xi)+\phi_0(\xi))]\psi_\tau(\xi) - d_\alpha \int_{-\infty}^{\xi_\tau^0} \frac{\psi_\tau'(y)}{(\xi-y)^\alpha}\, dy\,, \tag{3.3}$$

(for simplicity, we do not write the dependency of $\mathcal{R}$ in ψ_τ which implicit in the dependency on ϕ_τ and ϕ_0) here we have used the expression of h in (2.3) to write $h(\phi_\tau) - h(\phi_0) = [-c + (\phi_\tau + \phi_0)]\psi_\tau$. That can be solved subject to the initial conditions (see (3.1))

$$\psi(\xi_\tau^0) = 0\,, \quad \psi'(\xi_\tau^0) = \phi_\tau'(\xi_\tau^0) - \phi_0'(\xi_\tau^0)\,. \tag{3.4}$$

Using Laplace transform we can write the solution to (3.2)-(3.4) taking ϕ_τ and ϕ_0 as given. In order to do that more conveniently we can first shift the independent variable so that $\eta = \xi - \xi_\tau^0$ and let $\bar\psi_\tau(\eta) = \psi_\tau(\eta + \xi_\tau^0)$, so (3.2) and (3.3) read

$$\tau\bar\psi_\tau'' + \mathcal{D}_0^\alpha \bar\psi_\tau + \bar\psi_\tau = \mathcal{Q}(\phi_0,\phi_\tau,\eta)\,, \quad {}' = \frac{d}{d\eta} \tag{3.5}$$

$$\mathcal{Q}(\phi_0,\phi_\tau,\eta) = \mathcal{R}(\phi_0,\phi_\tau,\eta+\xi_\tau^0) - \tau\phi_0''(\eta+\xi_\tau^0) + \bar\psi_\tau(\eta) \tag{3.6}$$

where we add an subtract the term $\bar\psi_\tau$ for technical reasons outlined below. Then, (3.5)-(3.6) must be solved with initial conditions (see (3.4))

$$\bar\psi_\tau(0^+) = 0\,, \quad \bar\psi_\tau'(0^+) = \psi'(\xi_\tau^0)\,. \tag{3.7}$$

Employing the computation performed in Appendix C, but here with $a = 1$, the solution of (3.5)-(3.7) is given implicitly by

$$\bar\psi(\eta) = -\tau\bar\psi'(0^+)v'(\eta) - \int_0^\eta v'(y)\mathcal{Q}(\phi_0,\phi_\tau,\eta-y)\, dy \tag{3.8}$$

where $v(\eta)$ reads (see (C.12) and (C.13))

$$v(\eta) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\eta r}K_{\tau,\alpha}(r)\, dr + 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\,, \tag{3.9}$$

and s_1 is the solution of

$$\tau z^2 + z^\alpha + 1 = 0 \tag{3.10}$$

with positive imaginary part, and $\beta = \arg(s_1) \in (\pi/2,\pi)$ (see Lemma B.1 and Appendix C), and where

$$K_{\tau,\alpha}(r) = r^{\alpha-1}\tilde K_{\tau,\alpha}(r) \quad \text{with} \quad \tilde K_{\tau,\alpha}(r) = \frac{1}{(\tau r^2+1)^2 + 2(\tau r^2+1)r^\alpha\cos(\alpha\pi) + r^{2\alpha}}\,. \tag{3.11}$$

Then it is easy to see that

$$v'(\eta) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\eta r}r^\alpha\tilde K_{\tau,\alpha}(r)\, dr + 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1^2 + s_1^\alpha}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\,. \tag{3.12}$$

The reason to introduce the term $\bar\psi_\tau$ is that this implies that the resulting algebraic function when applying the Laplace transform to the left-hand side of (3.5) has poles away from the negative real axis. Without this term, 0 would be a pole of such function, but is also a branch point, thus making the computation of the inverse Laplace transform a little cumbersome.

We also need to get pointwise estimates on $|\bar{\psi}_\tau(\eta)|$. We shall do this directly from the expression obtained differentiating (3.8):

$$\bar{\psi}'_\tau(\eta) = -\tau\bar{\psi}'_\tau(0^+)v''(\eta) - v'(\eta)\mathcal{Q}(\phi_0,\phi_\tau,0) - \int_0^\eta v'(y)\frac{d\mathcal{Q}}{d\eta}(\phi_0,\phi_\tau,\eta-y)\,dy\,. \tag{3.13}$$

The following estimates hold:

Lemma 3.3 *If $r < 1/\tau^{\frac{1}{2-\alpha}}$, for some τ small enough, then there exists a $C>0$ independent of τ such that*

$$\left|\tilde{K}_{\tau,\alpha}(r) - \frac{1}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}\right| \le C\,\tau\,.$$

If on the contrary $r \ge 1/\tau^{\frac{1}{2-\alpha}}$, for some τ small enough, there exists a $C>0$ independent of τ such that

$$|\tilde{K}_{\tau,\alpha}(r)| \le C\tau^{\frac{2}{2-\alpha}}\,.$$

Proof. We leave the proof to the reader. One can convince him or herself by inspecting the functions involved and a formal dominant balance analysis that can be made rigorous by performing the calculus. ■

Regarding the second term on the right-hand side of (3.9) we have the following preliminary estimates that give exponential decay (observe that $\cos(\beta)<0$:

Lemma 3.4 *If τ is small enough then, there exists a $0<C(\tau)<1$ such that $C(\tau)\sim 2/(2+\alpha)$ as $\tau\to 0$ and*

$$\left|Re\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1+\alpha s_1^{\alpha-1}}\right)\right| \le C(\tau)e^{(|s_1|\cos\beta)\,\eta}$$

where $\beta = arg(s_1)\in(\pi/2,\pi)$ with $\beta\to\pi/(2-\alpha)$ and $|s_1| = O(1/\tau^{1/(2-\alpha)})$ as $\tau\to 0$. Similarly, estimates on derivatives with respect to η of this term differ by a factor $|s_1|^n$ where n is the order of the derivative.

Proof. This is proved by using Lemma B.1 and (B.6) of Appendix B. Details are left to the reader. ■

We next derive estimates that we apply to (3.8) and (3.13). Essentially, we get estimates on the uniform norms of v, v', $\mathcal{Q}$ and $d\mathcal{Q}/d\eta$, estimates on the integral of $|v'|$ over $[0,\eta]$ and on v''. We also need to get estimates on the integral terms of $\mathcal{Q}$ and $d\mathcal{Q}/d\eta$.

Lemma 3.5 *The following estimates hold for all $\eta>0$ and τ small enough:*

(i) There exist constants $C(\|\phi_0''\|_{L^\infty})>0$ and $C>0$ such that

$$|\mathcal{Q}(\phi_0,\phi_\tau,\eta)| \le (1+3|\phi_-|+|\phi_+|)\;|\bar{\psi}_\tau(\eta)| + \tau C(\|\phi_0''\|_{L^\infty}) \tag{3.14}$$

and

$$\left|\frac{d\mathcal{Q}}{d\eta}(\phi_0,\phi_\tau,\eta)\right| \le C\left(|\bar{\psi}_\tau(\eta)| + |\bar{\psi}'_\tau(\eta)| + \tau\eta^{1-\alpha}\right) + \tau\,\|\phi_0'''\|_{L^\infty}\,. \tag{3.15}$$

(ii) There exists a constant $C>0$ such that

$$|\mathcal{Q}(\phi_0,\phi_\tau,0)| \le \tau\,\left(C + |\phi_0''(\xi_\tau^0)|\right)\,.$$

(iii) The functions v, v' and v'' are uniformly bounded on $[0,\infty)$. the first one by a constant independent of τ, whereas the other two by an constant that becomes unbounded as $\tau \to 0^+$. Moreover, for all $\eta > 0$ there exists a constant $C > 0$

$$|v'(\eta)| \le C\left(\tau^{\frac{\alpha}{2(2-\alpha)}}\eta^{\frac{\alpha-2}{2}} + \tau^{\frac{1-\alpha}{2-\alpha}}\eta^{-\alpha} + \tau^{\frac{3-\alpha}{2-\alpha}}\right) + 2C(\tau)e^{(|s_1|\cos\beta)\,\eta}|s_1| \tag{3.16}$$

s_1 being the zero of (3.10) with positive imaginary part and $\beta \in (\pi/2,\pi)$ its principal argument, and

$$|v''(\eta)| \le C(\tau^{-1} + \tau^{-\frac{2}{2-\alpha}} + \tau^{-\frac{\alpha}{2-\alpha}} + \tau^{\frac{1-\alpha}{2-\alpha}})\,. \tag{3.17}$$

Proof. Statement *(i)* follows from the properties of ϕ_τ and ϕ_0. In order to estimate the integral term of $\mathcal{Q}$ we use the construction of the solutions ϕ_τ and ϕ_0 in the interval $(-\infty,\xi_\tau^0]$ and that $\lambda_\tau e^{\lambda_\tau y} - \lambda_0 e^{\lambda_0 y} = e^{\lambda_0 y}F(\tau,y)$ where $F(\tau,y)$ in uniformly bounded in $y < \xi_\tau^0$ $\tau > 0$ (see Lemma 3.2, (3.1)). One can show *(ii)* similarly, since

$$\mathcal{Q}(\phi_0,\phi_\tau,0) = -d_\alpha\mathcal{D}^\alpha(\xi_\tau^0) - \phi_0''(\xi_\tau^0)\,.$$

The integral term of $dQ/d\eta$ reads, here $\eta > 0$,

$$I := -d_\alpha\int_{-\infty}^{\xi_\tau^0}\frac{\psi_\tau'(y)}{(\xi-y)^{\alpha+1}}dy = -d_\alpha\int_{-\infty}^{0}\frac{\bar\psi_\tau'(y)}{(\eta-y)^{\alpha+1}}dy\,.$$

Integration by parts gives

$$I = -\frac{d_\alpha}{\alpha}\frac{\bar\psi_\tau'(0)}{\eta^\alpha} + \frac{d_\alpha}{\alpha}\int_{-\infty}^{0}\frac{\bar\psi_\tau''(y)}{(\eta-y)^\alpha}dy$$

and using Lemma 3.2, (3.1) gives the estimate.

The statement in *(iii)* about v and v' follows from (C.5) and (C.14) of Appendix C.

Let us get the estimate on $|v'(\eta)|$. Using the expression of $v'(\eta)$ in (3.12) and Lemma 3.4 we obtain a first estimate

$$\begin{aligned}|v'(\eta)| &\le \frac{1}{\pi}\int_0^\infty e^{-\eta r}r^\alpha\left|\tilde K_{\tau,\alpha}(r)\right|\,dr + 2\left|\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1^2 + s_1^\alpha}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\right|\\ &\le \frac{1}{\pi}\int_0^\infty e^{-\eta r}r^\alpha\left|\tilde K_{\tau,\alpha}(r)\right|\,dr + Ce^{(|s_1|\cos\beta)\,\eta})|s_1|\,,\end{aligned}$$

for some positive constant independent of τ, where s_1 is the zero of (3.10) with positive imaginary part and β its principal argument. We estimate the first term on the right-hand side of the inequality above assuming that τ is small enough and so that we can apply Lemma 3.3. Thus we first split the integral over r and apply this lemma:

$$\begin{aligned}&\int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-\eta r}r^\alpha\left|\tilde K_{\tau,\alpha}(r)\right|\,dr\\ &\le \int_0^{\tau^{-\frac{1}{2-\alpha}}}\frac{e^{-\eta r}r^{\alpha-\gamma}r^\gamma}{1+2r^\alpha\cos(\alpha\pi)+r^{2\alpha}}\,dr + \tau^{1-\frac{1}{2-\alpha}}C\int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-\eta r}r^{\alpha-1}\,dr\,,\end{aligned} \tag{3.18}$$

where $\gamma \in (\alpha,2\alpha)$ and

$$\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r}r^\alpha\left|\tilde K_{\tau,\alpha}(r)\right|\,dr \le \tau^{\frac{2}{2-\alpha}}C\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r}r^\alpha\,dr\,. \tag{3.19}$$

In order to estimate (3.18) we shall use that

$$\int_A^B e^{-\eta r} r^\sigma\, dr \le \int_0^\infty e^{-\eta r} r^\sigma\, dr = \frac{\Gamma(\sigma+1)}{\eta^{\sigma+1}} \quad \text{with } \sigma > -1\,,\ 0 \le A < B\,,\ \eta > 0 \tag{3.20}$$

and in each integral term we apply this as an estimate with different values of σ. For the first term on the right-hand side of (3.18) we first rescale $r = \tau^{-\frac{1}{2-\alpha}}\bar{r}$ and observe that for any $\gamma \in (\alpha, 2\alpha]$ the function $\bar{r}^\gamma/(\tau^{\frac{2\alpha}{2-\alpha}}a^2 + \tau^{\frac{\alpha}{2-\alpha}}2\bar{r}^\alpha\cos(\alpha\pi) + \bar{r}^{2\alpha})$ is uniformly bounded for $\bar{r} \in [0,1]$ by a constant independent of τ. Then, after this change of variables, the estimate and applying (3.20) with $A = 0$, $B = 1$ and $\sigma = \alpha - \gamma$ one gets:

$$\int_0^{\tau^{-\frac{1}{2-\alpha}}} \frac{e^{-\eta r} r^\alpha}{1 + 2\cos(\alpha\pi) r^\alpha + r^{2\alpha}}\, dr \le \tau^{\frac{\alpha-1}{2-\alpha}} C \int_0^1 e^{-\tau^{-\frac{1}{2-\alpha}}\eta\bar{r}} \bar{r}^{\alpha-\gamma}\, d\bar{r} \le \tau^{\frac{2\alpha-\gamma}{2-\alpha}} C(\alpha) \frac{1}{\eta^{\alpha-\gamma+1}}\,.$$

We obtain

$$\int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-\eta r} r^\alpha \left|\tilde{K}_{\tau,\alpha}(r)\right|\, dr \le \tau^{\frac{2\alpha-\gamma}{2-\alpha}} C(\alpha)\eta^{\gamma-\alpha+1} + \tau^{\frac{1-\alpha}{2-\alpha}} C\Gamma(\alpha)\eta^{-\alpha}$$

and, taking $\gamma = 3\alpha/2$, this gives the first two terms on the right-hand side of (3.16).

We further estimate (3.19) as follows:

$$\begin{aligned}
&\int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r} r^\alpha \left|\tilde{K}_{\tau,\alpha}(r)\right|\, dr \\
&\le \tau^{\frac{2}{2-\alpha}} C \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-\eta r} r^2 r^{\alpha-2}\, dr \le \tau^{\frac{2}{2-\alpha}} C \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty r^{\alpha-2}\, dr = \tau^{\frac{3-\alpha}{2-\alpha}} C\,,
\end{aligned}$$

where we use Lemma 3.3 in the first step. Putting the estimates together we obtain (3.16).

Let us get now the estimate on $|v''(\eta)|$. We differentiate (3.12) and estimate the last term as in Lemma 3.4, then

$$\begin{aligned}
|v''(\eta)| &\le \frac{|\sin(\alpha\pi)|}{\pi} \int_0^\infty e^{-r\eta} r^{\alpha+1} |\tilde{K}(r)| dr + 2\left|\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1^3 + s_1^{\alpha+1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)\right| \\
&\le \frac{|\sin(\alpha\pi)|}{\pi} \int_0^\infty e^{-r\eta} r^{\alpha+1} |\tilde{K}(r)| dr + C\, e^{(|s_1|\cos\beta)\eta} |s_1|^2\,.
\end{aligned} \tag{3.21}$$

We get estimates on the first term on the right-hand side by dividing the integral over the intervals $(0, \tau^{-1/(2-\alpha)})$ and $(\tau^{-1/(2-\alpha)}, \infty)$ and apply the estimates on $\tilde{K}$ of Lemma 3.3, then

$$\begin{aligned}
&\frac{|\sin(\alpha\pi)|}{\pi} \int_0^\infty e^{-r\eta} r^{\alpha+1} |\tilde{K}(r)| dr \\
&\le C\left(\int_0^{\tau^{-\frac{1}{2-\alpha}}} \frac{e^{-r\eta} r^{2\alpha} r^{1-\alpha}}{1 + 2r^\alpha\cos(\alpha\pi) + r^{2\alpha}} dr + \tau \int_0^{\tau^{-\frac{1}{2-\alpha}}} e^{-r\eta} r^{\alpha+1} dr + \tau^{\frac{2}{2-\alpha}} \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty e^{-r\eta} r^3 r^{\alpha-2} dr\right) \\
&\le C\left(\int_0^{\tau^{-\frac{1}{2-\alpha}}} r^{1-\alpha} dr + \tau^{-\frac{\alpha}{2-\alpha}} + \tau^{\frac{2}{2-\alpha}} \int_{\tau^{-\frac{1}{2-\alpha}}}^\infty r^{\alpha-2} dr\right) = C\left(\frac{\tau^{-1}}{2-\alpha} + \tau^{-\frac{\alpha}{2-\alpha}} + \frac{\tau^{\frac{1-\alpha}{2-\alpha}}}{1-\alpha}\right).
\end{aligned}$$

This together with (3.21) and the asymptotic behaviour of $|s_1|$ as $\tau \to 0^+$ (see Lemma 3.4) concludes the proof of (3.17). ■

We can now prove Theorem 3.1

Proof of Theorem 3.1. We start estimating $|\bar{\psi}_\tau(\eta)|$ and $|\bar{\psi}'_\tau(\eta)|$ directly from (3.8) and (3.13) and using the estimates of Lemma 3.5:

$$|\bar{\psi}_\tau(\eta)| \le \tau C(\|\phi_0''\|_{L^\infty}) \int_0^\eta |v'(y)|dy + \tau^2 C|v'(\eta)| + \int_0^\eta |v'(y)|\, |\bar{\psi}_\tau(\eta - y)|$$

$$|\bar{\psi}'_\tau(\eta)| \le \tau C(|v'(\eta)| + \tau|v''(\eta)|) + \int_0^\eta |v'(y)| \left(|\bar{\psi}_\tau(\eta - y)| + \tau(\eta - y)^{1-\alpha} + \tau\|\phi_0'''\|_{L^\infty}\right) dy$$
$$+ \int_0^\eta |v'(y)|\, |\bar{\psi}'_\tau(\eta - y)|dy| \,.$$

The result follows by using Gronwall's Lemma, which implies that

$$|\bar{\psi}_\tau(\eta)| \le A_1(\eta) + \int_0^\eta A_1(x)B(x)e^{\int_x^\eta B(s)ds}dx \tag{3.22}$$

$$|\bar{\psi}'_\tau(\eta)| \le A_2(\eta) + \int_0^\eta A_2(x)B(x)e^{\int_x^\eta B(s)ds}dx \tag{3.23}$$

where

$$A_1(\eta) = \tau C(\|\phi_0''\|_{L^\infty}) \int_0^\eta |v'(y)|dy + \tau^2 C|v'(\eta)| \,, \quad B(\eta) = |v'(\eta - x)|$$

$$A_2(\eta) = \tau C(|v'(\eta)| + \tau|v''(\eta)|) + \int_0^\eta |v'(y)| \left(|\bar{\psi}_\tau(\eta - y)| + \tau(\eta - y)^{1-\alpha} + \tau\|\phi_0'''\|_{L^\infty}\right) dy \,.$$

Observe that A_1, $\int_0^\eta B_1(y)dy$ and $\exp(\int_s^\eta B_1(s)ds)$ are uniformly bounded for $\eta < C\tau^{-1/(2-\alpha)}$ and, moreover, $|A_1(\eta)| \le \tau C$ for some $C > 0$. Using this in (3.22) implies $|\bar{\psi}_\tau(\eta)| \le \tau C$ for some $C > 0$ for all such η's. We can apply this last fact to (3.23) to conclude the proof, since $A_2 \le \tau C\eta^{1-\alpha}$, thus in this range of η's $A_2 \le \tau^{1/(2-\alpha)}C$. ■

Finally, we discuss the fact that in the tail travelling waves are monotone as long as τ is small enough. This does not imply that the waves are decreasing in the whole of the domain, however. The following result holds:

Theorem 3.2 *Let ϕ be a solution of (2.1)-(2.2) as constructed in Theorem 2.1, then there exist a $\bar{\xi}$ large enough such that if τ is small enough ϕ is monotone decreasing in $(\bar{\xi}, \infty)$.*

Proof. We only sketch the proof. It can be done by a bootstrap argument based on the behaviour of the solutions in the tail for τ small enough. For every δ, let $\xi_\delta \in \mathbb{R}$ be such that $\xi_\delta = \inf\{\xi :\ \phi(\xi) - \phi_+ = \delta\}$. Let us write (2.1) as follows:

$$h(\phi) = \mathcal{D}^\alpha_{\xi_\delta}\phi + \int_{-\infty}^{\xi_\delta} \frac{\phi'(y)}{(\xi - y)^\alpha}\, dy + \tau\phi'' \,. \tag{3.24}$$

Let $\psi(\xi) = \phi(\xi) - \phi_+$, then (3.24) reads

$$h(\phi) - h(\phi_+) - h'(\phi_+)\psi = \mathcal{D}^\alpha_{\xi_\delta}\psi + \int_{-\infty}^{\xi_\delta} \frac{\psi'(y)}{(\xi - y)^\alpha}\, dy + \tau\psi'' - h'(\phi_+)\psi \,,$$

where we use that $h(\phi_+) = 0$. It is convenient to shift the independent variable as follows $\zeta = \xi - \xi_\delta$ and let $\psi(\xi) = \Psi(\zeta)$. Then, (3.24) reads, rearranging terms,

$$\tau\Psi'' + \mathcal{D}^\alpha_0\Psi - h'(\phi_+)\Psi = h(\phi) - h(\phi_+) - h'(\phi_+)\Psi - \int_{-\infty}^0 \frac{\Psi'(y)}{(\zeta - y)^\alpha}\, dy \,, \tag{3.25}$$

Then, we express the solution implicitly as in Appendix C with $a = -h'(\phi_+) > 0$

$$\Psi(\zeta) = \bar{\psi}(0^+)v(\zeta) + \frac{\tau}{h'(\phi_+)}\bar{\psi}'(0^+)v'(\zeta) + \frac{1}{h'(\phi_+)}\int_0^\zeta v'(y)\mathcal{Q}(\zeta - y)\,dy\,, \tag{3.26}$$

$$Q(\zeta) := h(\phi(\zeta + \xi_\delta)) - h(\phi_+) - h'(\phi_+)\Psi(\zeta) - \int_{-\infty}^0 \frac{\Psi'(y)}{(\zeta - y)^\alpha}\,dy \tag{3.27}$$

(cf. (C.6)) where $v(\zeta)$ has been computed in Appendix C and reads (see (C.12) and (C.13))

$$v(\zeta) = -h'(\phi_+)\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\zeta r}K_\alpha(r)\,dr + 2\mathrm{Re}\left(e^{s_1\zeta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right), \tag{3.28}$$

(cf. (3.9)) where, as before, s_1 is the solution of

$$\tau z^2 + z^\alpha - h'(\phi_+) = 0 \tag{3.29}$$

with positive imaginary part, and $\beta = \arg(s_1) \in (\pi/2, \pi)$ (see appendixes B and C), and where

$$K_\alpha(r) = r^{\alpha-1}\tilde{K}_\alpha(r) \quad \text{with} \quad \tilde{K}_\alpha(r) = \frac{1}{(\tau r^2 - h'(\phi_+))^2 + 2(\tau r^2 - h'(\phi_+))r^\alpha\cos(\alpha\pi) + r^{2\alpha}}\,.$$

Observe that $v(\zeta)$ is the sum of a monotone (first term on the right-hand side of (3.28)) and a oscillatory term (second term on the right-hand side of (3.28)). On the other hand, the non-monotone contribution of $v'(\zeta)$ is given by the derivative of the exponential oscillatory term of $v(\zeta)$ if ζ is very large (thus if δ is very small), and the last term on the right hand side of (3.26) can be made arbitrarily small for δ small. Thus taking ζ large enough and τ small enough the small oscillations get damped by the algebraic decaying terms of the monotone part. Observe that $\mathrm{Re}\left(e^{s_1\zeta}\frac{\tau s_1+s_1^{\alpha-1}}{2\tau s_1+\alpha s_1^{\alpha-1}}\right)$ has infinitely many oscillations with frequency $\omega = \rho\sin\beta = \mathrm{Im}(s_1)$, but its amplitude decreases exponentially like $e^{(|s_1|\cos\beta)\zeta}$ as $\zeta \to \infty$ (recall that $|s_1| = O(\tau^{-1/(2-\alpha)})$ and $\cos(\beta) < 0$). ■

4 Asymptotic stability of monotone travelling waves

In this section we assume that the travelling waves found in Theorem 2.1 are monotone (decreasing) and we prove their dynamic stability. Existence of such waves is guaranteed for small enough values of τ as the analysis of the previous section suggests. The stability analysis is done in a similar way as for the KdV-Burgers equation and the Burgers equation (see e.g. [17], and also [1] for the corresponding fractional diffusion Burgers equation). We next outline the key ideas of the proof.

It is convenient to first change variables to $x \to \xi = x - ct$ in (1.1), so it becomes

$$\partial_t u + \partial_\xi(u^2 - cu) = \partial_\xi\mathcal{D}^\alpha u + \tau\partial_\xi^3 u\,. \tag{4.1}$$

We then look for solutions of (4.1) which are a small perturbation of a travelling wave and that in particular share the same far-field values. Let $u_0(\xi)$ be an initial datum and $\phi(\xi)$ a monotone travelling wave as constructed in Theorem 2.1, with a shift in ξ chosen such that

$$\int_{\mathbb{R}}(u_0(\xi) - \phi(\xi))\,d\xi = 0\,. \tag{4.2}$$

Observe that conservation of mass, a property satisfied by (4.1), implies that

$$\int_{\mathbb{R}} (u(t,\xi) - \phi(\xi))d\xi = 0\,, \quad \text{for all} \quad t \geq 0\,.$$

Now, the perturbation $U = u - \phi$ satisfies the equation

$$\partial_t U + \partial_\xi((2\phi - c)U) + \partial_\xi U^2 = \partial_\xi \mathcal{D}^\alpha U + \tau \partial_\xi^3 U\,. \tag{4.3}$$

The aim is to show that U tends to 0 in a suitable sense as t tends to ∞ for small enough $U_0 = u_0 - \phi$. We use integral estimates. For instance, testing (4.3) with U, we get

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \int_{\mathbb{R}} \phi' U^2 d\xi = -a_\alpha \|U\|^2_{\dot{H}^{(1+\alpha)/2}}\,, \tag{4.4}$$

where several integrations by parts have been carried out. Since we are assuming that $\phi' \leq 0$, the second term in (4.4) is non-positive. We next introduce the primitive of the perturbation and of the corresponding initial data

$$W(t,\xi) = \int_{-\infty}^{\xi} U(t,\eta)d\eta\,, \quad W_0(\xi) = \int_{-\infty}^{\xi} U_0(\eta)d\eta\,,$$

which satisfies the integrated version of (4.3),

$$\partial_t W + (2\phi - c)\partial_\xi W + (\partial_\xi W)^2 = \mathcal{D}^\alpha \partial_\xi W + \tau \partial_\xi^3 W\,, \tag{4.5}$$

and

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \int_{\mathbb{R}} \phi' W^2 d\,\xi + \int_{\mathbb{R}} (\partial_\xi W)^2 W\, d\xi = -a_\alpha \|W\|^2_{\dot{H}^{(1+\alpha)/2}}\,. \tag{4.6}$$

This integral identity has the crucial property that the term involving ϕ' is non-negative. In the cubic term (arising from the nonlinearity) we can estimate $|W|$ by the L^∞-norm, this factor can then be controlled by using the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

The right-hand side in (4.4) is obtained using Plancherel's theorem that, together with $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$, implies that

$$\int_{\mathbb{R}} \operatorname{sgn}(k)\, |k|^j\, |\hat{u}(k,t)|^2\, dk = 0 \quad j \in \mathbb{N}\,.$$

We observe that (as one can easily check based on (1.3) and (1.5))

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha) = \mathcal{F}(\mathcal{D}^\alpha \partial_x) = -(a_\alpha - i b_\alpha \operatorname{sgn}(k))|k|^{\alpha+1}\,, \tag{4.7}$$

and from this we can obtain in the same way as for (4.4) the right-hand side of (4.6).

The well-posedness result below and the fact that (1.1), or (4.1), is a third order equation requires that we work with $U \in H^2$, in fact we shall require that at least $U_0 \in H^3(\mathbb{R})$ since we need integral estimates of higher order. We assume for the moment that the following theorem holds, and we prove it in Section 5:

Theorem 4.1 *For every $U_0 \in H^s(\mathbb{R})$, $s \geq 3$ and assuming that $\phi \in H^{s+1}(\mathbb{R})$, there is a $T > 0$ such that (4.3) with initial data $U(\xi,0) = U_0(\xi)$ has a unique solution $U \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R}))$ satisfying*

$$\|U\|_{H^s} \leq C\|U_0\|_{H^s}\,.$$

The same result applies to (4.5) with initial condition $W(\xi,0) = W_0(\xi)$.

Then assuming this, we can locally perform further integral estimates of the consecutive differentiations of (4.5). Namely, from

$$\partial_t\partial_\xi^2 W + \partial_\xi^2((2\phi - c)\partial_\xi W) + \partial_\xi^2(\partial_\xi W)^2 = \partial_\xi^2\mathcal{D}^\alpha\partial_\xi W + \tau\partial_\xi^5 W\,. \tag{4.8}$$

we obtain, testing with $\partial_\xi^2 W$, the integral identity

$$\frac{1}{2}\frac{d}{dt}\|\partial_\xi^2 W\|_{L^2}^2 - \int_{\mathbb{R}}\phi'''(\partial_\xi W)^2 d\xi + 3\int_{\mathbb{R}}\phi'(\partial_\xi^2 W)^2 d\xi + \int_{\mathbb{R}}(\partial_\xi^2 W)^3 d\xi = -a_\alpha\|\partial_\xi^2 W\|_{\dot{H}^{(1+\alpha)/2}}^2\,. \tag{4.9}$$

Further, from the equation

$$\partial_t\partial_\xi^3 W + \partial_\xi^3((2\phi - c)\partial_\xi W) + \partial_\xi^3(\partial_\xi W)^2 = \partial_\xi^3\mathcal{D}^\alpha\partial_\xi W + \tau\partial_\xi^6 W\,, \tag{4.10}$$

we obtain testing now with $\partial_\xi^3 W$ the integral identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_\xi^3 W\|_{L^2}^2 + 2\int_{\mathbb{R}}\phi'''\partial_\xi W\,\partial_\xi^3 W d\xi - 3\int_{\mathbb{R}}\phi'''(\partial_\xi^2 W)^2 d\xi + 5\int_{\mathbb{R}}\phi'(\partial_\xi^3 W)^2 d\xi \\ + 5\int_{\mathbb{R}}\partial_\xi^2 W(\partial_\xi^3 W)^2 d\xi = -a_\alpha\|\partial_\xi^3 W\|_{\dot{H}^{(1+\alpha)/2}}^2\,.\end{aligned} \tag{4.11}$$

In order to justify the vanishing of the integral terms coming from the highest order term in each equation, we use Theorem 4.1 above, that allows to obtain these identities in $[0, T]$ provided the initial condition $W_0 \in H^{s+1}$ with $s \geq 3$. The proof of stability then uses a combination of the integral identities just obtained choosing the coefficients in such a way that the resulting functional is decreasing in time. The main point is that the terms with the wrong sign, coming in general from the nonlinear terms and the ones involving derivatives of ϕ, can be controlled by the dissipative ones via versions of the interpolation inequality

$$b^2\|g\|_{\dot{H}^1}^2 \leq b^{1+\alpha}\|g\|_{\dot{H}^{(1+\alpha)/2}}^2 + b^{3+\alpha}\|g\|_{\dot{H}^{(3+\alpha)/2}}^2\,, \quad b > 0 \tag{4.12}$$

that holds as a consequence of $(bk)^2 \leq |bk|^{1+\alpha} + |bk|^{3+\alpha}$, $k \in \mathbb{R}$ with $b > 0$. We shall also need the following one

$$\|g\|_{\dot{H}^1}^2 \leq \max\{1, 1/b\}\left(\|g\|_{\dot{H}^{(1+\alpha)/2}}^2 + b\|g\|_{\dot{H}^{(3+\alpha)/2}}^2\right)\,, \quad b > 0\,, \tag{4.13}$$

that follows from $(k)^2 \leq |k|^{1+\alpha} + |k|^{3+\alpha} \leq \min\{\tilde{b}|k|^{1+\alpha} + |k|^{3+\alpha}, |k|^{1+\alpha} + \tilde{b}|k|^{3+\alpha}\}$ for any $\tilde{b} > 1$.

After these preparations we can prove the following result.

Theorem 4.2 *Let ϕ be a travelling wave as in Theorem 2.1, and let $u_0(\xi)$ be an initial datum for (4.1), such that $W_0(\xi) = \int_{-\infty}^{\xi}(u_0(\eta) - \phi(\eta))d\eta$ satisfies $W_0 \in H^{s+1}$ with $s \geq 3$. Then if $\|W_0\|_{H^3}$ is small enough, the Cauchy problem for (4.1) with initial datum u_0 has a unique global solution with $u(t) \in H^{s-1}$ for all $t > 0$ converging to the travelling wave in the sense that*

$$\lim_{t\to\infty}\int_t^\infty \|u(\sigma,\cdot) - \phi\|_{H^2}^2 d\sigma = 0\,.$$

Note that (4.2), which can be translated into the condition $W_0(\pm\infty) = 0$, is part of the assumption $W_0 \in H^s$ in Theorem 4.2.

Proof. As mentioned earlier the integral identities (4.4)-(4.11) are justified by Theorem 4.1 in an interval $[0, T]$. Then, (4.4) and (4.6) imply the estimates

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 - C_0\|U\|_{L^2}^2 \leq -a_\alpha\|U\|_{\dot{H}^{(1+\alpha)/2}}^2 \,, \tag{4.14}$$

$$\frac{1}{2}\frac{d}{dt}\|W\|_{L^2}^2 - \|W\|_{L^\infty}\|\partial_\xi W\|_{L^2}^2 \leq -a_\alpha\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 \,, \tag{4.15}$$

with $C_0 = \|\phi'\|_{L^\infty}$. Now, (4.9) and (4.11) imply the estimates

$$\frac{1}{2}\frac{d}{dt}\|\partial_\xi^2 W\|_{L^2}^2 - C_1\|\partial_\xi W\|_{L^2}^2 - \left(\|\partial_\xi^2 W\|_{L^\infty} + 3C_0\right)\|\partial_\xi^2 W\|_{L^2}^2 \leq -a_\alpha\|\partial_\xi^2 W\|_{\dot{H}^{(1+\alpha)/2}}^2 \,, \tag{4.16}$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_\xi^3 W\|_{L^2}^2 - C_1\|\partial_\xi W\|_{L^2}^2 - 3C_1\|\partial_\xi^2 W\|_{L^2}^2 - \left(5\|\partial_\xi^2 W\|_{L^\infty} + 5C_0 + C_1\right)\|\partial_\xi^3 W\|_{L^2}^2 \leq -a_\alpha\|\partial_\xi^3 W\|_{\dot{H}^{(1+\alpha)/2}}^2 \,. \tag{4.17}$$

with $C_1 = \|\phi'''\|_{L^\infty}$ and where we choose the constant $X > 0$ below.

Then we can combine the estimate by choosing three positive constants, say A, B and C to obtain the functional (that can be seen as a function of t)

$$J = \|W\|_{L^2}^2 + A\|\partial_\xi W\|_{L^2}^2 + B\|\partial_\xi^2 W\|_{L^2}^2 + C\|\partial_\xi^3 W\|_{L^2}^2$$

that clearly satisfies that there exist constants C_* and C^* such that

$$C_*\|W\|_{H^3}^2 \leq J \leq C^*\|W\|_{H^3}^2 \,. \tag{4.18}$$

Combining these estimates we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{dJ}{dt} - \left(\|W\|_{L^\infty} + AC_0 + BC_1 + CC_1\right)\|W\|_{\dot{H}^1}^2 \\
&-B\left(\|\partial_\xi^2 W\|_{L^\infty} + 3C_0\right)\|W\|_{\dot{H}^2}^2 - C\left(5\|\partial_\xi^2 W\|_{L^\infty} + 5C_0 + C_1\right)\|W\|_{\dot{H}^3}^2 \\
&+a_\alpha\left(\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + A\|W\|_{\dot{H}^{(3+\alpha)/2}}^2 + B\|W\|_{\dot{H}^{(5+\alpha)/2}}^2 + C\|W\|_{\dot{H}^{(7+\alpha)/2}}^2\right) \leq 0 \,.
\end{aligned}$$

Then we can estimate as follows

$$(AC_0 + BC_1 + CC_1)\|W\|_{\dot{H}^1}^2 \leq \frac{a_\alpha}{2}\left(\|W\|_{\dot{H}^{(1+\alpha)/2}}^2 + \frac{A}{2}\|W\|_{\dot{H}^{(3+\alpha)/2}}^2\right) , \tag{4.19}$$

$$3(CC_1 + BC_0)\|W\|_{\dot{H}^2}^2 \leq \frac{a_\alpha}{2}\left(\frac{A}{2}\|W\|_{\dot{H}^{(3+\alpha)/2}}^2 + \frac{B}{2}\|W\|_{\dot{H}^{(5+\alpha)/2}}^2\right) , \tag{4.20}$$

$$C\left(5C_0 + C_1\right)\|W\|_{\dot{H}^3}^2 \leq \frac{a_\alpha}{2}\left(\frac{B}{2}\|W\|_{\dot{H}^{(5+\alpha)/2}}^2 + C\|W\|_{\dot{H}^{(7+\alpha)/2}}^2\right) . \tag{4.21}$$

In order to obtain this we use (4.12), this implies, identifying coefficients, that the following must be satisfied:

$$\frac{2}{a_\alpha}(AC_0 + BC_1 + CC_1) = \left(\frac{A}{2}\right)^{\frac{1-\alpha}{2}} ,$$

$$\frac{12}{a_\alpha A}(CC_1 + BC_0) = \left(\frac{B}{A}\right)^{\frac{1-\alpha}{2}} ,$$

$$B = C\left(\frac{2^{\alpha+3}}{a_\alpha}(5C_0 + C_1)\right)^{\frac{2}{1+\alpha}} .$$

This can be solved using the third equation to eliminate C from the second one that can then be solve for B/A. Then one can eliminate C and B from the first equation to solve for A, and recovering B and C from the second and third equations, one gets

$$A = \frac{1}{2^{\frac{3-\alpha}{1+\alpha}}} \left(\frac{a_\alpha L_1 L_2}{C_0 L_1 L_2 + C_1(L_1+1)} \right)^{\frac{2}{1+\alpha}}, \; B = \frac{A}{L_2}, \; C = \frac{B}{L_1} \tag{4.22}$$

where

$$L_1 = \left(\frac{2^{\alpha+3}}{a_\alpha} (5C_0 + C_1) \right)^{\frac{2}{1+\alpha}}, \; L_2 = \left(\frac{12}{a_\alpha} \left(C_0 + \frac{C_1}{L_1} \right) \right)^{\frac{2}{1+\alpha}}.$$

Finally, we can also estimate the terms that contain coefficients with L^∞ norms of W and/or its second derivative. Namely, the following hold easily from (4.13)

$$\|W\|^2_{\dot H^1} \le \max\{1, 2/A\} \left(\|W\|^2_{\dot H^{(1+\alpha)/2}} + \frac{A}{2} \|W\|^2_{\dot H^{(3+\alpha)/2}} \right) \tag{4.23}$$

$$B\|W\|^2_{\dot H^2} \le \frac{B}{A} \max\{1, 2A/B\} \left(\frac{A}{2} \|W\|^2_{\dot H^{(3+\alpha)/2}} + \frac{B}{2} \|W\|^2_{\dot H^{(5+\alpha)/2}} \right) \tag{4.24}$$

$$5C\|W\|^2_{\dot H^3} \le \frac{10C}{B} \max\{1, B/2C\} \left(\frac{B}{2} \|W\|^2_{\dot H^{(5+\alpha)/2}} + C\|W\|^2_{\dot H^{(7+\alpha)/2}} \right) \tag{4.25}$$

and hence the combined estimate reads:

$$\begin{aligned} &\frac{1}{2}\frac{dJ}{dt} + \left(\frac{a_\alpha}{2} - \max\{1, 2/A\} \|W\|_{L^\infty} \right) \left(\|W\|^2_{\dot H^{(1+\alpha)/2}} + \frac{A}{2} \|W\|^2_{\dot H^{(3+\alpha)/2}} \right) \\ &+ \left(\frac{a_\alpha}{2} - \max\{B/A, 2\} \|\partial_\xi^2 W\|_{L^\infty} \right) \left(\frac{A}{2} \|W\|^2_{\dot H^{(3+\alpha)/2}} + \frac{B}{2} \|W\|^2_{\dot H^{(5+\alpha)/2}} \right) \\ &+ \left(\frac{a_\alpha}{2} - \max\{10C/B, 5\} \|\partial_\xi^2 W\|_{L^\infty} \right) \left(\frac{B}{2} \|W\|^2_{\dot H^{(5+\alpha)/2}} + C\|W\|^2_{\dot H^{(7+\alpha)/2}} \right) \le 0 \,. \end{aligned} \tag{4.26}$$

By the Sobolev embedding and (4.18) we have

$$\|W\|_{L^\infty}, \; \|\partial_\xi^2 W\|_{L^\infty} \le \|W\|_{H^3} \le \sqrt{\frac{1}{C_*} J}\,.$$

then letting the initial data be small enough such that $J(0) < C_* a_\alpha^2 (\min\{1/5, B/10C, A/B, A/2\})^2/8$, this and (4.26) imply the existence of a $\lambda > 0$ and a $\bar\lambda > 0$, such that

$$\frac{dJ}{dt} \le -\lambda \left(\|W\|^2_{\dot H^{(1+\alpha)/2}} + A\|W\|^2_{\dot H^{(3+\alpha)/2}} + B\|W\|^2_{\dot H^{(5+\alpha)/2}} + C\|W\|^2_{\dot H^{(7+\alpha)/2}} \right) \le -\bar\lambda \|U\|^2_{H^2}$$

for all $t > 0$. Integration with respect to time concludes the proof. ■

5 The proof of Theorem 4.1

We now prove the well-posedness of the Cauchy problem for (4.5) for a given initial data $W(0,x) = W_0(x) \in H^s(\mathbb{R})$ with $s \ge 3$. In fact we show that the operators involved satisfy certain properties, collected in Lemma 5.1 below. Then, we can prove the existence of the Cauchy problem for (4.3) by applying Lemma 5.1 and then we can apply the lemma again (writing $(\partial_\xi W)^2 = U\partial_\xi W$ in (4.5)) to conclude local existence of the Cauchy problem associated to (4.5).

In the analysis we follow the semigroup approach for the Korteweg-de Vries equation by Pazy in [16, Section 8.5], which is a variant of Kato [10], Namely, one has to use [16, Theorem 6.4.3] and the fact that the conditions of the theorem can be relaxed for time independent and transport type operators, as is done in [16, Section 8.5] for the KdV equation. We can then use the following version of [16, Theorem 6.4.3] to conclude local existence:

Lemma 5.1 *Let X and Y be Banach spaces such that Y is densely and continuously embedded in X. For every $r > 0$, let $A(v)$ be a family of operators $A(v)$, $v \in B_r := \{v \in Y : \|v\|_Y \le r\}$ that satisfies the conditions*

(i) Each of the operators of family $A(v)$, with $v \in B_r$, generate a C_0 semigroup $T_v(t)$ in Y such that $\|T_v(t)\| \le \exp(\beta t)$ where $\beta \ge c_0\|v\|_Y$ with c_0 independent of v.

(ii) There is an isomorphism S from Y onto X such that, for every $v \in B_r$, $SA(v)S^{-1} - A(v)$ is a bounded operator in X and

$$\|SA(v)S^{-1} - A(v)\|_{X\to X} \le C_1 \quad \text{for all } v \in B_r\,.$$

(iii) For each $v \in B_r$, $D(A(v)) \subset Y$, $A(v)$ is a bounded linear operator from Y into X and

$$\|A(v_1) - A(v_2)\|_{Y\to X} \le C_2\|v_1 - v_2\|_X \quad \text{for all } v_1\,, v_2 \in B_r\,.$$

Then, there exists a $T > 0$ such that the quasilinear problem

$$\begin{cases} \partial_t u + A(u)u = 0 \quad \text{for } 0 \le t \le T\,, \\ u(0) = u_0 \in Y\,, \end{cases} \tag{5.1}$$

has a unique mild solution $u \in C([0,T], Y) \cap C^1([0,T], X)$.

In order to verify the conditions of the lemma, we shall split the homogeneous linear operators of (4.3) (and of (4.5)) into two operators. Namely, we take

$$A_0 \,:\, D(A_0) = H^3(\mathbb{R}) \mapsto L^2(\mathbb{R})\,,\; u \longmapsto \partial_\xi^3 u\,, \tag{5.2}$$

$$A_2 \,:\, D(A_2) = H^2(\mathbb{R}) \mapsto L^2(\mathbb{R})\,,\; u \longmapsto \partial_\xi \mathcal{D}^\alpha u\,. \tag{5.3}$$

In addition we define the following family of transport operators for $v \in B_r$:

$$A_1(v) \,:\, D(A_1(v)) = H^1(\mathbb{R}) \mapsto L^2(\mathbb{R})\,,\; u \mapsto v\partial_\xi u\,. \tag{5.4}$$

In order to show that the conditions of Lemma 5.1 are satisfied, we first derive some properties of these operators.

Lemma 5.2 *(i) A_0 is the infinitesimal generator of a C_0 group of isometries on $L^2(\mathbb{R})$.*

(ii) For every $v \in H^s(\mathbb{R})$ with $s \ge 3$, the operator $A_1(v)$ is well-defined with domain $D(A_1(v)) = H^1(\mathbb{R})$ (dense in $L^2(\mathbb{R})$). Moreover, the operator $-(A_1(v) + \beta I)$ is dissipative for all $\beta \ge \beta_0(v) = c_0\|v\|_{H^s}$ where c_0 is independent of v. Also, if $u \in H^3(\mathbb{R})$ and $\varepsilon > 0$, the estimate

$$\|A_1(v)\|_{L^2} \le \varepsilon\|\partial_x^3 u\|_{L^2} + C_1(\varepsilon, \|v\|_{H^s})\|u\|_{L^2} \tag{5.5}$$

holds for some positive constant C_1 depending on ε and on $\|v\|_{H^s}$.

(iii) For every $0 < \alpha < 1$, the operator A_2 is well-defined with domain $D(A_2) = H^2(\mathbb{R})$ (dense in $L^2(\mathbb{R})$). Moreover, A_2 is dissipative with

$$(A_2 u, u) = -a_\alpha \|u\|^2_{\dot{H}^{\frac{\alpha+1}{2}}} \leq 0 \quad \textit{for } u \in H^2(\mathbb{R})\,,$$

where $a_\alpha = \sin(\frac{\alpha\pi}{2}) > 0$. Finally, it satisfies for $u \in H^3(\mathbb{R})$ and $\varepsilon > 0$ the estimate

$$\|A_2 u\|_{L^2} \leq \varepsilon \|\partial_x^3 u\|_{L^2} + C_2(\varepsilon)\|u\|_{L^2} \tag{5.6}$$

with $C_2(\varepsilon) = (\varepsilon p)^{\frac{1}{1-p}} (\frac{p}{p-1})^{-1}$ and $p = \frac{3}{\alpha+1} > 1$.

Proof. The proofs of *(i)* and *(ii)* can be found in [16, Lemma 8.5.2 and Lemma 8.5.3] respectively. We next prove *(iii)*.

That A_2 is dissipative follows from Plancherel's formula and the fact that $|\hat{u}(k)|^2 = |\hat{u}(-k)|^2$ imply

$$\int_{\mathbb{R}} \operatorname{sgn}(k)|k|^\gamma |\hat{u}(k,t)|^2 \, dk = 0 \quad \text{for } \gamma > 0$$

(see [1]). Namely, for every $u \in H^2(\mathbb{R})$ and $0 < \alpha < 1$,

$$\begin{aligned}(A_2 u, u) &= \int_{\mathbb{R}} \left(\partial_x \mathcal{D}^\alpha u\right) u dx = -\int_{\mathbb{R}} \left(a_\alpha - i b_\alpha \operatorname{sgn}(k)\right) |k|^{\alpha+1} |\mathcal{F}(u)(k)|^2 dk \\ &= -a_\alpha \int_{\mathbb{R}} |k|^{\alpha+1} |\mathcal{F}(u)(k)|^2 dk = -a_\alpha \|u\|^2_{\dot{H}^{\frac{\alpha+1}{2}}} \leq 0\,.\end{aligned}$$

It remains to prove (5.6). Indeed, for every $u \in H^3(\mathbb{R})$ and $0 < \alpha < 1$, we obtain the estimate

$$\begin{aligned}\|A_2 u\|^2_{L^2} &= \int_{\mathbb{R}} \left||k|^{\alpha+1} \mathcal{F}(u)(k)\right|^2 \, dk = \int_{\mathbb{R}} \left(|k|^{\alpha+1} |\mathcal{F}(u)(k)|^{\frac{\alpha+1}{3}}\right)^2 \left(|\mathcal{F}(u)(k)|^{1-\frac{\alpha+1}{3}}\right)^2 \, dk \\ &\leq \left(\int_{\mathbb{R}} \left(|k|^3 |\mathcal{F}(u)(k)|\right)^2 \, dk\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |\mathcal{F}(u)(k)|^2 \, dk\right)^{\frac{p-1}{p}} = \|\partial_x^3 u\|^{\frac{2}{p}}_{L^2} \|u\|^{\frac{2(p-1)}{p}}_{L^2}\,,\end{aligned}$$

where we have used again Plancherel's formula, the fact that $(a_\alpha^2 + b_\alpha^2) = 1$, and Hölder's inequality with $p = 3/(\alpha+1) > 1$. Taking the square root of the last inequality and using Young's inequality [1] for some $\varepsilon > 0$, we infer that

$$\|A_2 u\|_{L^2} \leq \varepsilon \|\partial_x^3 u\|_{L^2} + C_2(\varepsilon)\|u\|_{L^2}$$

with $C_2(\varepsilon) = (\varepsilon p)^{\frac{1}{1-p}} (\frac{p}{p-1})^{-1}$ and $p = \frac{3}{\alpha+1} > 1$. ■

Lemma 5.3 *(i) For every $v \in H^s$, the operator $A_0 - A_1(v)$ is the infinitesimal generator of a C_0-semigroup $T_v(t)$ on L^2 satisfying $T_v(t) \leq \exp(\beta t)$ for every $\beta \geq \beta_0(v) = c_0 \|v\|_{H^s}$, where c_0 is a constant independent of v.*

[1] For positive real numbers a, b and ε, as well as $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the inequality

$$a\, b \leq \varepsilon\, a^p + C(\varepsilon)\, b^q \quad \text{with } C(\varepsilon) = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$$

holds.

(ii) For every $0 < \alpha < 1$ and $v \in H^s(\mathbb{R})$ with $s \geq 3$, the operator $A_1(v) - A_2$ is well-defined from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Moreover, the operator $A(v) = A_0 + A_2 - A_1(v)$ is the infinitesimal generator of a C_0-semigroup $S_v(t)$ on L^2 satisfying

$$\|S_v(t)\| \leq \exp(\beta t) \tag{5.7}$$

for every $\beta \geq \beta_0(v) := c_0\|v\|_{H^s}$, where c_0 is a constant independent of v.

Proof. The proof of the statement *(i)* can be found in [16, Lemma 8.5.3]. We then prove *(ii)*.

Due to $v \in H^s(\mathbb{R})$ with $s \geq 3$, $\partial_x v \in H^{s-1}(\mathbb{R})$ and $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ such that $\|\partial_x v\|_{L^\infty} \leq C\|\partial_x v\|_{H^{s-1}} \leq C\|v\|_{H^s}$. For every $u \in H^2(\mathbb{R})$,

$$((A_1(v) - A_2)u, u) \geq -c_0\|v\|_{H^s}\|u\|_{L^2}^2 + a_\alpha\|u\|_{\dot{H}^{\frac{\alpha+1}{2}}}^2 \geq -c_0\|v\|_{H^s}\|u\|_{L^2}^2 ,$$

since c_0 and a_α are positive constants. Therefore $-(A_1(v) - A_2 + \beta I)$ is dissipative for all $\beta \geq \beta_0(v) := c_0\|v\|_{H^s}$. A_0 is a skew-adjoint operator, whence $A_0 + A_2 - A_1(v) - \beta I$ is also dissipative for $\beta \geq \beta_0(v)$. Moreover, due to the estimates (5.5) and (5.6),

$$\begin{aligned}\|(A_1(v) - A_2 + \beta I)u\|_{L^2} &\leq \|A_1(v)u\|_{L^2} + \|A_2 u\|_{L^2} + |\beta|\|u\|_{L^2} \\ &\leq 2\varepsilon\|\partial_x^3 u\|_{L^2} + C_3(\beta, \varepsilon, \|v\|_{H^s})\|u\|_{L^2}\end{aligned}$$

holds with a positive constant $C_3(\beta, \varepsilon, \|v\|_{H^s}) := C_1(\varepsilon, \|v\|_{H^s}) + C_2(\varepsilon) + |\beta|$. Due to [16, Corollary 3.3.3] and the last estimate with $0 < \varepsilon < \frac{1}{2}$, we conclude that $A_0 + A_2 - A_1(v) - \beta I$ is the infinitesimal generator of a C_0-semigroup of contractions on X for every $\beta \geq \beta_0(v)$. Therefore $A_0 + A_2 - A_1(v)$ is the infinitesimal generator of a C_0-semigroup $T_v(t)$ satisfying (5.7). ■

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We need only to check that the assumptions of Lemma 5.1 are satisfied for the operator, in the notation of this section

$$A(u)u = 2\phi' u + (2\phi - c + 2u)\partial_\xi u - \partial_x \mathcal{D}^\alpha u - \tau\partial_x^3 u .$$

We first observe that the second operator term can be seen as the sum of three operators of the form A_1 and the results of the previous lemmas apply. The first term has not been analysed in the previous lemmas, but since $\phi' \in H^{s+1}(\mathbb{R})$ the operator is bounded and adding it to the ones of the form A_1 preserves the properties shown above. Thus, Lemma 5.2 and 5.3 show that *(i)* holds with A_1 given by

$$A_1(v)u = (2\phi - c + 2w)\partial_x u \quad \text{with} \quad v := 2\phi - c + 2w ,$$

we only observe that the constants that depend on $\|v\|_{H^s}$ in these lemmas now depend on $\|w\|_{H^s}$, c, $\|\phi\|_\infty$ and $\|\phi'\|_\infty$.

Let us show that *(ii)* holds. We proceed as in [16]. For $s \geq 3$ (for $s \geq 3/2$, in fact) the operator

$$f \to \Lambda^s f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(ix \cdot \xi)(1 + \xi^2)^{\frac{s}{2}} \hat{f}(\xi)\, d\xi ,$$

is an isomorphism from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$. We notice that for $u, v \in H^s(\mathbb{R})$

$$(\Lambda^s A(v)\Lambda^{-s} - A(v))u = (\Lambda^s v - v\Lambda^s)\Lambda^{-s}\partial_x u + 2(\Lambda^s \phi' \Lambda^{-s} u - \phi' u) , \tag{5.8}$$

since for $u \in H^s$

$$\Lambda^s \partial_x^3 \Lambda^{-s} u = \partial_x^3 u \quad \text{and} \quad \Lambda^s \partial_x \mathcal{D}^\alpha \Lambda^{-s} u = \partial_x \mathcal{D}^\alpha u \,,$$

(see [16] for details). Therefore, for u, $v \in H^s$ and the multiplication operator $u \mapsto vu$, we deduce from (5.8) and [16, Lemma 8.5.4] that

$$\|(\Lambda^s v - v\Lambda^s)\Lambda^{1-s}\Lambda^{-1}\partial_x u\|_{L^2} \leq C\|v\|_{H^s}\|u\|_{L^2} \,.$$

It is easy to also show that

$$\|\Lambda^s \phi' \Lambda^{-s} u - \phi' u\|_{L^2} \leq (\|\phi'\|_{H^s} + \|\phi'\|_\infty)\|u\|_{L^2} \leq C\|u\|_{L^2} \,.$$

This estimate and $H^s(\mathbb{R})$ being dense in $\mathbb{R}$, implies that $\|SA(v)S^{-1} - A(v)\|_{L^2\to L^2} \leq C(\|w\|_{H^s} + c + \|\phi'\|_\infty + 1) \leq C$ for $w \in B_r \subset H^s$ and *(ii)* is satisfied with $S = \Lambda^s$.

It remains to show *(iii)*. Observe that for $s \geq 3$ and $0 < \alpha < 1$,

$$H^3(\mathbb{R}) = D(A(v)) \supset H^s(\mathbb{R}) \quad \text{for every} \quad v \in L^\infty(\mathbb{R}) \,,$$

and also

$$\begin{aligned} \|A(v)u\|_{L^2} &\leq \|2\phi' u\|_{L^2} + \|v\partial_x u\|_{L^2} + \|\partial_x \mathcal{D}^\alpha u\|_{L^2} + \|\partial_x^3 u\|_{L^2} \\ &\leq 2\|\phi'\|_{L^\infty}\|u\|_{L^2} + \|v\|_{L^\infty}\|\partial_x u\|_{L^2} + \|u\|_{\dot{H}^{\alpha+1}} + \|\partial_x^3 u\|_{L^2} \leq C(1 + \|v\|_\infty)\|u\|_{H^s} \,. \end{aligned}$$

Therefore, for $w \in B_r$, $A(v)$ is a bounded operator from $H^s(\mathbb{R})$ into $L^2(\mathbb{R})$. Moreover, if $v_1, v_2 \in B_r$ and $u \in H^s(\mathbb{R})$, then

$$\|(A(v_1) - A(v_2))u\|_{L^2} = \|(w_1 - w_2)\partial_x u\|_{L^2} \leq \|w_1 - w_2\|_{L^2}\|\partial_x u\|_{L^\infty} \leq C\|w_1 - w_2\|_{L^2}\|u\|_{H^s}$$

and *(iii)* holds as well, since $v_1 - v_2 = w_1 - w_2$. ■

Acknowledgements The authors wish to thank Michael Kaltenbäck for helpful comments. They also gratefully acknowledge the hospitality and financial support of Vienna University of Technology via its International Office, the financial support of the Spanish MINECO through project MTM2011-24109 and the hospitality of ICMAT.

Appendix

A The linear problem (2.8) on $(-\infty, \xi_0]$

In this appendix we show that the only solutions of the linear problem (2.8) are exponential functions in suitable weighted spaces. We shall assume without loss of generality that $\xi_0 = 0$ throughout this section. We use the approach introduced for *Wiener-Hopf* integral equations of the form

$$W(\xi) - \int_0^\infty K(\xi - y)W(y)dy = 0 \qquad \xi \geq 0 \,, \tag{A.1}$$

which are related to the Fredholm property by conditions on its symbol, see [21]. We use the result by Krein [13, 14] that extends the method to equations with L^1-integrable kernels. Namely:

Theorem A.1 (Krein (1958&62)) *Let $K \in L^1(\mathbb{R})$. If the symbol $a(k) := 1 - \int_{\mathbb{R}} e^{-ixk} K(x)\, dx$ $(= 1 - \sqrt{2\pi}\mathcal{F}[K])$ is elliptic, i.e. $\inf_{s\in\mathbb{R}} |a(s)| > 0$, and the winding number of the curve $\{a(s) : s \in (-\infty, \infty)\}$ around the origin is a non-negative integer r, then (A.1) has exactly r linearly independent solutions in any of the Lebesgue spaces $L^p(\mathbb{R}_+)$, with $1 \le p \le \infty$.*

We observe that have adapted Theorem A.1 from the original result by Krein that is stated for $\sqrt{2\pi}\mathcal{F}(-k)$ instead of $\mathcal{F}(k)$.

It is not obvious that (2.8) can be transformed into a Wiener-Hopf equation, i.e. to the form (A.1). In particular, we will investigate the problem on weighted spaces, such that it is admissible to consider the integrated equation and compute its symbol.

For a generalisation of the Wiener-Hopf method to other spaces we refer to [5] and for generalisations to convolution kernels being distributions we refer to [7].

In order to write (2.8) as a Wiener-Hopf equation we first change variables so that it is posed in $\mathbb{R}_+$ rather than in $\mathbb{R}_-$:

Lemma A.1 *If $V \in H^3(\mathbb{R}_+)$ is a solution of the integral equation*

$$0 = \tau V(\xi) + \int_\xi^\infty \int_y^\infty \mathcal{D}_-^\alpha[V](z)\, dz\, dy - h'(\phi_-) \int_\xi^\infty \int_y^\infty V(z)\, dz\, dy \tag{A.2}$$

where $\mathcal{D}_-^\alpha[V](\xi) := -d_\alpha \int_\xi^\infty \frac{V'(y)}{(y-\xi)^\alpha}\, dy$, then $v(\xi) := V(-\xi)$ for $\xi \in \mathbb{R}_-$ is a solution of (2.8).

Moreover, if $v \in H^3(\mathbb{R}_-)$ is a solution of (2.8) whose primitives are integrable, then $V(\xi) := v(-\xi)$ for $\xi \in \mathbb{R}_+$ is a solution of (A.2).

Proof. Due to a Sobolev embedding $H^3(\mathbb{R}_-) \hookrightarrow C_b^2(\mathbb{R}_-)$, a solution $v \in H^3(\mathbb{R}_-)$ has a representative in $C_b^2(\mathbb{R}_-)$, such that equation (2.8) holds pointwise. We perform the change of variables in (2.8) $V(-\xi) = v(\xi)$, such that $V \in H^3(\mathbb{R}_+) \hookrightarrow C_b^2(\mathbb{R}_+)$, $\xi \to -\xi \in \mathbb{R}_+$ and $y \to -y$ inside the integral term, to get

$$0 = \tau V''(\xi) + \mathcal{D}_-^\alpha[V](\xi) - h'(\phi_-)V(\xi) \qquad \forall \xi \in \mathbb{R}_+\,. \tag{A.3}$$

Finally, $V \in H^3(\mathbb{R}_+) \hookrightarrow C_b^2(\mathbb{R}_+)$ implies that V has a representative in $C_b^2(\mathbb{R}_+)$, such that

$$\lim_{\xi\to+\infty} V(\xi) = 0\,, \quad \lim_{\xi\to+\infty} V'(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to+\infty} V''(\xi) = 0\,.$$

Integrating (A.3) twice under the assumption that the primitives of V are integrable and reverting the change of variables yields (A.2). ∎

Lemma A.2 *Suppose that $\mu > 0$ and let*

$$V \in H_\mu^3(\mathbb{R}_+) = \{f \in H^3(\mathbb{R}_+) : \; f(\xi) = e^{-\mu\xi} g(\xi) \quad \text{for some} \quad g \in H^3(\mathbb{R}_+)\}$$

be a solution of (A.2). Then, the corresponding equation for W where $V(\xi) = e^{-\mu\xi}W(\xi)$ and $W \in H^3(\mathbb{R}_+)$ can be written in the form (A.1) with the L^1-integrable kernel

$$K(z) := -(\theta(-z)e^{\mu z}(-z)^{\alpha-1}) + h'(\phi_-)\, (\theta(-z)e^{\mu z}) * (\theta(-z)e^{\mu z}) \tag{A.4}$$

having support on the negative real line $\mathbb{R}_-$, and has symbol

$$a_\mu(k) = \frac{\tau(\mu - ik)^2 + (\mu - ik)^\alpha - h'(\phi_-)}{(\mu - ik)^2}\,. \tag{A.5}$$

Proof. Let $V(\xi) = e^{-\mu\xi}W(\xi)$ with $W \in H^3(\mathbb{R}_+)$, then (A.2) becomes, after multiplying by $e^{\mu\xi}$,

$$0 = \tau W(\xi) + e^{\mu\xi} \int_\xi^\infty \int_y^\infty \mathcal{D}_-^\alpha[We^{-\mu\cdot}](z)\,dz\,dy - e^{\mu\xi}h'(\phi_-) \int_\xi^\infty \int_y^\infty W(z)e^{-\mu z}\,dz\,dy\,. \quad \text{(A.6)}$$

We have to extract alternative representations for the integral operators in (A.6). The first integral operator satisfies

$$\begin{aligned} e^{\mu\xi} \int_\xi^\infty \int_y^\infty \mathcal{D}_-^\alpha[We^{-\mu\cdot}](z)\,dz\,dy &= \int_\xi^\infty e^{\mu(\xi-y)} \int_y^\infty e^{\mu(y-z)}\mathcal{D}_-^\alpha[We^{\mu(z-\cdot)}](z)\,dz\,dy \\ &= (\theta(-\cdot)e^{\mu\cdot}) * (\theta(-\cdot)e^{\mu\cdot}) * \mathcal{D}_-^\alpha[We^{\mu(z-\cdot)}](z)\,. \end{aligned}$$

Observe that

$$\mathcal{D}_-^\alpha[We^{\mu(z-\cdot)}](z) = -d_\alpha \int_z^\infty \frac{\left(W(\sigma)e^{\mu(z-\sigma)}\right)'}{(\sigma-z)^\alpha}\,d\sigma = -d_\alpha\Big([\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}]*W' - [\mu\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}]*W \Big)\,.$$

The convolution kernel $(\theta(-\cdot)e^{\mu\cdot})$ is L^1 integrable and its Fourier transform satisfies $\mathcal{F}[\theta(-\cdot)e^{\mu\cdot}](k) = (\mu - ik)^{-1}/\sqrt{2\pi}$. We use the identities $\mathcal{F}[f * g](k) = \sqrt{2\pi}\mathcal{F}[f](k)\,\mathcal{F}[g](k)$, $\mathcal{F}[fe^{\mu\cdot}](k) = \mathcal{F}[f](k + i\mu)$, and

$$\mathcal{F}\left[\frac{\theta(-\xi)}{(-\xi)^\alpha}\right](k) = \mathcal{F}\left[\frac{\theta(\xi)}{\xi^\alpha}\right](-k) = \frac{(-ik)^{\alpha-1}}{d_\alpha\sqrt{2\pi}}$$

for $k \in \mathbb{C}$, to compute

$$\begin{aligned} \mathcal{F}\Big[e^{\mu\cdot}\mathcal{D}^\alpha[We^{-\mu\cdot}](.)\Big](k) &= -d_\alpha\mathcal{F}\Big[[\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}] * W' - [\mu\theta(-\cdot)e^{\mu\cdot}(-\cdot)^{-\alpha}] * W\Big] \\ &= -d_\alpha\sqrt{2\pi}\Big(\mathcal{F}[\theta(-\cdot)(-\cdot)^{-\alpha}](k + i\mu)\Big)\Big((ik - \mu)\mathcal{F}[W](k)\Big) \\ &= (\mu - ik)^\alpha \mathcal{F}[W](k)\,. \end{aligned}$$

Therefore, the first integral operator is a pseudo-differential operator with

$$\mathcal{F}\left[e^{\mu\xi} \int_\xi^\infty \int_y^\infty \mathcal{D}_-^\alpha[We^{-\mu\cdot}](z)\,dz\,dy\right](k) = (\mu - ik)^{\alpha-2}\,\mathcal{F}[W](k)\,. \quad \text{(A.7)}$$

The second integral operator satisfies

$$-e^{\mu\xi}h'(\phi_-) \int_\xi^\infty \int_y^\infty W(z)e^{-\mu z}\,dz\,dy = -h'(\phi_-)\,(\theta(-.)e^{\mu\cdot}) * (\theta(-.)e^{\mu\cdot}) * W\,,$$

whence the integral operator is a pseudo-differential operator with

$$\mathcal{F}\left[-e^{\mu\xi}h'(\phi_-) \int_\xi^\infty \int_y^\infty W(z)e^{-\mu z}\,dz\,dy\right](k) = -h'(\phi_-)\,(\mu - ik)^{-2}\,\mathcal{F}[W](k)\,. \quad \text{(A.8)}$$

Thus the linear operator in (A.6) is a pseudo-differential operator with symbol (A.5).

It remains to justify that (A.6) is a Wiener-Hopf equation with some L^1 integrable kernel. Indeed, inverting the symbols (A.7) and (A.8) allows to write (A.6) as $\tau W(x) - K * W(x) = 0$ with K given by (A.4), which has support on the negative real line $\mathbb{R}_-$ and is L^1 integrable. ■

Theorem A.2 *Suppose that* $0 < \mu < \min\{\lambda, h'(\phi_-)/(2-\alpha)\}$*, where* λ *is the unique positive real root of (2.9). Then, all solutions of (2.8) that are in the space*

$$L^\infty_w(\mathbb{R}_-) = \{f \in L^\infty(\mathbb{R}_+) : \ f(\xi) = e^{\mu\xi} g(\xi) \quad \text{for some} \quad g \in L^\infty(\mathbb{R}_-)\}$$

are given by the one-parameter family $\{be^{\lambda\xi} : \ b \in \mathbb{R}\}$.

Proof. Let us see that the conditions of Theorem A.1 are satisfied by the symbol (A.5). The symbol a_μ gives a closed curve $s \to a_\mu(s)$ for $s \in \mathbb{R}$, since $\lim_{s\to\pm\infty} a_\mu(s) = \tau$. The ellipticity follows from the fact that the numerator of (A.5) only vanishes identically at $s = 0$ and $\mu = \lambda$ (by assumption $0 < \mu < \lambda$) and the denominator of

$$|a_\mu(s)|^2 = \frac{|\tau(\mu - is)^2 + (\mu - is)^\alpha - h'(\phi_-)|^2}{(\mu^2 - s^2)^2 + 4\mu^2 s^2}$$

does not vanish.

Moreover, the winding number of the closed curve is a well-defined integer. In order to compute the winding number around the origin we add the number of times that the curve crosses the negative real line in the anticlockwise direction and subtract the number of times it does it in the clockwise one.

There is a crossing at $s = 0$, since

$$a_\mu(0) = \frac{\tau\mu^2 + \mu^\alpha - h'(\phi_-)}{\mu^4} < 0 \quad \mu \in (0, \lambda)\,.$$

Let us see that this is the only one. In order to do that we compute

$$\mathrm{Re}(a_\mu(s)) = \tau + \frac{(\mu^2+s^2)^{\frac{\alpha}{2}}\left((\mu^2 - s^2)\cos(\Theta_{s,\mu}\alpha) + 2s\mu\sin(\Theta_{s,\mu}\alpha)\right) - (\mu^2 - s^2)h'(\phi_-)}{\mu^4 + s^4 + 2\mu^2 s^2}\,, \tag{A.9}$$

and

$$\mathrm{Im}(a_\mu(s)) = \frac{(\mu^2+s^2)^{\frac{\alpha}{2}}\left(2s\mu\cos(\Theta_{s,\mu}\alpha) - (\mu^2 - s^2)\sin(\Theta_{s,\mu}\alpha)\right) - 2s\mu h'(\phi_-)}{\mu^4 + s^4 + 2\mu^2 s^2}\,.$$

We observe that when the curve crosses the real line then $\mathrm{Im}(a\mu(s)) = 0$, imposing this condition and using (2.6) gives

$$h'(\phi_-) = (\mu^2+s^2)^{\frac{\alpha+2}{2}} \frac{\sin(\Theta_{s,\mu}\alpha)}{2s\mu} > 0$$

and substituting this expression into (A.9) gives

$$\mathrm{Re} = \tau + (\mu^2+s^2)^{\frac{\alpha}{2}} \frac{\sin(\Theta_{s,\mu}\alpha)}{2s\mu} > 0\,,$$

thus the curve crosses the negative real line only once. It remains to determine whether the crossing is in the clockwise or anticlockwise direction. We compute[2]

$$\frac{d}{ds}\mathrm{Re}(a_\mu(s))\big|_{s=0} = \frac{3h'(\phi_-)}{\mu^2} > 0$$

[2]We give the full expressions of the derivatives for completeness:

$$\frac{d}{ds}\mathrm{Re}(a_\mu(s)) = \frac{(2-\alpha)(\mu^2+s^2)^{\frac{\alpha}{2}}\Big((\mu^2 - 3s^2)\mu\sin(\Theta_{s,\mu}\alpha) - (3\mu^2 - s^2)s\cos(\Theta_{s,\mu}\alpha)\Big) + (3\mu^2 - s^2)h'(\phi_-)}{\mu^4 + s^4 + 2\mu^2 s^2}$$

and, under the assumption on μ,

$$\frac{d}{ds}\mathrm{Im}(a_\mu(s))|_{s=0} = \frac{(2-\alpha)\mu^\alpha - h'(\phi_-)}{\mu} < 0\,.$$

Thus the curve $a_\mu(s)$ runs once around the origin in the anticlockwise sense, i.e. the winding number is 1. Applying Theorem A.1 and changing from W to V and then to the original variable imply the statement. ■

B The roots of (2.9), (3.10) and (3.29)

In this appendix we show that (2.9) has exactly one real positive root two complex conjugate roots with negative real part. We prove the result for the more general algebraic equation:

$$f(z) = z^2 + az^\alpha - b \quad \text{for} \quad a\,, b > 0\,, \quad \alpha \in (0,1)\,. \tag{B.1}$$

In order to prove this we use a version of Rouche's theorem as in [4], where it is shown that

$$g(z) = z^2 + az^\alpha + b \quad \text{for} \quad a\,, b > 0\,, \quad \alpha \in (0,1) \tag{B.2}$$

has exactly two complex conjugate roots with negative real part. We observe that (3.10) and (3.29) are of this form, so they have two complex conjugate roots with negative real part.

Lemma B.1 *For any positive values of a, b and any value $\alpha \in (0,1)$. Assume that z is the principal part of z^α ($-\pi < arg(z) < \pi$), then (B.1) has exactly one real positive root and two complex conjugate roots with negative real part, and (B.2) has exactly two complex conjugate roots with negative real part on the principal branch.*

Proof. The statement about (B.2) has been shown in [4] (Theorem 13), we do not prove it here. In fact the proof for (B.1) can be done along the same lines, as follows.

First, it is easy to see that the unique positive real root of (B.1) is the only root with positive real part (see argument following (2.9)). Indeed, we argue by contradiction and assume that there exists a $z_0 \in \mathbb{C}$ that solves $f(z) = 0$ and that $\mathrm{Re}(z_0) > 0$. Since clearly $\bar{z}_0$ must also solve $f(z) = 0$ we can assume that $\arg(z_0) \in (0, \pi/2)$. Then, inspection of $f(z_0)$ shows that $\mathrm{Im}(f(z_0)) > 0$, which contradicts the assumption $f(z_0) = 0$.

It is also easy to show by simple inspection of $f(z)$ that there are neither purely imaginary roots of (B.1) nor negative real ones.

Since $f(z) = 0$ implies $f(\bar{z}) = 0$ we can restrict ourselves to the open sector

$$Q := \{z \in \mathbb{C} : \ \arg(z) \in (\pi/2, \pi)\}\,.$$

It then remains to show that there is only one $z \in Q$ such that $f(z) = 0$. In order to do that we use a version of Rouche's theorem by T.Estermann [6]. This theorem says that if f and l are regular functions on a simply connected region $\Omega \subset \mathbb{C}$ and if $|f - l| < |f| + |l|$ on $\partial\Omega$ then

and

$$\frac{d}{ds}\mathrm{Im}(a_\mu(s))|_{s=0} = -\frac{(\mu^2+s^2)^{\frac{\alpha}{2}}(2-\alpha)\left((3s^2-\mu^2)\mu\cos(\Theta_{s,\mu}\alpha) + (s^2-3\mu^2)s\sin(\Theta_{s,\mu}\alpha)\right) - 2\mu(\mu^2-3s^2)h'(\phi_-)}{\mu^4+s^4+2\mu^2 s^2}\,.$$

f and h have the same number of zeros in Ω counted with their multiplicity. We shall then apply this to f as in (B.1), which we shall compare to l given by

$$l(z) = z^2 + i\,.$$

Let $z \in Q \cap A_R$ where, for any given $R > 0$, A_R denotes the ring $\{z \in \mathbb{C} : \ 1/R < |z| < R\}$. One can check that if R is large enough so that for all $\theta \in [\pi/2, \pi]$ $z = Re^{\theta i}$ then

$$|f(z)| + |l(z)| > 2R^2 - aR^\alpha - b - 1 > aR^\alpha + b + 1 > |f(z) - l(z)|\,.$$

In order to prove the strict inequality on the rest of the boundary of $Q \cap A_R$, we argue by contradiction and assume that $|f - l| = |f| + |l|$ in this region. Thus, in particular, there exists a $L > 0$ such that $f = -L\,l$ there. Then for $z = e^{i\theta}/R$ with $\theta \in (\pi/2, \pi)$ we obtain $|\mathrm{Im}(l)| > |\mathrm{Im}(f)|$, but $|\mathrm{Re}(l)| < |\mathrm{Re}(f)|$ if R is sufficiently large, and this contradicts $f = -L\,l$. Finally, if $\theta \in \{\pi/2, \pi\}$ then $\mathrm{Im}(-l) = -c < 0$ and $\mathrm{Im}(f) = a|z|\sin(\alpha\theta) > 0$, and this also contradicts $f = -L\,lh$. Since we can take R as large as we want, this concludes the proof. ■

Let us for completeness compute the two term expansion of the roots of (2.9) for very small values of τ (this can be made rigorous be applying the implicit function theorem): A regular expansion gives the real root, in this case it is easy to obtain by inserting the ansatz $\lambda = \lambda_0 + \tau\lambda_1 + O(\tau^2)$, and one gets that

$$\lambda = h'(\phi_-)^{\frac{1}{\alpha}} - \frac{\tau}{\alpha} h'(\phi_-)^{\frac{3-\alpha}{\alpha}} + O(\tau^2)\,. \tag{B.3}$$

The complex conjugated roots are obtained by first performing the scaling $\lambda = \tau^{-\frac{1}{2-\alpha}}\bar{\lambda}$, and inserting the ansatz $\bar{\lambda} = \bar{\lambda}_0 + \tau^{\frac{1}{2-\alpha}}\bar{\lambda}_1$ in the rescaled equation $\bar{\lambda}^2 + \lambda - \tau^{\frac{1}{2-\alpha}} = 0$. To leading order one gets three zeros, namely $\bar{\lambda}_0 = 0$, $e^{i\pi/(\alpha-2)}$ and $e^{-i\pi/(\alpha-2)}$. The first one corresponds to the real one already found, from the other two one then gets (in the original scaling):

$$\lambda = e^{\pm i\pi\frac{1}{\alpha-2}}\frac{1}{\tau^{\frac{1}{2-\alpha}}} + \frac{h'(\phi_-)}{2e^{\pm i\pi\frac{1}{\alpha-2}} + \alpha e^{\pm i\pi\frac{(\alpha-1)}{\alpha-2}}} + O(\tau^{\frac{1}{2-\alpha}})\,. \tag{B.4}$$

A similar approach can be used to compute the expansion of the zeros of

$$\tau z^2 + az^\alpha + b = 0 \tag{B.5}$$

provided that a and b are of order 1 as $\tau \to 0$. In that case the zeros are approximated by

$$z = a^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}}\frac{1}{\tau^{\frac{1}{2-\alpha}}} - \frac{b}{2a^{\frac{1}{\alpha-2}}e^{\pm i\pi\frac{1}{\alpha-2}} + a^{\frac{1}{\alpha-2}+1}\alpha e^{\pm i\pi\frac{(\alpha-1)}{\alpha-2}}} + O(\tau^{\frac{1}{2-\alpha}}) \quad \text{as} \quad \tau \to 0^+\,. \tag{B.6}$$

C Computation of the linear problems (2.23), (3.5) and (3.25)

In this appendix we give a way of solving implicitly equations of the type (3.5) and (3.25) for a given inhomogeneity and initial conditions on the unknown and its derivative. The method is by using the Laplace transform and the computations can be found in e.g. [4] and [8], we follow the latter.

Given the initial value problem

$$\tau\psi'' + \mathcal{D}_0^\alpha\psi + a\psi = Q(\eta)\,, \quad {}' = \frac{d}{d\eta} \tag{C.1}$$

subject to

$$\psi(0^+) = C_0\,, \quad \psi'(0^+) = C_1\,. \tag{C.2}$$

we apply the Laplace transform, $\mathcal{L}$ to get

$$\mathcal{L}(\psi)(s) = \frac{1}{\tau s^2 + s^\alpha + a}\left(\mathcal{L}(Q)(s) + (\tau s + s^{\alpha-1})\psi(0^+) + \tau\psi'(0^+))\right)\,, \tag{C.3}$$

we recall that $\mathcal{L}(f)(s) = \int_0^\infty e^{-s\eta} f(\eta)d\eta$. And using that $\mathcal{L}(f*g)(s) = \mathcal{L}(f)(s)\,\mathcal{L}(g)(s)$ then:

$$\psi = \psi(0^+)\mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right) + \tau\bar{\psi}'(0^+)\mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right) + \mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right) * \mathcal{Q}\,.$$

For simplicity, we let

$$v(\eta) = \mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right) \quad \text{and} \quad \tilde{v}(s) = \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a} \tag{C.4}$$

and observe that, since

$$\frac{1}{\tau s^2 + s^\alpha + a} = \frac{1}{a}(1 - s\tilde{v}(s))$$

then

$$\mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right)(\eta) = -\frac{1}{a}v'(\eta)\,.$$

We also observe that:

$$\lim_{\eta\to 0^+} v(\eta) = \lim_{s\to\infty} s\tilde{v}(s) = 1 \quad \text{and} \quad \lim_{\eta\to 0^+} v'(\eta) = 0\,. \tag{C.5}$$

We can write the expression of ψ in terms of v instead to get

$$\psi(\eta) = \psi(0^+)v(\eta) - \frac{\tau}{a}\bar{\psi}'(0^+)v'(\eta) - \frac{1}{a}\int_0^\eta v'(y)Q(\eta - y)\,dy\,. \tag{C.6}$$

For $a > 0$, let us sketch the computation of $v(\eta)$, we recall that since this is the inverse Laplace transform of $\tilde{v}(s)$, we have to compute:

$$v(\eta) = \frac{1}{2\pi i}\int_{Br} e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\,ds \tag{C.7}$$

where $Br \subset \mathbb{C}$ is a Bromwich contour:

$$Br := \{s :\ \mathrm{Re}(s) = \sigma \geq 1\ \&\ \mathrm{Im}(s) \in (-\infty, \infty)\} \tag{C.8}$$

moreover, we restrict to the principal representation of s, namely, here $\arg(s) \in (-\pi, \pi]$. We follow the approach in [8], although they do it in some more detail for a different example and in the analogous case the estate the formulae. The results of [4] about the zeros of $\tau z^2 + z^\alpha + a$ apply here to the poles of the integrand in (C.7) for $a > 0$, thus, there exist two zeros that are complex conjugates and have negative real part, let them be denoted by s_1 and s_2. Then the contribution to the integral of these poles can be computed away from the Riemann surface cut (since $\alpha \in (0,1)$) that is the negative part of the real line. One can then split the integral as follows:

$$v(\eta) = \frac{1}{2\pi i} \lim_{\delta\to 0} \int_{Ha(\delta)} e^{s\eta} \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\, ds + \sum_{s=s_1,s_2} \text{Res}\left(e^{s\eta} \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a} \right) \tag{C.9}$$

where $Ha(\delta)$ is the Hankel path in $\mathbb{C}$

$$Ha(\delta) = \{s = -r + i\delta\,,\ r > 0\} \cup \{s = -r - i\delta\,,\ r > 0\} \cup \{s = \delta e^{i\beta}\,,\ \beta \in [-\pi/2, \pi/2]\} \tag{C.10}$$

It is easy to see by splitting the first integral term of (C.9) on these three contours that the one corresponding to the semicircle tends to 0 as δ tends to 0. The contribution of the other two is symmetric and gives:

$$v(\eta) = -\frac{1}{\pi} \int_0^\infty e^{-\eta r} \text{Im}\left(\frac{\tau(re^{i\pi}) + (re^{i\pi})^{\alpha-1}}{\tau(re^{i\pi})^2 + (re^{i\pi})^\alpha + a} \right) dr + \sum_{s=s_1,s_2} \text{Res}\left(e^{s\eta} \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a} \right). \tag{C.11}$$

We compute the integrand and residues, and get (using trigonometry)

$$\text{Im}\left(\frac{\tau(re^{i\pi}) + (re^{i\pi})^{\alpha-1}}{\tau(re^{i\pi})^2 + (re^{i\pi})^\alpha + a} \right) = -\frac{a r^{\alpha-1} \sin(\alpha\pi)}{(\tau r^2 + a)^2 + 2(\tau r^2 + a) r^\alpha \cos(\alpha\pi) + r^{2\alpha}}$$

and (using that s_1 and s_2 are complex conjugates)

$$\sum_{s=s_1,s_2} \text{Res}\left(e^{s\eta} \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a} \right) = 2\text{Re}\left(e^{s_1\eta} \frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}} \right).$$

We then write $v(\eta)$ as

$$v(\eta) = \frac{a \sin(\alpha\pi)}{\pi} \int_0^\infty e^{-\eta r} K(r)\, dr + 2\text{Re}\left(e^{s_1\eta} \frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}} \right), \tag{C.12}$$

where

$$K(r) = r^{\alpha-1} \tilde{K}(r) \quad \text{with} \quad \tilde{K}(r) = \frac{1}{(\tau r^2 + a)^2 + 2(\tau r^2 + a) r^\alpha \cos(\alpha\pi) + r^{2\alpha}}. \tag{C.13}$$

That the integral term is bounded follows from application of Watson Lemma (see [20]), since $\tilde{K}$ is C^∞ near $r = 0$, $\tilde{K}(0) = 1 \neq 0$, $\alpha - 1 > -1$ and clearly there exist non-negative constants C and b such that $|K(r)| < Ce^{br}$. Then the integral is bounded and moreover if η is large enough the following approximation holds

$$\int_0^\infty e^{-\eta r} K(r) dr \sim \sum_{n=0}^\infty \frac{\tilde{K}^{n)}(0)\Gamma(\alpha+n)}{n!\,\eta^{\alpha+n}} \quad \text{as} \quad \eta \to \infty.$$

One can compute the derivatives of $\tilde{K}$ and show that the odd order ones are zero at $r = 0$ and the even order ones do not vanish there; a two-term expansion reads:

$$\int_0^\infty e^{-\eta r} K(r) dr \sim \frac{\Gamma(\alpha)}{a^2} \frac{1}{\eta^\alpha} - \frac{4\tau\Gamma(\alpha+2)}{a^3} \frac{1}{\eta^{\alpha+2}} + O\left(\frac{1}{\eta^{\alpha+4}} \right) \quad \text{as} \quad \eta \to \infty. \tag{C.14}$$

References

[1] F. Achleitner, S. Hittmeir, C. Schmeiser. On nonlinear conservation laws with a nonlocal diffusion term. *J. Differential Equations*, 250(4): 2177–2196, 2011.

[2] B. Alvarez-Samaniego, P. Azerad. Existence of travelling-wave solutions and local well-posedness of the Fowler equation. *Discrete Contin. Dyn. Syst. Ser. B*, 12(4): 671–692, 2009.

[3] J.L. Bona and M.E. Schonbeck. Travelling-wave solutions of the Korteweg-de Vries-Burgers equations. *Proc. Royal Soc. Edinburgh A, 101, pp 207-226, 1985.*

[4] H. Beyer, S. Kempfle. Definition of physically consistent damping laws with fractional derivatives. *Z. Angew. Math. Mech. 75*, no. 8, 623–635, 1995.

[5] R. Duduchava. Wiener-Hopf equations with the transmission property. *Integral Equations Operator Theory*, 15(3): 412–426, 1992.

[6] T. Estermann. Complex Numbers and Functions. *London: Athlone -Oxford University Press*, 1962.

[7] S. G. Gindikin and L. R. Volevich. Distributions and convolution equations. Gordon and Breach Science Publishers, 1992.

[8] R. Gorenflo, F. Mainardi. Fractional calculus: integral and differential equations of fractional order. *Fractals and fractional calculus in continuum mechanics (Udine, 1996)*, 223–276, CISM Courses and Lectures, 378, Springer, Vienna, 1997.

[9] H. Jafari, V. Daftardar-Gejji. Solving a system of nonlinear fractional differential equations using Adomian decomposition. *Journal of Computational and Applied Mathematics*, 196 (2): 644–651, 2006.

[10] T. Kato. Quasi-linear equations of evolution, with applications to partial differential equations. *Spectral Theory and Differential Equations.* Springer, Lecture Notes in Mathematics. Volume 448: 25–70, 1975.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*, volume 204 of *Mathematical Studies*. Amsterdam, Netherlands: Elsevier, 2006.

[12] A. Kluwick, E. A. Cox, A. Exner, C. Grinschgl. On the internal structure of weakly nonlinear bores in laminar high Reynolds number flow. *Acta Mechanica* 210: 135–157, 2010.

[13] M. G. Kreĭn. Integral equations on the half-line with a kernel depending on the difference of the arguments. *Uspehi Mat. Nauk.*, 13(5):3–120, 1958.

[14] M. G. Kreĭn. Integral equations on the half-line with a kernel depending on the difference of the arguments. *American Mathematical Society Translations, Series 2, Volume 22*, American Mathematical Society, 1962.

[15] Elliott H. Lieb and M. Loss *Analysis*. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.

[16] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[17] Robert L. Pego. Remarks on the stability of shock profiles for conservation laws with dissipation. *Trans. Amer. Math. Soc.* 291(1): 353–361, 1985.

[18] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[19] N. Viertl. Viscous regularisation of weak laminar hydraulic jumps and bores in two layer shallow water flow. *TU Wien*, Dissertation, 2005.

[20] G. N. Watson. The harmonic functions associated with the parabolic cylinder. *Proceedings of the London Mathematical Society* 2(17): 116–148, 1918.

[21] N. Wiener and E. Hopf. Über eine Klasse singulärer Integralgleichungen. *Sitz.Ber.Preuss.Akad.Wiss.Berlin*, XXXI: 696-706, 1931.