

On the one-dimensional Gelfand and Borg–Levinson spectral problems for discontinuous coefficients

Mourad Sini

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail: sini@math.sci.hokudai.ac.jp

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Abstract

In this paper, we deal with the inverse spectral problem for the equation $-(pu')' + qu = \lambda\rho u$ on a finite interval $(0, h)$. We give some uniqueness results on q and ρ from the Gelfand spectral data, when the coefficients p and ρ are piecewise Lipschitz and q is bounded. We also prove an equivalence result between the Gelfand spectral data and the Borg–Levinson spectral data. As a consequence, we obtain similar uniqueness results if we consider the Borg–Levinson spectral data. Finally, we consider the inverse problem from the nodes and give uniqueness results on ρ and in the case where the coefficients p, q and ρ are smooth we give a uniqueness result on both q and ρ .

1. Introduction and statement of the result

Let p and ρ be bounded and positive measurable functions, q be a bounded measurable function and $h > 0$ is a constant. We denote by λ_i and $e_i, i \in \mathbb{N}$, the associated eigenvalues and $L^2_\rho(0, h)$ -orthonormal eigenfunctions of the Sturm–Liouville problem:

$$\begin{cases} -(pu')' + qu - \lambda\rho u = 0 & \text{in } (0, h), \\ u(0) = 0, \\ u(h) = 0, \end{cases} \quad (1)$$

where $L^2_\rho(0, h)$ is the $L^2(0, h)$ -space with the scalar product $(f, g) := \int_0^h f(x)g(x)\rho(x) dx$.

The Gelfand inverse spectral problem consists in the reconstruction of some of the coefficients p, q and ρ from the spectral data $(\lambda_i, |p(e_i)'(0)|)_{i=1}^\infty$. This last sequence is called the Gelfand spectral data. If in (1), we take the Dirichlet boundary condition on $x_0 = 0$ and the Neumann one on $x_0 = h$ then we have another sequence, which we denote by $(\mu_i)_{i \in \mathbb{N}}$. The sequence $(\lambda_i, \mu_i)_{i \in \mathbb{N}}$ is called the Borg–Levinson spectral data. We call the Borg–Levinson problem the identification of some of the coefficients p, q and ρ from the Borg–Levinson spectral data.

It is well known that when these coefficients are regular (p and ρ are $C^{1,1}(0, h)$, for example) then by reducing the equation $-(pu')' + qu - \lambda ru = 0$ to the normal form, i.e. where $p = r = 1$, we can prove uniqueness of one of the three coefficients from the Borg–Levinson spectral data or from the Gelfand spectral data. The aim is to prove similar results for discontinuous coefficients. The case where $q = 0$ is quite well known. There are many works related to these cases. We refer the reader to [Mc1], [Mc2] or [HM] for details and references.

In the following work, we consider the case where $q \neq 0$ and prove uniqueness of $q \in L^\infty(0, h)$ and h when p and ρ are fixed and assumed to be piecewise Lipschitz. We also prove uniqueness of $\rho \in L^\infty(0, h)$ if p and q are fixed in $L^\infty(0, h)$. The method of analysis to prove uniqueness of q is the BC method, see [Bel2] and the first chapter of [KaKuLa] for the one-dimensional version of this method. One of the main tools of the BC method is the boundary approximate controllability for the corresponding hyperbolic problem. To prove this boundary approximate controllability, for the kind of discontinuities we mentioned above, we approximate the corresponding hyperbolic problem given by irregular coefficients by hyperbolic problems given by regular coefficients and take advantage of the equivalence between the space and the time variables in this one-dimensional case.

Let us now give the results.

Theorem 1. *Let p_j and ρ_j , $j = 1, 2$ be bounded and positive and q_j , $j = 1, 2$, be bounded and measurable functions. We denote by λ_i^j and e_i^j , $i \in \mathfrak{N}$, the associated eigenvalues and orthonormal eigenfunctions on $\Omega_j := (0, h_j)$.*

We suppose that the boundary spectral data are the same, i.e.

$$\lambda_i^1 = \lambda_i^2 \quad \text{and} \quad |p_1(e_i^1)'(0)| = |p_2(e_i^2)'(0)| \quad \text{for all } i \in \mathfrak{N}. \quad (2)$$

- (A1) *If we fix p_j and ρ_j with piecewise Lipschitz regularity, then we have uniqueness of q_j and h_j .*
 (A2) *If in addition p_j (or ρ_j) are piecewise constant, then fixing ρ_j (or p_j) we have uniqueness of both p_j (or ρ_j), q_j and h_j .*
 (B) *If we fix p_j and q_j bounded then we have uniqueness of ρ_j and h_j .*

The condition $q_1 = q_2$ means that $q_j = q$ a.e. in $(0, h_j)$ where q is defined on $(0, H)$ with $H \geq h_j$, $j = 1, 2$ and similarly for p_j and r_j . In the proof of this theorem we assume that $q_j > 0$ but the results holds also in the general case by shifting the spectrum. Since we can change the sign of $p_j(e_i^j)'(0)$, then the absolute value in (2) can be deleted.

The proofs of the parts (A1), (A2) and (B) are given in sections 2, 3 and 4, respectively. This is a continuation of [S] where we proved similar results for piecewise analytic coefficients. In (A2), we prove uniqueness of two coefficients, but we assumed that p (or ρ) are piecewise constants. It is interesting to see if we can consider more general discontinuities for p or ρ .

In section 5, we use the gauge transformation introduced in [Ku] for the inverse spectral problems and the Alessandrini identity to prove the equivalence between the Gelfand and the Borg–Levinson data. This equivalence enables us to state the results of theorem 1 considering the Borg–Levinson spectral data. To our knowledge these kinds of results are also new with respect to the literature, see [Mc1] and [Mc2].

Section 6 deals with the inverse nodal problem for the equation $-(pu')' + qu - \lambda\rho u = 0$ in $(0, h)$. This problem has been proposed in [Mc] for the case where $p = \rho = 1$ where uniqueness result of q is given. In [HaMc1], we find reconstruction algorithms for the identification of q . This case was considered later in [ShTs], [S] and [LSY] where we find other algorithms of reconstructions and information are given on the smoothness of q . In

[HaMc2], the case $q = 0$ was considered and uniqueness results on p or ρ were established. In our case we are just concerned with the uniqueness, but considering two of the three coefficients p, q and ρ . We are motivated by the question asked in [HaMc1], see also [HaMc2], on uniqueness of two coefficients from the nodes. This section follows the spirit of this last reference. The subset of nodes used has the following property: for $n \in \mathbb{N}$, if this set contains the j th node x_j^n of the n th eigenvalue then it also contains the $(j + 1)$ th or the $(j - 1)$ th node of this eigenfunction. We state the results we obtained in the following theorem.

Theorem 2. *Assume that p and ρ are BV functions and q is bounded. Suppose also that we know the previous subset of the nodes of the related eigenfunctions of the spectral problem (1).*

- (A) *Then fixing p and q we prove uniqueness, up to a multiplicative constant, of ρ .*
- (B) *If in addition we take p and ρ in $H^2(0, h)$ and q continuous almost everywhere, then fixing p we prove uniqueness of ρ , up to a multiplicative constant, and also uniqueness of q , up to an additive one-dimensional subspace generated by the function ρ .*

These results complete the work [HaMc2] and give a positive answer for the possibility to prove uniqueness of two coefficients from the nodes.

2. Proof of the part (A1) of theorem 1

Let us define the following hyperbolic problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{\rho} \frac{\partial}{\partial x} \left(\rho \frac{\partial u}{\partial x} \right) + \frac{q}{\rho} u = 0 & \text{in } (0, T) \times (0, h), \\ u(t, 0) = f(t), & u(t, h) = 0, \quad t \in (0, T), \\ u(0, x) = \frac{\partial u}{\partial t}(0, x) = 0, & x \in (0, h), \end{cases} \tag{3}$$

where $f \in H^1(0, T)$ such that $f(0) = 0$ and T is positive constant.

This problem has one and only one solution in $C[(0, T), H^1(0, h)] \cap C^1[(0, T), L^2(0, h)]$ satisfying

$$\max_{0 \leq t \leq T} \left[\|u(t)\|_{H^1(0,h)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(0,h)} \right] \leq c(T) (\|f\|_{H^1(0,h)}), \tag{4}$$

see [KaKuLa] or [LaLiTr].

We set $u^f(t, x)$ this solution. Using the basis $e_i, i \in \mathbb{N}$, we have the spectral representation of $u^f(t, x)$:

$$u^f(t, x) = \sum_{i=1}^{\infty} u_i^f(t) e_i(x)$$

where

$$u_i^f(t) := \int_0^h u^f(t, x) e_i(x) \rho(x) dx \tag{5}$$

satisfy the Cauchy problem

$$\begin{cases} \frac{d^2 u_i^f}{dt^2} = -\lambda_i u_i^f + p \frac{de_i}{dx}(0) f(t), & \text{in } (0, T), \\ u_i^f(0) = \frac{du_i^f}{dt}(0) = 0. \end{cases} \tag{6}$$

Hence

$$u_i^f(t) = \int_0^t f(t') \frac{\sin \sqrt{\lambda_i}(t - t')}{\sqrt{\lambda_i}} dt' \left(p \frac{de_i}{dx} \right) (0), \tag{7}$$

since $\lambda_i > 0$ for all $n \in \mathbb{N}$.

Since the coefficients p, q and ρ are not regular, even if we take f regular we cannot differentiate twice with respect to the time variable in (5) to justify (6). To prove that $u_i^f(t, x)$ satisfy (6), we start by taking the derivative of (5), using (4), to get

$$\frac{du_i^f}{dt} = \int_0^h \frac{du^f}{dt}(t, x)e_i(x)\rho(x) dx.$$

We take any function $\phi \in C_0^\infty(0, T)$ then

$$\int_0^T \frac{du_i^f}{dt} \frac{d\phi}{dt} dt = \int_0^T \int_0^h \frac{\partial u^f(t, x)}{\partial t} \frac{\partial(\phi(t)e_i(x))}{\partial t} \rho(x) dx dt.$$

Using the definition of the weak solution of (3), we deduce that

$$\int_0^T \frac{du_i^f}{dt} \frac{d\phi}{dt} dt = \int_0^T \int_0^h p \frac{\partial u^f(t, x)}{\partial x} \frac{\partial(\phi(t)e_i(x))}{\partial x} + qu^f(t, x)e_i(x)\phi(t) dx dt.$$

An integration by parts gives

$$\int_0^T \frac{du_i^f}{dt}(t) \frac{d\phi}{dt}(t) dt = \lambda_i \int_0^T \int_0^h u^f(t, x)e_i(x)\phi(t)\rho(x) dx dt - \int_0^T p \frac{de_i}{dx}(0)f(t)\phi(t) dt.$$

Hence in the distribution sense we have the first equation of (6). The initial conditions of (6) come from the ones of (3) which is justified by (4).

2.1. Domain of influence for the problem (3)

The object of this section is to prove, for general bounded coefficients p, q and ρ , that the domain of influence of the solution of the problem (3) is given by $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) \leq t\}$, where $v(x) := \sqrt{\frac{\rho}{p}}(x)$ is the velocity of propagation.

Let $(p_n), (q_n)$ and (ρ_n) be three sequences of regular coefficients given by $p_n = r_n \star \tilde{p}, q_n = r_n \star \tilde{q}$ and $\rho_n = r_n \star \tilde{\rho}$, where r_n is a mollifier sequence and \tilde{p}, \tilde{q} and $\tilde{\rho}$ are given by extending p, q and ρ by zero to $\mathfrak{R} \setminus (0, h)$, respectively. Then (p_n, q_n, ρ_n) tend to (p, q, ρ) in $(L^s(0, h))^3 \forall s, 1 \leq s < \infty$ and the sequences (p_n, q_n, ρ_n) are bounded from below and above in $L^\infty(0, h)$, i.e. $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma$, where δ and γ are constants.

Let u_n^f be the solution of (3) when we replace (p, q, ρ) by (p_n, q_n, ρ_n) .

We set

$$y = g_n(x) := \frac{1}{h_n} \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt, \quad \tilde{u}_n^f(t, y) := \left(\frac{\tilde{\rho}_n(y)\tilde{\rho}_n(y)}{\tilde{\rho}_n(0)\tilde{\rho}_n(0)} \right)^{\frac{1}{4}} u_n^f(t, y),$$

$$\tilde{p}_n(y) := p_n(g_n^{-1}(y)), \quad \tilde{q}_n(y) := q_n(g_n^{-1}(y)),$$

$$\tilde{\rho}_n(y) := \rho_n(g_n^{-1}(y)), \quad c_n(y) := \frac{\frac{\partial^2}{\partial y^2}(\tilde{p}_n(y)\tilde{\rho}_n(y))}{\tilde{p}_n(y)\tilde{\rho}_n(y)} + h_n \frac{\tilde{q}_n}{\tilde{\rho}_n}$$

and

$$h_n := \int_0^h \sqrt{\frac{\rho_n}{p_n}}(t) dt$$

then $\tilde{u}_n^f(y)$ satisfy

$$\begin{cases} \frac{\partial^2 \tilde{u}_n^f}{\partial t^2} - \frac{1}{h_n^2} \frac{\partial^2 \tilde{u}_n^f}{\partial y^2} + c_n \tilde{u}_n^f = 0 & \text{in } (0, T) \times (0, 1), \\ \tilde{u}_n^f(t, 0) = f(t), & \tilde{u}_n^f(t, 1) = 0, \\ \tilde{u}_n^f(0, y) = \frac{\partial \tilde{u}_n^f}{\partial t}(0, y) = 0, & y \in (0, 1). \end{cases} \tag{8}$$

Hence by the finiteness of the velocity of the propagation, which is here equal to $\frac{1}{h_n}$, we deduce that the support of \tilde{u}_n^f is included in $\{(t, y) \in (0, T) \times (0, 1) / y \leq \frac{1}{h_n}t\}$ then the support of u_n^f is included in $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt \leq t\}$.

In the appendix we prove that u_n^f tends to u^f in $L^2[(0, T) \times (0, h)]$. Hence, since (p_n, q_n, ρ_n) tend to (p, q, ρ) in $L^2(0, h)$ and $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma$ we deduce that the support of u^f is included in $\{(t, y) \in (0, T) \times (0, 1) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt \leq t\}$.

Indeed, let $(t_0, x_0) \in \{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt > t\}$. Then there exists $\epsilon > 0$, such that $\int_0^x \sqrt{\frac{\rho}{p}}(t) dt > t$, for all (t, x) satisfying $|t - t_0| < \epsilon$ and $|x - x_0| < \epsilon$. A simple computation shows that

$$\left| \int_0^x \sqrt{\frac{\rho}{p}}(t) dt - \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt \right| \leq c(\|p_n - p\|_{L^2(0,h)}^2 + \|\rho_n - \rho\|_{L^2(0,h)}^2),$$

where c is independent of n and x .

Hence we can find $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) dt > t, \forall (t, x) / |t - t_0| < \epsilon$ and $|x - x_0| < \epsilon$ (taking ϵ smaller if necessary). Then $u_n^f(t, x) = 0 \forall (t, x) / |t - t_0| < \epsilon$ and $|x - x_0| < \epsilon$ for all $n \geq n_0$. This implies that $u^f(t, x) = 0 \forall (t, x) / |t - t_0| < \epsilon$ and $|x - x_0| < \epsilon$. Finally, $u^f(t, x) = 0$ in $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt > t\}$.

We denote by $\Gamma_T = \{x \in (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) dt \leq T\}$.

2.2. Boundary controllability

The exact boundary controllability for the problem (3) is to find for every $z(x) \in L^2(\Gamma_T)$ a function (i.e. a control) $f(t) \in L^2(0, T)$ such that $u^f(x, T) = z(x)$. In the case where $p = \rho = 1$, this result is known and the proof uses the Green function of the non-perturbed part of the equation, given by $q = 0$, which is explicitly known in this case (see, for example, [KaKuLa], chapter 1). Another proof is given in ([Bel2]) (see also ([BK])). The authors considered the case where $p = 1, q = 0$ and $\rho \in C^2[0, h]$. In both these cases if $T \leq \int_0^h \sqrt{\frac{\rho}{p}}(s) ds$ then we have uniqueness of the control.

In our general case, we prove the following approximate boundary controllability, which is enough for the inverse spectral problem we are considering. As we said in the introduction, to prove this theorem, we add two arguments to the usual proof in the multidimensional case (see [KaKuLa] theorem 3.10). The first is to approximate the irregular coefficients by regular ones to justify the integration by parts. The second is to use the equivalence between the time variable and the space variable to consider the unique continuation of the hyperbolic problem as a usual Cauchy problem with time-dependent coefficients.

Theorem 3. *The linear subspace $\{u^f(x, T), f \in H_0^1(0, T)\}$ is dense in $L_\rho^2(\Gamma_T)$.*

Proof of theorem 3. Let $\psi \in L_\rho^2(\Gamma_T)$ such that $\int_0^h u^f \psi \rho dx = 0$ for all $f \in H_0^1(0, T)$. Let us take $f \in C_0^\infty[0, T]$. We take also $\psi_n \in C_0^\infty[0, h]$ such that ψ_n tends to ψ in $L^2(0, h)$. We recall that u_n^f is the solution of (3) replacing p, q and ρ by p_n, q_n and ρ_n .

We define the initial boundary value problem as:

$$\begin{cases} \frac{\partial^2 v_n}{\partial t^2} - \frac{1}{\rho_n} \frac{\partial}{\partial x} \left(p_n \frac{\partial v_n}{\partial x} \right) + \frac{q_n}{\rho_n} v_n = 0 & \text{in } (0, T) \times (0, h), \\ v_n(t, 0) = v_n(t, h) = 0, \\ v_n(T, x) = 0, & \frac{\partial v_n}{\partial t}(T, x) = \psi_n(x) \in (0, h). \end{cases} \tag{9}$$

This problem is called the dual problem of problem (3). In the appendix, we prove that v_n tends to v in $L^2((0, T) \times (0, h))$, where v satisfies (9) replacing p_n, q_n, ρ_n and ψ_n by p, q, ρ and ψ , respectively.

Since the coefficients p_n, q_n, ρ_n are in $C^\infty(0, h)$ and $\psi_n \in C_0^\infty(0, h)$, which guarantee the compatibility conditions, then $v_n \in C^\infty((0, T) \times (0, h))$.

By an integration by parts, one finds that

$$\begin{aligned} 0 &= \int_0^h \int_0^T u_n^f \left[\rho_n \frac{\partial^2 v_n}{\partial t^2} - \frac{\partial}{\partial x} \left(p_n \frac{\partial v_n}{\partial x} \right) + q_n v_n \right] \\ &\quad - v_n^f \left[\rho_n \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial}{\partial x} \left(p_n \frac{\partial u_n}{\partial x} \right) + q_n u_n \right] dt dx \\ &= \int_0^h u_n^f(T, x) \psi_n \rho_n dx + \int_0^T f p_n \frac{\partial v_n}{\partial x}(0, t) dt. \end{aligned}$$

We have

$$\begin{aligned} \int_0^h u_n^f(T, x) \psi_n \rho_n dx - \int_0^h u^f(T, x) \psi \rho dx &= \int_0^h u_n^f(T, x) \psi_n \rho_n dx - \int_0^h u_n^f(T, x) \psi_n \rho dx \\ &\quad + \int_0^h u_n^f(T, x) \psi_n \rho dx - \int_0^h u_n^f(T, x) \psi \rho dx \\ &\quad + \int_0^h u_n^f(T, x) \psi \rho dx - \int_0^h u^f(T, x) \psi \rho dx. \end{aligned}$$

We know that the sequence $u_n^f(t, x)$ is bounded in the energy norm from the energy inequality (see the appendix), then in particular in $C([0, T], H^1(0, h))$ hence $u_n^f(T, x)$ is bounded in $C[0, h]$. Since ψ_n is bounded in $L^2(0, h)$, then from the inequality

$$\left| \int_0^h u_n^f(T, x) \psi_n (\rho_n - \rho) dx \right| \leq \int_0^h |u_n^f(T, x)|^2 |\psi_n|^2 dx \int_0^h (\rho_n - \rho)^2 dx$$

we deduce that $\int_0^h u_n^f(T, x) \psi_n (\rho_n - \rho) dx$ tends to zero when n tends to ∞ . Arguing similarly for the other terms we deduce that $\int_0^h u_n^f(T, x) \psi_n \rho_n dx$ tends to zero since $\int_0^h u^f(T, x) \psi \rho dx = 0$.

This means that $\int_0^T f p_n \frac{\partial v_n}{\partial x}(0, t) dt$ tends to zero as n tends to ∞ .

We can say also that v_n satisfies the following problem:

$$\begin{cases} \frac{\partial}{\partial x} \left(p_n \frac{\partial v_n}{\partial x} \right) - \rho_n \frac{\partial^2 v_n}{\partial t^2} - q_n v_n = 0 & \text{in } (0, T) \times (0, h), \\ v_n(t, 0) = 0, \quad p_n \frac{\partial v_n}{\partial x}(t, 0) := \xi_n(t), & t \in (0, T), \\ v_n(T, x) = 0, \quad v_n(0, x) := \eta_n(x) \in (0, h), \end{cases} \tag{10}$$

where ξ_n tends weakly to zero in $L^2(0, T)$ since we took f arbitrary in a dense set of $L^2(0, T)$. We look at this problem as a Cauchy problem where the time variable is x and the space variable is t .

Since v_n is a solution of (9), then the boundness of v_n in $C([0, T], H^1(0, h))$ and of $\frac{\partial v_n}{\partial t}$ in $C([0, T], L^2(0, h))$ is given by the energy inequality applied to (9). Then we take a subsequence, which we denote also by v_n , and find a function \tilde{v} such that v_n tends to \tilde{v} weakly in $H^1((0, T), L^2(0, h)) \cap L^2((0, T), H^1(0, h))$. Hence, $v_n(0, x)$ and $v_n(T, x)$ tend to $\tilde{v}(0, x)$ and $\tilde{v}(T, x)$, respectively, in $L^2(0, h)$. As a consequence, we have $\tilde{v}(T, x) = 0$ since $v_n(T, x) = 0$ and $\tilde{v}(0, x) \in H_0^1(0, h)$ since $v_n(0, x) \in H_0^1(0, h)$, see (9) and $v_n(0, x)$ tends weakly to $v(0, x)$ in $H^1(0, h)$.

Now take $\phi \in H^1((0, T) \times (0, h))$ such that $\phi(t, h) = 0$, for $t \in (0, T)$ and $\phi(0, x) = \phi(T, x) = 0$, for $x \in (0, h)$. Multiplying in (10) by $\phi(t, x)$ and integrating by parts, we find

$$\int_0^T \int_0^h -p_n \frac{\partial v_n}{\partial x} \frac{\partial \phi}{\partial x} + \rho_n \frac{\partial v_n}{\partial t} \frac{\partial \phi}{\partial t} - q_n v_n \phi dx dt = - \int_0^T \xi_n(t) \phi(0, t) dt. \tag{11}$$

Now going to the limit in the previous equality using the properties of v_n and ξ_n , we have

$$\int_0^T \int_0^h -p \frac{\partial \tilde{v}}{\partial x} \frac{\partial \phi}{\partial x} + \rho \frac{\partial \tilde{v}}{\partial t} \frac{\partial \phi}{\partial t} - q \tilde{v} \phi \, dx \, dt = 0.$$

From the definition of a weak solution of the hyperbolic problem, we deduce that \tilde{v} satisfy the following problem:

$$\begin{cases} \frac{\partial}{\partial x} (p \frac{\partial \tilde{v}}{\partial x}) - \rho \frac{\partial^2 \tilde{v}}{\partial t^2} - q \tilde{v} = 0 & \text{in } (0, T) \times (0, h), \\ \tilde{v}(t, 0) = 0, \quad p \frac{\partial \tilde{v}}{\partial x}(t, 0) = 0, & t \in (0, T), \\ \tilde{v}(T, x) = 0, \quad \tilde{v}(0, x) = \eta(x) \in (0, h), \end{cases} \tag{12}$$

where $\eta \in H_0^1(0, h)$.

We write $v_n := v_n^1 + v_n^2$, where v_n^1 is the solution of (10) when we take $\xi_n(t) = 0$ and v_n^2 the solution when $\eta_n(x) = 0$. Arguing as in section 2.1, we see that the support of v_n^1 is included in $\{(t, x) \in (0, T) \times (0, h) / t \leq \int_0^x \sqrt{\frac{\rho_n}{p_n}}(t) \, dt\}$. Since ξ_n tends weakly to zero in $L^2(0, T)$, then it is bounded. Arguing as for v_n , we deduce that v_n^2 tends to zero in $L^2((0, T) \times (0, h))$. Since in this case the coefficients depend on the time variable, a little change is needed since the boundness in the energy norm requires Lipschitz regularity (see [La]). For this we state the problem satisfied by v_n^1 on the parts where the coefficients p and ρ are Lipschitz and we proceed by steps.

Since v_n tends to \tilde{v} in $L^2((0, T) \times (0, h))$, this means that v_n^1 tends to \tilde{v} in $L^2((0, T) \times (0, h))$. Now arguing as in section 2.1, we deduce that the support of \tilde{v} is included in $\{(t, x) \in (0, T) \times (0, h) / t \leq \int_0^x \sqrt{\frac{\rho}{p}}(t) \, dt\}$.

Hence $\tilde{v}(t, x) = 0$ in $\{(t, x) \in (0, T) \times (0, h) / \int_0^x \sqrt{\frac{\rho}{p}}(t) \, dt < t\}$. Since v_n tends to v hence $v = \tilde{v}$ then $\psi(x) = \frac{\partial \tilde{v}}{\partial t}(T, x) = 0$. □

Remark 2.1. It is in this last step, i.e. to characterize the domain of influence for the problem (12) (where the coefficients are time dependent), where we need the piecewise Lipschitz regularity of the coefficients p and ρ .

Another possible way to prove this boundary approximate controllability is to use the approach of [AvIv] where this question is related to a ‘good’ distribution of the eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$.

Let us now give the following theorem which gives the projections of the eigenfunctions from the Gelfand spectral data.

Theorem 4. *Let (p_j, q_j, ρ_j) be two families of coefficients, $j = 1, 2$. Let $(\lambda_i^j, e_i^j)_{i=1}^\infty$ be the eigenelements of the two Sturm–Liouville operators corresponding to the two families of coefficients.*

Then we have

$$\begin{aligned} \left(\lambda_i^1, p_1 \frac{d}{dx} (e_i^1)(0) \right)_{i \in \mathbb{N}} &= \left(\lambda_i^2, p_2 \frac{d}{dx} (e_i^2)(0) \right)_{i \in \mathbb{N}} \Rightarrow \int_{\Gamma_T} (e_i^1)^2 \rho_1 \, dx \\ &= \int_{\Gamma_T} (e_i^2)^2 \rho_2 \, dx, \quad \forall T > 0. \end{aligned}$$

Proof of theorem 4. Let $\alpha_k(t), k \in \mathbb{N}$, be a dense set of $H_0^1(0, T)$. From the previous theorem and the energy inequality we deduce that the finite combinations of the functions $u^{\alpha_k}(T, x)$ form a dense set in $L^2_{\rho_j}(\Gamma_T)$. Since the Fourier coefficients of u^{α_k} are determined by λ_k^j and $p_j \frac{d}{dx} (e_k^j)(0)$, then the products $\int_{\Gamma_T} u^{\alpha_i} u^{\alpha_k} \rho_j \, dx, (i, k) \in \mathbb{N}^2$ and $j = 1, 2$ are also determined by $(\lambda_i^j, p_j \frac{d}{dx} (e_i^j)(0))_{i \in \mathbb{N}}$. But maybe $\int_{\Gamma_T} u^{\alpha_i} u^{\alpha_k} \rho_j \, dx \neq \delta_{i,k}, (i, k) \in \mathbb{N}^2$. By

the Gram–Schmidt orthogonalization procedure we can find an orthonormal basis of $L^2_{\rho_j}(\Gamma_T)$ given by combinations of u^{α_i} , i.e. $v_s = \sum_{i=1}^{n(s)} d_{s,i} u^{\alpha_i}$. By linearity we have $v_s = u^{\beta_s}$ where $\beta_s = \sum_{i=1}^{n(s)} d_{s,i} \alpha_i$.

For every function $a \in L^2_{\rho_j}(0, h_j)$, we set

$$P_{\Gamma_T} a(x) = \begin{cases} a(x), & x \in \Gamma_T, \\ 0, & x \in (0, h_j) \setminus \Gamma_T, \end{cases}$$

then, $P_{\Gamma_T} a$ can be written as

$$P_{\Gamma_T} a(x) = \sum_{i \in \mathbb{N}} \int_{\Gamma_T} a(x) v_i(T, x) \rho_j(x) dx \cdot v_i(T, x),$$

hence

$$\int_{\Gamma_T} e_i^j(x) e_l^j(x) \rho_j(x) dx = \sum_{s \in \mathbb{N}} \int_{\Gamma_T} e_i^j(x) v_s(T, x) \rho_j(x) dx \int_{\Gamma_T} e_l^j(x) v_s(T, x) \rho_j(x) dx. \tag{13}$$

But $\int_{\Gamma_T} e_i^j(x) v_s(T, x) \rho_j(x) dx$ satisfies

$$\begin{cases} \frac{d^2 u}{dt^2} = -\lambda_i^j u + p_j \frac{\partial e_i^j}{\partial x}(0) \beta_s(t), & \text{in } (0, T), \\ u_i^f(0) = \frac{du_i^f}{dt}(0) = 0, \end{cases} \tag{14}$$

then

$$\int_{\Gamma_T} e_i^j(x) v_s(T, x) \rho_j(x) dx = \int_0^T \beta_s(t') \frac{\sin \sqrt{\lambda_i}(T - t')}{\sqrt{\lambda_i}} dt \left(p \frac{\partial e_i^j}{\partial x} \right) (0).$$

Hence the sequence $(\lambda_i^j, |p_j \frac{d}{dx}(e_i^j)(0)|)_{i \in \mathbb{N}}$ determines the infinite matrix given by the entries $M_{i,l} := \int_{\Gamma_T} e_i^j e_l^j \rho_j dx$, $(i, l) \in \mathbb{N}^2, \forall T > 0$. In particular, taking $i = l$, the spectral data $(\lambda_i^j, |p_j \frac{d}{dx}(e_i^j)(0)|)_{i \in \mathbb{N}}$ determine the diagonal part of the matrix $M_{i,l}$, i.e. $M_{i,i} = \int_{\Gamma_T} (e_i^j)^2 \rho_j dx$, $i \in \mathbb{N}$. □

Let us set Γ_T^j , $j = 1, 2$, the domain of influence related to p_j, q_j, ρ_j . Since $p_1 = p_2 =: p$ and $\rho_1 = \rho_2 =: \rho$ then $\Gamma_T^1 = \Gamma_T^2 =: \Gamma_T$. The asymptotics of the eigenvalues give the equality $h_1 = h_2 =: h$. From the previous theorem, we deduce that $\int_{\Gamma_T} (e_i^1)^2 \rho dx = \int_{\Gamma_T} (e_i^2)^2 \rho dx$ for every $T > 0$. This implies that $\sqrt{p(g^{-1}(y))\rho(g^{-1}(y))} (e_i^1(g^{-1}))^2 = \sqrt{p(g^{-1}(y))\rho(g^{-1}(y))} (e_i^2(g^{-1}))^2$ for all $y \leq \int_0^h g(x) dx$, $j = 1, 2$. Hence $|e_i^1(x)| = |e_i^2(x)|$ in $(0, h)$. We recall that the eigenfunction of the first eigenvalue has one sign. Replacing in the equations satisfied by this first eigenvalue and its associated eigenfunction, we deduce that $q_1 = q_2$.

A procedure to reconstruct q . The proof of this previous part gives a procedure to compute q from the spectral data. The steps can be given as follows:

(1a) Choose any basis α_i of $H_0^1(0, T)$ and for $T \leq \int_0^h \sqrt{\frac{\rho}{p}} ds$ fixed, compute the sum:

$$r_{i,j}(T) := \int_{\Gamma_T} u^{\alpha_i}(T, x) u^{\alpha_j}(T, x) \rho(x) dx = \sum_{l \in \mathbb{N}} (u^{\alpha_i}(T, x), e_l) \times (u^{\alpha_j}(T, x), e_l)$$

where $(u^{\alpha_i}(T, x), e_l) = \int_0^T \alpha_i(t') \frac{\sin \sqrt{\lambda_l}(T-t')}{\sqrt{\lambda_l}} dt \left(p \frac{de_l}{dx} \right) (0)$.

(1b) By the Gram–Schmidt orthogonalization procedure, construct $\beta_s(T)$, $s = 1, 2, \dots$ from $r_{i,j}(T)$.

(2) Compute

$$\int_{\Gamma_T} (e_1(x))^2 \rho(x) \, dx := \sum_{s \in \mathbb{N}} \left[\int_0^T \beta_s(t') \frac{\sin \sqrt{\lambda_1}(t-t')}{\sqrt{\lambda_1}} \, dt \left(p \frac{de_1}{dx} \right) (0) \right]^2.$$

(3) Compute the values of $e_1(x)$ as a derivative of the integral in the step 2.

(4) Compute $q(x) := \frac{(p(e_1)')'(x) + \lambda_1 \rho e_1(x)}{e_1(x)}$.

Remark 2.2. (1) Step (4) can be replaced by

(4') Compute $\int_0^h q(x) e_1(x) \xi_n(x) \, dx = \int_0^h e_1(x) \xi_n(x) \rho(x) \, dx (\lambda_1 + \omega_n)$, where (ξ_n, ω_n) are the eigenelements of the problem $-(pu')' = \lambda \rho u$ in $(0, h)$ with Dirichlet boundary conditions. In the case where $p = 1$, we can choose an explicit basis given by $\sin n(\frac{x}{h})$ to compute the Fourier coefficients of $q(x) e_1(x)$.

(2) The advantage of this procedure is that we take into account the three coefficients where p and ρ can be discontinuous. For the case $q = 0$ or $p = \rho = 1$, one can find, among others, procedures based on the Gelfand–Levitan representation, see [CCPR] chapter 3 for a review of these procedures.

3. Proof of the part (A2): Uniqueness of two coefficients

In this section we prove uniqueness of two coefficients among p, q and ρ . In [S], we proved uniqueness of p (or ρ) and q if p (or ρ) is piecewise constant, q is piecewise analytic and ρ (or p) is piecewise indefinitely differentiable. Here we show how to prove the same result for q in $L^\infty(0, h)$, p (or ρ) piecewise constant and ρ (or p) piecewise Lipschitz.

Indeed, let ρ be piecewise Lipschitz and fixed. As in [S], we prove that the Gelfand spectral data implies the values of p on $x = 0$. Since p is piecewise constant then we know p on $(0, c_1)$, where c_1 is the first discontinuity point. Let us now prove that q is known on $(0, c_1)$. To do this, we prove as in the previous section that $\int_{\Gamma_t} (e_i)^2(x) \rho(x) \, dx$ is known for every t . This implies that $|e_i(x)|$ is known on $(0, c_1)$. From the equations satisfied by these eigenfunctions we deduce that $q(x)$ is known on $(0, c_1)$.

Now, proceeding by layer stripping, since p, q and ρ are known on $(0, c_1)$, then we deduce that for the Sturm–Liouville problems stated on $(0, c_1)$, we have the Gelfand data. Then step by step, we prove uniqueness of p (or ρ) and q and hence of h .

4. Proof of the part (B)

Let p_j and q_j be fixed in $L^\infty(0, h_j)$, i.e. $p_1 = p_2$ and $q_1 = q_2$. We set $Au := \frac{1}{\rho}((-pu')' + \frac{q}{\rho}u)$. Let $k > 0$ be a continuous function. We set $\phi_i := ke_i$, then $A_k \phi_i := kAk^{-1}\phi_i = \lambda_i \phi_i$ and $A_k u = -\frac{k^2}{\rho}(k^{-2}pu')' + kA(k^{-1}) \cdot u$. We also have $[k^{-2}p\phi_i'](0) = k^{-1}(0)p(e_i)'(0)$. Now for $j = 1, 2$, we take $k := k_j$ such that k_j^{-1} is an $H^1(0, h)$ -solution of the elliptic problem:

$$\begin{cases} A_j(k_j^{-1}) = 0, & \text{in } (0, h_j), \\ k_j^{-1}(0) = 1, & k_j^{-1}(h) = 0. \end{cases} \tag{15}$$

then k_j^{-1} is positive by the maximum principle, see ([PW], chapter 1, theorem 3).

Since p and q are fixed then $k_1 = k_2$. Using this transformation we have the spectral data of the operators given by taking $p_j := k^{-2}p, q_j := 0, \rho_j := k^{-2}\rho_j$ and $h_j := h_j$. Now we use the transformation $g(x) := \int_0^x k^2(t)p^{-1}(t) \, dt$ to transform the last operators to the case,

where $p_j := 1$, $q_j := 0$, $\rho_j := k^{-4}(g^{-1})\rho_j(g^{-1})p(g^{-1})$ and $h_j := \int_0^{h_j} k^2(t)p^{-1}(t) dt$. This operator has been studied in [Bell] where it is proved that

$$k^{-4}(g^{-1})\rho_1(g^{-1})p(g^{-1}) = k^{-4}(g^{-1})\rho_2(g^{-1})p(g^{-1}). \quad (16)$$

Hence we deduce that $\rho_1 = \rho_2$ and $h_1 = h_2$.

Remark 4.1. In [BK], the case where $p = 1$ and $q = 0$ is studied and a procedure to reconstruct ρ from the spectral data is given.

Since in our case, we used the gauge function k , which can be constructed by solving a direct elliptic problem, and the Liouville type transformation, which is explicit, to transform this general case to the case studied in [BK], then a procedure to reconstruct ρ from the spectral data can be obtained. In [BK], ρ is assumed to be $C^2[0,h]$. This condition is assumed to prove the boundary controllability and not to give the reconstruction procedure. Hence, this procedure is applicable in our case.

5. Equivalence of the Gelfand data and the Borg–Levinson data

In the introduction we called the Borg–Levinson data the following sequence $(\lambda_n, \mu_n)_{n \in \mathbb{N}}$, where $(\lambda_n)_{n \in \mathbb{N}}$ is the sequence of the eigenvalues of the Sturm–Liouville problem where we take as boundary conditions the Dirichlet one on $x_0 = 0$ and $x_0 = h$, while $(\mu_n)_{n \in \mathbb{N}}$ is the sequence of the eigenvalues of the Sturm–Liouville problem where we take as boundary conditions the Dirichlet one on $x_0 = 0$ and the Neumann one on $x_0 = h$.

The aim of this section is to prove the equivalence of the Gelfand data and the Borg–Levinson data. More precisely, in section 5.1, we prove that the Gelfand spectral data imply the Borg–Levinson data for general bounded coefficients p, q and ρ . In section 5.2, we prove that the Borg–Levinson data imply the Gelfand data assuming that p and ρ are BV -functions and $p(0)\rho(0)$ is known.

5.1. Case A: Gelfand data imply Borg–Levinson data

Suppose that for two families of coefficients p_j, q_j, ρ_j , $j = 1, 2$, we have $(\lambda_i^1, |p_1(e_i^1)'(0)|)_{i \in \mathbb{N}} = (\lambda_i^2, |p_2(e_i^2)'(0)|)_{i \in \mathbb{N}}$. Now for $j = 1, 2$, we take k_j such that k_j^{-1} is the $H^1(0, h)$ -solution of the elliptic problem:

$$\begin{cases} A_j(k_j^{-1}) = 0, & \text{in } (0, h), \\ k_j^{-1}(0) = 1, & p_j(k_j^{-1})'(h) = 0. \end{cases} \quad (17)$$

Then k_j^{-1} is positive by the maximum principle, see [PW].

With this choice of k_j , we have $A_{k_j}^j u = -\frac{k_j^2}{\rho_j}(k_j^{-2}p_j u)'$. We set $\phi_i^j := k_j e_i^j$, hence:

$$\begin{cases} -(k_j^{-2}p_j(\phi_i^j))' = \lambda_i^j \rho_j k_j^{-2} \phi_i^j, & \text{in } (0, h), \\ \phi_i^j(0) = \phi_i^j(h) = 0, \\ |k_1^{-2}p_1(\phi_i^1)(0)| = |k_2^{-2}p_2(\phi_i^2)(0)|, \\ \int_0^h k_j^{-2} \rho_j (\phi_i^j)^2 dx = 1. \end{cases} \quad (18)$$

In ([S], remark 4), it is proved that $|k_1^{-2}p_1(\phi_i^1)'(h)| = |k_1^{-2}p_1(\phi_i^1)'(h)|, i \in \aleph$. We set $\psi_i^j = k_j^{-2}p_j(\phi_i^j)'$, then ψ_i^j satisfy the following Neumann problem:

$$\begin{cases} -(k_j^2\rho_j^{-1}(\psi_i^j)')' = \lambda_i^j p_j^{-1}k_j^2\psi_i^j, & \text{in } (0, h), \\ k_j^2\rho_j^{-1}(\psi_i^j)'(0) = k_j^2\rho_j^{-1}(\psi_i^j)'(h) = 0, \\ |\psi_i^1(0)| = |\psi_i^2(0)|, & |\psi_i^1(h)| = |\psi_i^2(h)|, \\ \int_0^h k_j^2\rho_j^{-1}(\psi_i^j)^2 dx = \lambda_i^1 = \lambda_i^2. \end{cases} \tag{19}$$

Let us now define the following elliptic problems:

$$\begin{cases} \text{Find } u_j \in H^1(0, h) \text{ such that:} \\ (k_j^2\rho_j^{-1}(u_j)')' + \lambda_i^j p_j^{-1}k_j^2u_j = 0, & \text{in } (0, h), \\ k_j^2\rho_j^{-1}(u_j)'(0) = a, & k_j^2\rho_j^{-1}(u_j)'(h) = b. \end{cases} \tag{20}$$

The functionals $(a, b) \longrightarrow \Lambda_j^N(\lambda) := (u^j(0), u^j(h)), j = 1, 2$, are called the Neumann to Dirichlet operators.

In ([S], section 2.3), it is proved that these two Neumann to Dirichlet operators are equal for all $\lambda \in \aleph \setminus \{\lambda_i^j; j = 1, 2; i \in \aleph\}$.

Remark 5.1. For general coefficients (p, q, ρ) and for the Dirichlet problem, we have no proof of equality of the Dirichlet to Neumann map if we have equality of the Gelfand spectral data. This is the reason why we transformed this Dirichlet spectral problem to the Neumann spectral problem for which we have a proof, see [CK] or [S]. In the previous part, we proved equality of the Neumann–Dirichlet operators from the Dirichlet Gelfand spectral data. Using the Alessandrini identity we deduce the equality of the Dirichlet–Neumann operators.

Let us now use the Alessandrini identity [A1]. Let u_j be an $H^1(0, h)$ -solution of (20), then we have the following equality:

$$\int_0^h (k_1^2\rho_1^{-1} - k_2^2\rho_2^{-1})u_1'u_2' - \lambda(k_1^2p_1^{-1} - k_2^2p_2^{-1})u_1u_2 dx = ((\Lambda_1^N - \Lambda_2^N)(a, b)) \cdot (c, d),$$

where $(k_1^2\rho_1^{-1}u_1'(0), k_1^2\rho_1^{-1}u_1'(h)) = (a, b)$ and $(k_2^2\rho_1^{-1}u_2'(0), k_2^2\rho_1^{-1}u_2'(h)) = (c, d)$.

Since

$$\Lambda_1^N(\lambda) = \Lambda_2^N(\lambda) \quad \text{for all } \lambda \in \aleph \setminus \{\lambda_i^j; j = 1, 2; i \in \aleph\},$$

then

$$\int_0^h (k_1^2\rho_1^{-1} - k_2^2\rho_2^{-1})u_1'u_2' - \lambda(k_1^2p_1^{-1} - k_2^2p_2^{-1})u_1u_2 dx = 0 \tag{21}$$

for every $H^1(0, h)$ -solution of (20).

We define the mixed-mixed operators as

$$(a, b) \longrightarrow \Lambda_j^M := (u_j(0), k_j^2\rho_j^{-1}u_j'(h)), \quad j = 1, 2$$

We use also the Alessandrini identity for the mixed problems as

$$\int_0^h (k_1^2\rho_1^{-1} - k_2^2\rho_2^{-1})u_1'u_2' - \lambda(k_1^2p_1^{-1} - k_2^2p_2^{-1})u_1u_2 dx = ((\Lambda_1^M - \Lambda_2^M)(a, b)) \cdot (c, d),$$

for all $H^1(0, h)$ -solution of (20), where $(u_1(0), k_2^2p_2^{-1}u_1'(0)) = (a, b)$ and $(u_2(0), k_2^2p_2^{-1}u_2'(0)) = (a, b)$.

From (21), we deduce that $(\Lambda_1^M - \Lambda_2^M)(a, b) = 0$ for all $(a, b) \in \mathfrak{N}^2$ and $\lambda \in \mathfrak{N} \setminus (S_1 \cup S_2)$, where S_j are the set of the mixed eigenvalues of the operators given by $-\frac{p_j}{k_j^2} \left(\frac{k_j^2}{\rho_j} u'\right)'$ with the mixed homogeneous boundary conditions. We denote these eigenvalues by $\bar{\mu}_i^j$.

It is known that if $\Lambda_1^M = \Lambda_2^M, \forall \lambda \in \mathfrak{N} \setminus (S_1 \cup S_2)$, then $S_1 = S_2$ (see [S], lemma 2.5). This means that $\bar{\mu}_i^1 = \bar{\mu}_i^2, i \in \mathfrak{N}$. Set $\psi_i^{j,M}$ the corresponding eigenfunctions.

Using the reverse transformation: $\phi_i^{j,M} = k_j^2 \rho_j^{-1} (\psi_i^{j,M})'$, we find that the mixed eigenvalues given by the boundary conditions, $\phi_i^{j,M}(0) = 0, k_j^{-2} p_j (\phi_i^{j,M})'(0) = 0$, are also the same $\bar{\mu}_i^1 = \bar{\mu}_i^2$. Now using the reverse gauge transformation defining $e_i^{j,M} := k_j^{-1} \phi_i^{j,M}$, then we find that $\mu_i^j = \bar{\mu}_i^j$. Hence, we have equality of the mixed eigenvalues for the operators given by (p_j, q_j, ρ_j) .

5.2. Case B: Borg–Levinson data imply Gelfand data

In this part we suppose that p and ρ are BV -functions and that $p(0)\rho(0)$ is known.

Suppose that $\mu_i^1 = \mu_i^2$ and $\lambda_i^1 = \lambda_i^2$, for the mixed and Dirichlet operators given by the differential expressions $-\frac{1}{\rho}(pu')' + \frac{q}{\rho}$. We are going to prove that the Borg–Levinson spectral data imply the Gelfand spectral data.

First we transform this problem to the case where $q = 0$ as in the case (A) by introducing the gauge function k solution of the problem $-(p(k^{-1})')' + qk^{-1} = 0$ in $(0, h)$ and $k^{-1}(0) = 1$ and $p(k^{-1})'(h) = 0$. Hence μ_i^j and λ_i^j are the mixed and Dirichlet spectrum for the operator $-\frac{1}{\rho k^{-2}}(pk^{-2}u')'$. Secondly, we transform this problem to the case where the equations are $-\frac{1}{\tilde{p}}(\tilde{p}u')'$, where $\tilde{p} = p(g^{-1}(y))\rho(g^{-1}(y))k^{-4}(g^{-1}(y))$ and $g(x) = \int_0^x \sqrt{\frac{\rho}{p}}(t) dt$. From [An], we deduce uniqueness of \tilde{p} since $k^{-4}(0)\rho(0)p(0) =: \tilde{p}(0)$ is known and \tilde{p} is a BV -function. In particular, one has equality of the Gelfand data. Now, using the reverse transformations we deduce the equality of the Gelfand data for the expression $-\frac{1}{\rho}(pu')' + \frac{q}{\rho}u$.

Remark 5.2. The additional condition on $p(0)\rho(0)$ is natural since in the case $q = 0$, we have uniqueness just up to a multiplicative constant.

6. Uniqueness from the nodes: proof of theorem 2

In this section we suppose that p and ρ are BV -functions continuous from the right and at $x = h$.

Let N be the set of the nodes of the eigenfunctions e_i . As in section 4 we introduce the gauge function $k(x)$ solution of the problem $-(p(k^{-1})') + qk^{-1} = 0$ in $(0, h)$ with the boundary conditions $k^{-1}(0) = k^{-1}(h) = 1$. The function $k(x)$ is in $H^1(0, h)$ and $k > 0$. We set $\phi_i := ke_i$. We know that $\phi_i, i \in \mathfrak{N}$, are the eigenfunctions of the operators $A_k u := \rho^{-1}k^{-2}(pk^{-2}u')'$ on $(0, h)$. Hence the set of nodes of the functions $\phi_i, i \in \mathfrak{N}$, is exactly N since $k > 0$. We recall the following result which gives the asymptotic of the eigenvalues (see [HaMc2]):

$$\left| \sqrt{\lambda_n} \int_0^h \sqrt{\frac{\rho}{p}}(x) dx - n\pi \right| \leq \frac{1}{4} V \left(\ln \left(\frac{p\rho}{k^4} \right) \right), \tag{22}$$

where $V(f)$ is the total variations of f .

Since p and q are fixed then k is also known. Hence the only unknown is ρ .

In [HaMc2], the authors proved that for the equation $-(pu')' = \lambda\rho u$ with p and ρ are BV -functions continuous from the right and at $x = h$, uniqueness of ρ , up to a multiplicative

constant, holds from a dense subset of the nodes if p is fixed. We recall that this set of nodes has the following property: for $n \in \mathbb{N}$, if this set contains the j th node x_j^n of the n th eigenvalue then it also contains the $(j + 1)$ th or the $(j - 1)$ th node of this eigenfunction.

Hence using this result we deduce that ρk^2 and hence ρ is unique up to a multiplicative constant from this dense subset of nodes.

Let us now consider the uniqueness of both q and ρ . We suppose in addition to the previous part that the coefficient p and ρ are in $H^2(0, h)$ and q is continuous almost every where. In this case we have the following asymptotics of the eigenvalues:

$$\left| \sqrt{\lambda_n} \int_0^h \sqrt{\frac{\rho}{p}}(x) dx - n\pi \right| = O\left(\frac{1}{n}\right), \tag{23}$$

see ([CCPR], p 82) and [HaMc2] for more references about the asymptotics of the eigenvalues.

We prove uniqueness of the coefficients on the points of continuity.

The method of proof is the one given in [HaMc2].

Let x be a point of continuity of the coefficients q and ρ . Set $x_{l(n)}^n$ as a sequence of this dense set tending to the point x .

Consider the equations $-(pu'_j)' + q_j u_j = \lambda_j^n \rho_j u_j$ in $(x_{l(n)}^n, x_{l(n)+1}^n)$. We can consider λ_j^n as the first eigenvalue for the operators replacing $(0, h)$ by $(x_{l(n)}^n, x_{l(n)+1}^n)$. Multiplying the $(j = 1)$ -equation by u_2 and the $(j = 2)$ -equation by u_1 , integrating over $I_n := (x_{l(n)}^n, x_{l(n)+1}^n)$ and subtracting the results we find

$$\int_{I_n} [(q_1 - q_2) + (\lambda_n^2 \rho_2 - \lambda_n^1 \rho_1)] u_1^n u_2^n dx = 0.$$

Hence there exist x'_n and x''_n in I_n such that

$$[(q_1 - q_2) + (\lambda_n^2 \rho_2 - \lambda_n^1 \rho_1)](x'_n) \geq 0 \tag{24}$$

and

$$[(q_1 - q_2) + (\lambda_n^2 \rho_2 - \lambda_n^1 \rho_1)](x''_n) \leq 0 \tag{25}$$

since $u_1^n u_2^n$ can be chosen to be a positive function in $(x_{l(n)}^n, x_{l(n)+1}^n)$.

When n tends to infinity, x'_n and x''_n tend to x . Dividing the inequalities (24) and (25) by (λ_n^2) , going to the limit when n tends to infinity and using the fact that $\sqrt{\frac{\lambda_n^1}{\lambda_n^2}}$ tends to a constant R , which is given by (22) or (23), we deduce that $\rho_2(x) = R^2 \rho_1(x)$. Since x is arbitrary we deduce that $\rho_2(x) = R^2 \rho_1(x)$ in every point of continuity of the coefficients. From (22), we deduce that

$$R = \frac{\int_0^h \sqrt{\frac{\rho_2}{p}}(x) dx}{\int_0^h \sqrt{\frac{\rho_1}{p}}(x) dx}.$$

Now (24) and (25) can be rewritten as

$$[(q_1 - q_2) + (\lambda_n^2 R^2 - \lambda_n^1) \rho_1](x'_n) \geq 0 \tag{26}$$

and

$$[(q_1 - q_2) + (\lambda_n^2 R^2 - \lambda_n^1) \rho_1](x''_n) \leq 0. \tag{27}$$

But $(\lambda_n^2 R^2 - \lambda_n^1) = \frac{1}{R^2} [R^2 \lambda_n^2 - R^2 \lambda_n^1]$, where $R_j := \int_0^h \sqrt{\frac{\rho_j}{p}}(x) dx$.

Or from the bounds of the eigenvalues (23), we deduce that the sequence $\frac{1}{R^2} [R^2 \lambda_n^2 - R^2 \lambda_n^1]$ is bounded. Hence we can take a converging subsequence. We set α the limit of this subsequence. Hence from (26) and (27), we deduce using the relative subsequence for λ_n^j ,

that $(q_1 - q_2 - \alpha\rho_1)(x) = 0$. Finally we have $q_1 - q_2$ is in the one-dimensional subspace generated by ρ_1 .

Remark 6.1. (1) Using this gauge transformation, we deduce similar asymptotics for the eigenvalues for the equation $-(pu')' + qu = \lambda\rho u$ as those given in [HaMc2]. (2) If we take p and ρ discontinuous the asymptotics (23) are not valid (see [HaMc2]). Hence we cannot deduce the boundness of the sequence $\frac{1}{R_1^2}[R_2^2\lambda_n^2 - R_1^2\lambda_n^1]$.

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Appendix

Let p_n, q_n and ρ_n be three sequences of functions in $L^\infty(0, h)$ such that $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma \forall n$ in \mathfrak{N} . Let u_n be the solution of the following hyperbolic problem:

$$\begin{cases} \rho_n \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial}{\partial x} \left(p_n \frac{\partial u_n}{\partial x} \right) + q_n u_n = 0 & \text{in } (0, T) \times (0, h), \\ u_n(t, 0) = f_1(t), u_n(t, h) = f_2(t), & t \in (0, T), \\ u_n(0, x) = \psi_0(x), \frac{\partial u_n}{\partial t}(0, x) = \psi_1(x), & x \in (0, h), \end{cases} \tag{A.1}$$

where ψ_0, ψ_1 are in $C_0^\infty[0, h]$ and f_i is in $C_0^\infty[0, T], i = 1, 2$.

The aim of this part of the appendix is to prove that u_n is bounded in the energy norm.

Let $y(x) := g_n(x) = \frac{1}{h_n} \int_0^x p_n^{-1}(t) dt$. We set $v_n(t, y) = u_n(t, g_n^{-1}(y)), \tilde{\rho}_n(y) := \rho_n(g_n^{-1}(y)), \tilde{p}_n(y) = p_n(g_n^{-1}(y)), \tilde{q}_n(y) = q_n(g_n^{-1}(y))$ and $h_n = \int_0^h p_n(t) dt$. Hence v_n satisfies the following problem:

$$\begin{cases} \frac{\partial^2 v_n}{\partial t^2} - \frac{h_n^2}{\tilde{p}_n(y)\tilde{\rho}_n(y)} \frac{\partial^2 v_n}{\partial y^2} + \frac{\tilde{q}_n(y)}{\tilde{\rho}_n(y)} v_n = 0 & \text{in } (0, T) \times (0, 1), \\ v_n(t, 0) = f_1(t), u_n(t, 1) = f_2(t), & t \in (0, T), \\ v_n(0, x) = \psi_0(g_n^{-1}(y)), \frac{\partial v_n}{\partial t}(0, x) = \psi_1(g_n^{-1}(y)), & y \in (0, 1). \end{cases} \tag{A.2}$$

Now let $F(t, y)$ be a regular function on $(0, T) \times (0, 1)$ such that $F(t, 0) = f_1(t)$ and $F(t, 1) = f_2(t), t \in (0, T)$.

We set $\bar{v}_n = F(t, y) - v_n(t, y)$, then \bar{v}_n satisfies the following problem:

$$\begin{cases} E_n \frac{\partial^2 \bar{v}_n}{\partial t^2} - \frac{\partial^2 \bar{v}_n}{\partial y^2} + B_n \bar{v}_n = G_n(t, y) & \text{in } (0, T) \times (0, 1), \\ \bar{v}_n(t, 0) =, \quad \bar{v}_n(t, 1) = 0, & t \in (0, T), \\ v_n(0, x) = -\psi_0(g_n^{-1}(y)) + F(0, y), & y \in (0, 1), \\ \frac{\partial v_n}{\partial t}(0, x) = -\psi_1(g_n^{-1}(y)) + \frac{\partial F}{\partial t}(0, y), & y \in (0, 1), \end{cases} \tag{A.3}$$

where $G_n(t, y) := E_n \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial y^2} + B_n F$ and $E_n = \frac{1}{h_n^2} \tilde{p}_n(y)\tilde{\rho}_n(y), B_n = \frac{1}{h_n^2} \tilde{p}_n(y)\tilde{\rho}_n(y)$.

It is clear that $G_n(t, y) \in L^2((0, T) \times (0, 1))$ and $\|G_n\|_{L^2((0,T)\times(0,1))} \leq c, \forall n \in \mathfrak{N}$. Since $p_n \geq \gamma > 0, \forall n \in \mathfrak{N}$, then $\psi_0(g_n^{-1}(y))$ and $\psi_1(g_n^{-1}(y))$ are bounded in $H^1(0, 1)$ and $L^2(0, 1)$, respectively. Mimicking the proof given in [La], section IV.2, (2.15) and section IV, (3.17), we find that \bar{v}_n is in $C([0, T], H^1(0, 1)) \cap C^1([0, T], L^2_{E_n}(0, 1))$ with a bounded norm. Note that the sequence E_n is bounded from below and above. Since $F(t, y)$ is regular then we

have $v_n \in C([0, T], H^1(0, 1)) \cap C^1([0, T], L^2_{E_n}(0, 1))$ with a bounded norm. Now using the reverse transformation we deduce that $v_n \in C([0, T], H^1(0, h)) \cap C^1([0, T], L^2(0, h))$ with a bounded norm.

Proposition 7.1. *Let (p_n, q_n, ρ_n) be three sequences of functions in $L^\infty(0, h)$ satisfying $0 < \delta \leq p_n, q_n, \rho_n \leq \gamma \forall n$ in \mathbb{N} and (p, q, ρ) in $L^\infty(0, h)$ such that (p_n, q_n, ρ_n) tends to (p, q, ρ) in $(L^2(0, h))^3$. Let u_n and u be the solutions (A.1) with the coefficients (p_n, q_n, ρ_n) and (p, q, ρ) , respectively. Then u_n tends to u in $L^2((0, T) \times (0, h))$.*

Proof of proposition 7.1. From the energy inequality, we deduce that u_n is bounded in $L^2((0, T), H^1(0, h)) \cap H^1((0, T), L^2(0, h))$ as in the previous part of this appendix. Then we can take a subsequence, denoted also by u_n , and a function \bar{u} such that u_n tends to \bar{u} in $H^1((0, T), L^2(0, h))$ weakly. Hence $\|u_n - \bar{u}\|_{L^2(0, h)}(t)$ is bounded in $H^1(0, T)$ then, up to a subsequence, $\int_0^T \|u_n - \bar{u}\|_{L^2(0, h)}^2(t) dt$ is convergent. This means that $u_n - \bar{u}$ is convergent in $L^2((0, T) \times (0, h))$ and the limit is zero.

Using the fact that u_n tends weakly to \bar{u} in $L^2([0, T], H^1(0, h)) \cap H^1([0, T], L^2(0, h))$ and the definition of the weak solution of the hyperbolic problem, we deduce that \bar{u} is solution of (A.1) replacing p_n, q_n and ρ_n respectively by p, q and ρ . Hence $\bar{u} = u$ by uniqueness. \square

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