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About embedded eigenvalues for a spectral problem arising in the study of elastic surface waves in a topographical waveguide

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SUMMARY

In this paper, we are interested with the spectral study of an operator given by an elastic topographical waveguide, a deformed half-space, of which the cross-section is a local perturbation of a homogeneous half-plane. We look for guided waves propagating more rapidly than Rayleigh waves (which mathematically would correspond to embedded eigenvalues) and prove that there are no guided waves propagating more rapidly than S-waves. Thanks to the boundary of the deformed half-plane and some reduced equations, these eventual eigenmodes must locally vanish. Adapting to our case a unique continuation principle for the elasticity system, we conclude that these eigenmodes vanish everywhere.

KEY WORDS: waveguide; spectral analysis; linear elasticity system

1. INTRODUCTION

1.1. The mathematical framework

The next sections are devoted to the investigation of a situation that occurs in the elastic waveguides. We are interested by the topographical waveguide in an isotropic and heterogeneous medium with free boundary. Let $\tilde{\Omega}=\{(x,y,z)\in\mathbb{R}^3; g(x)>z\}$ be the guide of which $\Omega=\{(x,z)\in\mathbb{R}^2; g(x)>z\}$ is the section, g being a Lipschitz function vanishing for $|x|\geqslant a$. Here a is a fixed positive real number (Figure 1).

Let $U(x,y,z,t)=(U_1(x,y,z,t),U_2(x,y,z,t),U_3(x,y,z,t))$ be the displacement field, then the propagation of an elastic wave is governed by the following system of equations:

$$\text{(I)}\quad \begin{cases} \rho\,\frac{\partial^2 U}{\partial t^2} = \operatorname{div}(\sigma(U)) \\ \sigma_{ij}(U) = \lambda \operatorname{div}(U)\delta_{ij} + 2\mu\varepsilon_{ij}(U) \quad \text{(stress tensor)} \\ \varepsilon_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \end{cases}$$

where $x_1=x,\ x_2=y,\ x_3=z$ and $\delta_{ij},\ i,j=1,2,3$, is Kronecker's symbol.

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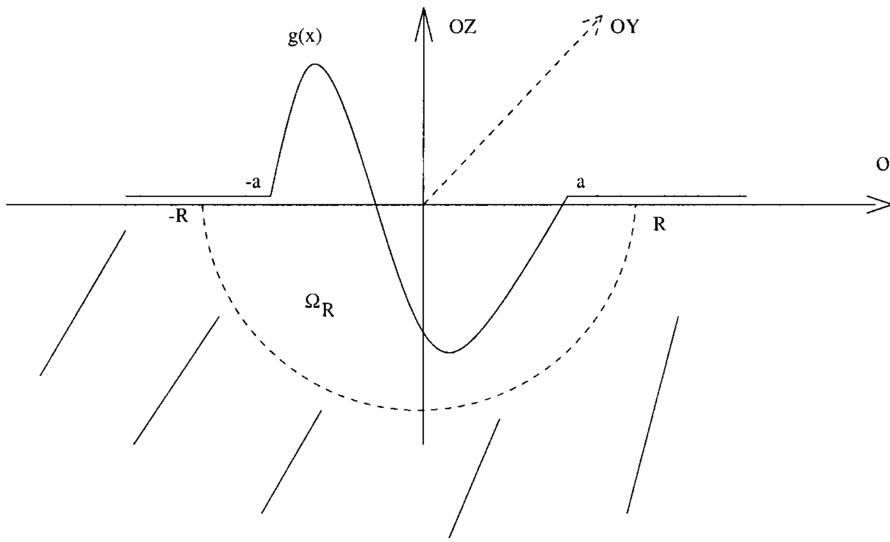

Figure 1. An example of a waveguide.

The free boundary implies that the trace of the normal component of the stress tensor vanishes on the boundary of the guide, i.e. $\sigma(U)\tilde{v}|_{\partial\tilde{\Omega}} = 0$, where $\tilde{v} = (v_1, 0, v_2)^\top$ is the exterior normal vector to $\tilde{\Omega}$ ($v = (v_1, v_2)^\top$ denotes the exterior normal vector to Ω). The Lamé coefficients λ, μ, and the density ρ satisfy the following conditions:

$$\begin{cases} \lambda(x,y,z) = \lambda(x,z), \mu(x,y,z) = \mu(x,z), \rho(x,y,z) = \rho(x,z) \\ \lambda(x,z) = \lambda_\infty, \mu(x,z) = \mu_\infty, \rho(x,z) = \rho_\infty \text{ if } (x,z) \in \Omega \backslash \Omega_R \\ \lambda(x,z) \geqslant \lambda_-, \mu(x,z) \geqslant \mu_-, \rho(x,z) \geqslant \rho_- \end{cases}$$

where $\lambda_-, \mu_-, \rho_-, \lambda_\infty, \mu_\infty, \rho_\infty$ are positive constants, $\Omega_R = \Omega \backslash \{(x,z) \in \mathbb{R}^2_- / x^2 + z^2 \geqslant R\}$ and $R \geqslant \max(a, \sup_{x\in\mathbb{R}} |g(x)|)$.

We consider the coefficients $\lambda(x,z)$, $\mu(x,z)$ and $\rho(x,z)$ in $W^{1,\infty}(\Omega)$ and piecewise $W^{2,\infty}(\Omega_R)$ i.e. there exists a decomposition (finite or not) of Ω_R into a family of subdomains $(\Omega^i_R)_{i\in I}$, $I \subset \mathbb{N}$, such that $\Omega^i_R \cap \Omega^j_R = \emptyset$, $i \neq j$, $\overline{\Omega_R} = \bigcup_{i\in I} \overline{\Omega^i_R}$ and we suppose that $\lambda_{|\Omega^i_R}(x,z), \mu_{|\Omega^i_R}(x,z)$ and $\rho_{|\Omega^i_R}(x,z)$ are in $W^{2,\infty}(\Omega^i_R)$ and can be extended to $W^{2,\infty}$ functions in an open set containing $\overline{\Omega^i_R}$. The extensions need not agree with the values of $\lambda(x,z)$, $\mu(x,z)$ and $\rho(x,z)$ outside Ω^i_R. The set $W^{m,\infty}$ is the Sobolev space of functions which are bounded with their m first derivatives, where $m \in \mathbb{N}$.

A guided wave is a particular solution of (I) of the following form:

$$\text{(II)} \quad U(x,y,z,t) = \hat{u}(x,z) \exp i(\beta y - \omega t), \quad (\omega, \beta) \in \mathbb{R}^2$$

with the field $\hat{u}(x,z)$ belonging to $(H^1(\Omega))^3$.

This definition enables us to describe a wave propagating in the (Oy) direction and having the velocity ω/β. We say that ω and β are the pulsation and the wave propagation number, respectively. The fact that $\hat{u}(x,z) \in (H^1(\Omega))^3$ means that the wave transverse energy is finite.

Plugging (II) into (I) and introducing the new field $u(x,z)=(u_1,u_2,u_3):=(\hat{u}_1, i\hat{u}_2, \hat{u}_3)$, the investigation of the guided waves is reduced to the analysis of the following spectral problems with the parameter $\beta\in\mathbb{R}$ (see Reference [1] for the case where the coefficients are constants):

$$\begin{cases} \text{Find } u\in(H^1(\Omega))^3, u\neq 0,\ \omega\in\mathbb{R} \text{ such that} \\ A_\beta(u)=\omega^2 u \text{ in } \Omega \\ \sigma^\beta(u)\tilde{v}|_{\partial\Omega}=0 \end{cases}$$

where

$$A_\beta u := \frac{1}{\rho}\begin{cases} -\frac{\partial}{\partial x}(\lambda+2\mu)\frac{\partial u_1}{\partial x} - \frac{\partial}{\partial z}\mu\frac{\partial u_1}{\partial z} - \frac{\partial}{\partial x}\lambda\frac{\partial u_3}{\partial z} - \frac{\partial}{\partial z}\mu\frac{\partial u_3}{\partial x} - \beta\frac{\partial}{\partial x}(\lambda u_2) \\ \quad -\beta\mu\frac{\partial}{\partial x}u_2 + \beta^2\mu u_1 \\ -\frac{\partial}{\partial x}\mu\frac{\partial u_2}{\partial x} - \frac{\partial}{\partial z}\mu\frac{\partial u_2}{\partial z} + \beta\frac{\partial}{\partial x}\mu u_1 + \beta\frac{\partial}{\partial z}\mu u_3 + \beta\lambda\frac{\partial}{\partial x}u_1 + \beta\mu\frac{\partial}{\partial z}u_3 \\ \quad +(\lambda+2\mu)\beta^2 u_2 \\ -\frac{\partial}{\partial x}(\mu)\frac{\partial u_3}{\partial x} - \frac{\partial}{\partial z}(\lambda+2\mu)\frac{\partial u_3}{\partial z} - \frac{\partial}{\partial x}\lambda\frac{\partial u_1}{\partial z} - \frac{\partial}{\partial z}\mu\frac{\partial u_1}{\partial x} \\ \quad -\beta\frac{\partial}{\partial z}(\lambda u_2) - \beta\mu\frac{\partial}{\partial z}(u_2) + \beta^2\mu u_3 \end{cases} \tag{1}$$

and setting $\mathrm{div}^\beta(u)=(\frac{\partial u_1}{\partial x}+\beta u_2+\frac{\partial u_3}{\partial z})$, we get:

$$\sigma^\beta(u)\tilde{v}|_{\partial\Omega} := \begin{cases} (\lambda\,\mathrm{div}^\beta u + 2\mu\frac{\partial u_1}{\partial x})v_1 + \mu(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x})v_2 \\ \mu(\frac{\partial u_2}{\partial x}-\beta u_1)v_1 + \mu(\frac{\partial u_2}{\partial z}-\beta u_3)v_2 \\ \mu(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x})v_1 + (\lambda\,\mathrm{div}^\beta u + 2\mu\frac{\partial u_3}{\partial z})v_2 \end{cases} \tag{2}$$

For the sequel, we define the following subspace:

$$\mathscr{D}(A_\beta)=\{u\in(H^1(\Omega))^3/A_\beta u\in(L^2(\Omega))^3 \text{ and } \sigma^\beta(u)\tilde{v}|_{\partial\Omega}=0\}$$

where $\sigma^\beta(u)\tilde{v}|_{\partial\Omega}$ belongs to $(H^{-1/2}(\partial\Omega))^3$.

1.2. Presentation of the result

The case of constant coefficients $(\lambda_\infty,\mu_\infty,\rho_\infty)$ was considered in Reference [1] where the function g were assumed to be non-negative. It has been proved that the corresponding operator which we denote by $(A_\beta^\infty, \mathscr{D}(A_\beta^\infty))$ is self-adjoint, positive and that its essential spectrum is characterized by $\sigma_{\mathrm{ess}}(A_\beta^\infty)=[\beta^2 c_{\mathrm{R}}^2, +\infty[$ where c_{R} is the velocity of the Rayleigh wave that propagates in a perfect half-space [2,3]. It is the unique solution in $]0,c_{\mathrm{s}}[$ of the so-called Rayleigh equation:

$$4\left(1-\frac{z^2}{c_{\mathrm{p}}^2}\right)^{1/2}\left(1-\frac{z^2}{c_{\mathrm{s}}^2}\right)^{1/2} - \left(2-\frac{z^2}{c_{\mathrm{s}}^2}\right)^{1/2} = 0 \tag{3}$$

where $c_{\mathrm{s}}=(\mu/\rho)^{1/2}$ (resp. $c_{\mathrm{p}}=((\lambda+2\mu)/\rho)^{1/2}$) is the velocity of the transverse wave or S-wave (resp. longitudinal wave or P-wave) of the homogeneous medium.

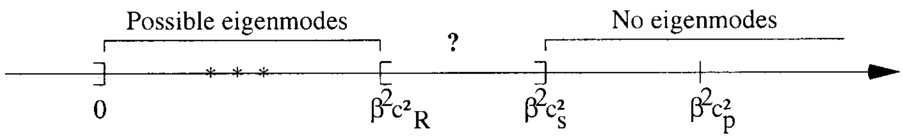

Figure 2. The structure of the spectrum.

Using the same ideas as in References [1, 4] the same result occurs, namely, the operator $(A_\beta, \mathscr{D}(A_\beta))$ is self-adjoint, positive and its essential spectrum is characterized by $\sigma_{ess}(A_\beta) = [\beta^2 c_R^2, +\infty[$. Denote by $\sigma_p(A_\beta)$ the point spectrum of $(A_\beta, \mathscr{D}(A_\beta))$ (cf. Reference [5]).

In Reference [1], the existence of guided waves with velocity strictly less than c_R was established. From a mathematical point of view, this concerns the discrete spectrum. Using the min–max principle, the authors have shown that for a large family of geometric perturbations of the half-space (in other words, for a large family of functions g) and for large values of β, the discrete spectrum of $(A_\beta, \mathscr{D}(A_\beta))$ is not empty. In their paper, the question of existence of embedded eigenvalues in the essential spectrum was still open. In this article, we partially answer to this question. Indeed, we shall prove the following theorem.

Theorem
If the function g and the coefficients λ, μ and ρ satisfy the conditions of Section 1.1, then for every non-negative β, we have

$$\sigma_p(A_\beta) \cap]\beta^2 c_s^2, +\infty[= \emptyset$$

In Figure 2 we give the spectral properties of $(A_\beta, \mathscr{D}(A_\beta))$.

Comments

(1) Some extensions of this theorem can be proved: the boundary $\partial\Omega$ need not be defined by a univoque function g and inclusions in Ω can be considered. We do not detail these situations to load not the text.
(2) To the author's knowledge there are no results available, up to now, concerning the existence of eigenvalues in the part $[\beta^2 c_R^2, \beta^2 c_s^2]$ of the spectrum, even in the constant coefficients case, see the remarks at the end of the proof.
(3) A spectral analysis of the perturbed elastic half-space was presented in Reference [6]. A limiting absorption principle as well as some qualitative results about the eigenvalues were established. In particular, the following fact was pointed out, namely, if eigenvalues exist, their set must be discrete. The method developed here confirms this and claims that this set is empty. The situation in Reference [6] corresponds to the case where $\beta = 0$.
(4) An analysis of the elastic guided waves in the whole space was presented in Reference [4, 7]. The authors of Reference [4] gave some family of Lamé coefficients and density for which the discrete spectrum of the operator is not empty as well as a proof of absence of eigenvalues in the part $]\beta^2 c_p^2, +\infty[$ of the operator's essential spectrum for piecewise constant coefficients.
A family of coefficients for which the corresponding operator have eigenvalues in the part $[\beta^2 c_s^2, \beta^2 c_p^2]$ of the essential spectrum was given in Reference [7]. The method

used in Reference [7] is based on a decomposition of the operator into a direct sum of two Schrödinger-type operators having $\beta^2 c_p^2$ as infinimum of their essential spectrums. With the help of the min–max principle, they proved the existence of eigenvalues in $[\beta^2 c_s^2, \beta^2 c_p^2]$.

In the present situation, this decomposition is no longer valid because the boundary conditions cannot be decomposed. Precisely, in our paper, we shall take advantage of the boundary to prove that there is no eigenvalue in $]\beta^2 c_s^2, +\infty[$. We prove this result by contradiction. In first, we suppose the existence of an eigenvalue and show that the boundary conditions lead to a contradiction of which we deduce that the eigenfunctions vanish in $\mathbb{R}^2_-$. We use the partial Fourier transform in the direction (ox) in order to reduce the original problem to a one-dimensional problem with parameter β_1 (i.e. the dual variable of x given by the Fourier transform). This argument has been used in References [8, 9] where the authors studied the Schrödinger and the acoustical operator, respectively. Hence, taking into account the just mentioned reduction, we have obtained a differential system of order three. In Section 2.1.1, we start by decomposing the obtained system in a differential system of order two and a scalar equation. We remark that, with respect to the parameter, the solution of the 2-system as function of a complex variable has two incompatible behaviours near some poles. This enables us to deduce that the eigenfunction must vanish in $\mathbb{R}^2_-$. This is done in Section 2.1.2.

The classical procedure is to use a unique continuation principle to conclude that the eigenfunction vanish everywhere. With regard to the elasticity system, a unique continuation principle can be found in References [10–12] which requires a C^2 regularity of the coefficients. We adapt this principle to our operator for a piecewise $W^{2,\infty}$ regularity.

A part of the theorem was announced in Reference [13].

2. PROOF OF THE THEOREM

The proof is organized as follows. In Section 2.1, we start by the case where g is non-negative and $R=0$. We prove that the eigenfunction vanishes in $\Omega_0 = \mathbb{R}^2_- \subset \Omega$. In Section 2.2, we show how to prove the same thing in the general case. In Section 2.3, we deduce that the eigenfunction vanishes everywhere.

2.1. Part a: g is non-negative and R = 0

Let $\omega^2 > \beta^2 c_s^2$ be an eigenvalue of $(A_\beta, \mathscr{D}(A_\beta))$ and $v \in \mathscr{D}(A_\beta)$ be an associated eigenfunction, i.e. $A_\beta v = \omega^2 v$, $v \neq 0$. In this part, g is assumed to be non-negative, hence $\mathbb{R}^2_- \subset \Omega$ where $\mathbb{R}^2_- = \{(x,z) \in \mathbb{R}^2; z<0\}$. If we set $u = v_{|_{\mathbb{R}^2_-}}$, the function u has the following properties:

$$\begin{cases} u \in (H^1(\mathbb{R}^2_-))^3 \\ A_\beta(u) = \omega^2 u \text{ in } \mathbb{R}^2_- \\ \sigma^\beta(u)\tilde{\nu}|_{z=0} = \sigma^\beta(v)\tilde{\nu}|_{z=0} \text{ in } (H^{-1/2}(\mathbb{R}))^3 \end{cases}$$

We use the same notation for A_β as in (1).

We will proceed in several steps.

2.1.1. First step: decomposition: Let $\mathscr{F}_x : L^2(\mathbb{R}^2_-) \to L^2(\mathbb{R}^2_-)$, be the Fourier transform with respect to x and let $\mathscr{R}$ be the rotation of $\mathbb{R}^3$ defined by the following matrix:

$$M = \begin{pmatrix} \frac{\beta_1}{|\xi|} & \frac{\beta}{|\xi|} & 0 \\ -\frac{\beta}{|\xi|} & \frac{\beta_1}{|\xi|} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where β_1 is the dual variable of x and $|\xi|^2 = \beta^2 + \beta_1^2$.

We consider the mapping $\mathscr{L} : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $\mathscr{L}(u_1, u_2, u_3) = (iu_1, u_2, u_3)$. Next we set $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3) = \mathscr{R}\mathscr{L}\mathscr{F}_x(u)$ and $\bar{u}_{1,3} = (\bar{u}_1, \bar{u}_3)$.

We put $\sigma^{\beta_1,\beta}(\bar{u}) = (\sigma_1^{\beta_1,\beta}(\bar{u}_{1,3}), \sigma_2^{\beta_1,\beta}(\bar{u}_2), \sigma_3^{\beta_1,\beta}(\bar{u}_{1,3}))$ where $\sigma_1^{\beta_1,\beta}(\bar{u}_{1,3}) = \mu \mathrm{d}\bar{u}_1/\mathrm{d}z - \mu|\xi|\bar{u}_3$, $\sigma_2^{\beta_1,\beta}(\bar{u}_2) = \mu \mathrm{d}\bar{u}_2/\mathrm{d}z$, $\sigma_3^{\beta_1,\beta}(\bar{u}_{1,3}) = (\lambda+2\mu)\,\mathrm{d}\bar{u}_3/\mathrm{d}z + |\xi|\lambda\bar{u}_1$ and we note $\sigma_{1,3}^{\beta_1,\beta}(\bar{u}_{1,3}) = (\sigma_1^{\beta_1,\beta}(\bar{u}_{1,3}),$ $\sigma_3^{\beta_1,\beta}(\bar{u}_{1,3}))$.

The vector function $\sigma^{\beta_1,\beta}(\bar{u})$ is deduced from the stress tensor $\sigma^{\beta}(u)$ by the relation $\sigma^{\beta_1,\beta}(\bar{u}) = \mathscr{R}\mathscr{L}\mathscr{F}_x(\sigma^{\beta}(u))$.

We recall that $L^2_s(\mathbb{R})$ is the Fourier transform of the Sobolev space $H^s(\mathbb{R})$. Let g belonging to $C^{1,1}(\mathbb{R})$ (the space of Lipschitz real-valued functions for which the first derivative is also Lipschitz). Since the coefficients are $W^{1,\infty}(\Omega)$, we deduce that u belongs to $H^2(\Omega)$ (cf. Reference [14]). The set $(C_0^{\infty}(\overline{\mathbb{R}^2_-}))^3$ (of indefinitly differentiable functions with compact support in $\overline{\mathbb{R}^2_-}$) being dense in $(H^2(\mathbb{R}^2_-))^3$, we have the following lemma.

Lemma
The functions $\beta_1 \to \mathscr{R}\mathscr{L}(\mathscr{F}_x(\sigma^{\beta}(u)\tilde{v}|_{z=0}))$ and $\beta_1 \to \sigma^{\beta_1,\beta}(\mathscr{R}\mathscr{L}(\mathscr{F}_x(u)))|_{z=0}$ are in $(L^2_{1/2}(\mathbb{R}))^3$. Furthermore, we have the following equality:

$$\mathscr{R}\mathscr{L}(\mathscr{F}_x(\sigma^{\beta}(u)\tilde{v}|_{z=0})) = \sigma^{\beta_1,\beta}(\mathscr{R}\mathscr{L}(\mathscr{F}_x(u)))|_{z=0}$$

Now, set $\beta_1 \to \mathscr{R}\mathscr{L}(\mathscr{F}_x(\sigma^{\beta}(u)\tilde{v}|_{z=0})) = (c_1(\beta_1), c_2(\beta_1), c_3(\beta_1))$, $c_{1,3}(\beta_1) = (c_1(\beta_1), c_3(\beta_1))$,

$$A_{\mathrm{P-SV}}(\bar{u}_{1,3}) = \frac{1}{\rho}\begin{cases} -\mu\,\frac{\mathrm{d}^2\bar{u}_1}{\mathrm{d}z^2} + |\xi|(\lambda+\mu)\,\frac{\mathrm{d}\bar{u}_3}{\mathrm{d}z} + |\xi|^2(\lambda+2\mu)\bar{u}_1 \\ -(\lambda+2\mu)\,\frac{\mathrm{d}^2\bar{u}_3}{\mathrm{d}z^2} - |\xi|(\lambda+\mu)\,\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}z} + |\xi|^2\mu\bar{u}_3 \end{cases}$$

and

$$A_{\mathrm{SH}}(\bar{u}_2) = c_s^2\left(-\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}z^2} + |\xi|^2\bar{u}_2\right)$$

Then $\bar{u}_{1,3}$ satisfies the following assertions:

$$\begin{cases} \bar{u}_{1,3} \in (H^2(\mathbb{R}_-, L^2(\mathbb{R}))^2 \\ A_{\mathrm{P-SV}}(\bar{u}_{1,3}) = \omega^2\bar{u}_{1,3} \text{ in } (L_2(\mathbb{R}_-))^2 \text{ for almost every } \beta_1 \in \mathbb{R} \\ \sigma_{1,3}^{\beta_1,\beta}(\bar{u}_{1,3})|_{z=0} = c_{1,3}(\beta_1) \text{ in } (L^2_{1/2}(\mathbb{R}))^2 \end{cases} \tag{4}$$

Similarly, $\bar{u}_2$ satisfies the following assertions:

$$\begin{cases} \bar{u_2} \in (H^2(\mathbb{R}_-, L^2(\mathbb{R})) \\ A_{\mathrm{SH}}(\bar{u}_2) = \omega^2 \bar{u}_2 \ \text{in} \ L_2(\mathbb{R}_-) \ \text{for almost every} \ \beta_1 \in \mathbb{R} \\ \sigma_2^{\beta_1,\beta)}(\bar{u}_2)|_{z=0} = c_2(\beta_1, \beta) \ \text{in} \ L^2_{1/2}(\mathbb{R}) \end{cases} \quad (5)$$

where $\mathbb{R}_- =]-\infty, 0[$.

2.1.2. Second step: $u = 0$ in $\mathbb{R}^2_-$: Since the support of $\sigma^\beta(v)\tilde{v}|_{z=0}$ is compact, the functions $\beta_1 \to c_{1,3}(\beta_1)$ and $\beta_1 \to c_2(\beta_1)$ are analytic functions on $\mathbb{R}$. In fact, the function $\beta_1 \to \mathscr{F}_x(\sigma^\beta(u)\,\tilde{v}|_{z=0})$ is analytic on $\mathbb{C}$ but the elements of the matrix M cannot be extended to the whole plane $\mathbb{C}$ as an analytic function. Choosing as determination of $\sqrt{z}$ the one given by $Re\sqrt{z} > 0$ if $z \in \mathbb{C}\backslash]-\infty, 0]$, we can conclude that these elements of M can be extended as analytic functions in the strip $S_\beta = \{z \in \mathbb{C} / -\beta < \operatorname{Im} z < \beta\}$. This implies that the functions $\beta_1 \to c_i(\beta_1)$, $i = 1,2,3$, can be extended to analytic functions on S_β. We shall return to this point later.

Now, the function $\bar{u}_2(\beta_1, \cdot)$ verifies the following problem:

$$(\mathrm{SH}) \begin{cases} -c_\mathrm{s}^2 \left(\frac{\mathrm{d}^2 \bar{u}_2}{\mathrm{d}z^2} - |\xi|^2 \bar{u}_2 \right) = \omega^2 \bar{u}_2 \ \text{on}]-\infty, 0[\\ \mu \frac{\mathrm{d}\bar{u}_2}{\mathrm{d}z}|_{z=0} = c_2(\beta_1) \end{cases}$$

As $\omega^2 > \beta^2 c_\mathrm{s}^2$, we see that $\omega^2 > (\beta^2 + \beta_1^2) c_\mathrm{s}^2$, $\forall \beta_1^2 \in [0, \omega^2/c_\mathrm{s}^2 - \beta^2[$ and, in this case, the unique solution of $-c_\mathrm{s}^2 \mathrm{d}^2 \bar{u}_2/\mathrm{d}z^2 = (\omega^2 - |\xi|^2 c_\mathrm{s}^2)\bar{u}_2$ in $L^2(\mathbb{R}_-)$ is $\bar{u}_2 = 0$. Then we obtain $c_2(\beta_1) = \mu \mathrm{d}\bar{u}_2/\mathrm{d}z|_{z=0} = 0$ for almost every $\beta_1^2 \in [0, \omega^2/c_\mathrm{s}^2 - \beta^2[$. The function $\beta_1 \to c_2(\beta_1)$ being analytic on $\mathbb{R}$, we deduce therefore that $c_2(\beta_1) = 0$, $\forall \beta_1 \in \mathbb{R}$. Thus $\bar{u}_2(\beta_1)$ satisfies the following assertions:

$$\begin{cases} \bar{u}_2 \in H^2(\mathbb{R}_-) \\ -c_\mathrm{s}^2 (\frac{\mathrm{d}^2 \bar{u}_2}{\mathrm{d}z^2} - |\xi|^2 \bar{u}_2) = \omega^2 \bar{u}_2 \ \text{on} \ ]-\infty, 0[\\ \mu \frac{\mathrm{d}\bar{u}_2}{\mathrm{d}z}|_{z=0} = 0 \end{cases}$$

As this problem has only the trivial solution, we see that $\bar{u}_2 = 0$ in $L^2(\mathbb{R}_-)$ for almost every $\beta_1 \in \mathbb{R}$.

The vector function $(\bar{u}_1, \bar{u}_3)$ verifies the following relations:

$$(\mathrm{P-SV}) \begin{cases} -\mu \frac{\mathrm{d}^2 \bar{u}_1}{\mathrm{d}z^2} + |\xi|(\lambda + \mu) \frac{\mathrm{d}\bar{u}_3}{\mathrm{d}z} + |\xi|^2 (\lambda + 2\mu) \bar{u}_1 = \rho \omega^2 \bar{u}_1 \\ -(\lambda + 2\mu) \frac{\mathrm{d}^2 \bar{u}_3}{\mathrm{d}z^2} - |\xi|(\lambda + \mu) \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}z} + |\xi|^2 \mu \bar{u}_3 = \rho \omega^2 \bar{u}_3 \\ (\mu \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}z} - \mu |\xi| \bar{u}_3)|_{z=0} = c_1(\beta_1) \\ ((\lambda + 2\mu) \frac{\mathrm{d}\bar{u}_3}{\mathrm{d}z} + |\xi| \lambda \bar{u}_1)|_{z=0} = c_3(\beta_1) \end{cases}$$

In the following, we are going to prove that $c_1(\beta_1) = c_3(\beta_1) = 0$ for every $\beta_1 \in \mathbb{R}$.

Deriving the first equation of this system with respect to z, multiplying the second by $|\xi|$ and summing up the resulting equations, we obtain the following identity:
$-c_\mathrm{s}^2 (\frac{\mathrm{d}^2 V}{\mathrm{d}z^2} - |\xi|^2 V) = \omega^2 V$ in $]-\infty, 0[$, where $V = \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}z} + |\xi| \bar{u}_3$.

As in the case of $\bar{u}_2$, we verify that $V(\beta_1, z) = 0$ for almost every $\beta_1^2 \in [0, \omega^2/c_s^2 - \beta^2[$, i.e.

$$\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}z} = -|\xi|\bar{u}_3 \text{ for almost every } \beta_1^2 \in \left[0, \frac{\omega^2}{c_s^2} - \beta^2\right[\tag{6}$$

Substituting the latter identity in the first equation of this system, we conclude that $\bar{u}_1(\beta_1, z)$ verifies the following equation:

$$-c_p^2 \frac{\mathrm{d}^2\bar{u}_1}{\mathrm{d}z^2} = (\omega^2 - |\xi|^2 c_p^2)\bar{u}_1 \tag{7}$$

Now, since $\omega^2 > \beta^2 c_s^2$, we distinguish two cases:

Case 1: $\omega^2 > \beta^2 c_p^2$. We have $\omega^2 > |\xi|^2 c_p^2$ when $\beta_1^2 \in [0, (\omega^2/c_p^2) - \beta^2[$. From (7), we conclude that $\bar{u}_1(\beta_1, z) = 0$ for almost every $\beta_1^2 \in [0, \omega^2/c_p^2 - \beta^2[$ and by the relation $\mathrm{d}\bar{u}_1/\mathrm{d}z = -|\xi|\bar{u}_3$ that $\bar{u}_3(\beta_1, \beta, z) = 0$ for almost every $\beta_1^2 \in [0, \omega^2/c_p^2 - \beta^2[$. Hence, we deduce that $c_1(\beta_1) = c_3(\beta_1) = 0$ for almost every $\beta_1^2 \in [0, \frac{\omega^2}{c_p^2} - \beta^2[$. Now, the analyticity on $\mathbb{R}$ of $c_{1,3}(\beta_1) = (c_1(\beta_1), c_3(\beta_1))$ yields $c_{1,3}(\beta_1) = 0$ for all β_1 in $\mathbb{R}$.

Case 2: $\beta^2 c_s^2 < \omega^2 \leqslant \beta^2 c_p^2$. As we have $\omega^2 < |\xi|^2 c_p^2$ for all β_1 in $\mathbb{R}\backslash\{0\}$, then the unique solution belonging to $L^2(\mathbb{R}_-)$ of (7) has the form $\bar{u}_1(\beta_1, z) = r(\beta_1)\mathrm{e}^{\alpha_p z}$ where $\alpha_p = \sqrt{|\xi|^2 - \frac{\omega^2}{c_p^2}}$.

Let $\beta_1^2 \in]0, \omega^2/c_s^2 - \beta^2[$, then $\omega^2 > |\xi|^2 c_s^2$ and we have:

$$\bar{u}_3(\beta_1, z) = -\frac{1}{|\xi|}\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}z} = -\frac{\alpha_p}{|\xi|} r(\beta_1)\mathrm{e}^{\alpha_p z} \tag{8}$$

Furthermore, we set

$$\tilde{u} := \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} := \mathscr{R}^{-1}\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = \begin{pmatrix} \frac{\beta_1}{|\xi|} & \frac{-\beta}{|\xi|} & 0 \\ \frac{\beta}{|\xi|} & \frac{\beta_1}{|\xi|} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} \tag{9}$$

and

$$\begin{pmatrix} \bar{c}_1(\beta_1) \\ \bar{c}_2(\beta_1) \\ \bar{c}_3(\beta_1) \end{pmatrix} := \mathscr{L}(\mathscr{F}_x(\sigma^\beta(u)\tilde{v}|_{z=0}))$$

Hence, in contrast to the functions $\beta_1 \to c_i(\beta_1)$, $i = 1, 2, 3$, the functions $\beta_1 \to \bar{c}_i(\beta_1)$ can be extended as analytic functions on the whole complex plane $\mathbb{C}$ and

$$\begin{pmatrix} c_1(\beta_1) \\ c_2(\beta_1) \\ c_3(\beta_1) \end{pmatrix} := \mathscr{R}\begin{pmatrix} \bar{c}_1(\beta_1) \\ \bar{c}_2(\beta_1) \\ \bar{c}_3(\beta_1) \end{pmatrix}$$

Since $\bar{u}_2 = 0$, then from (7) and (8) we obtain:

$$\begin{cases} \tilde{u}_1(\beta_1, z) = \frac{\beta_1}{|\xi|}\bar{u}_1 = \frac{\beta_1 r(\beta_1)}{|\xi|}\,\mathrm{e}^{\alpha_\mathrm{p} z} \\ \tilde{u}_2(\beta_1, z) = \frac{\beta}{|\xi|}\bar{u}_1 = \frac{\beta r(\beta_1)}{|\xi|}\,\mathrm{e}^{\alpha_\mathrm{p} z} \\ \tilde{u}_3(\beta_1, z) = \bar{u}_3 = -\frac{\alpha_\mathrm{p} r(\beta_1)}{|\xi|}\,\mathrm{e}^{\alpha_\mathrm{p} z} \end{cases} \tag{10}$$

On one hand, we deduce from $c_1(\beta_1) = (\mu \mathrm{d}\bar{u}_1/\mathrm{d}z - \mu|\xi|\bar{u}_3)|_{z=0} = 2\mu\alpha_\mathrm{p} r(\beta_1)$, that

$$r(\beta_1) = \frac{c_1(\beta_1)}{2\mu\alpha_\mathrm{p}} \tag{11}$$

and we can write

$$\tilde{u}_1(\beta_1, 0) = \frac{\beta_1 r(\beta_1)}{|\xi|} = \frac{\beta_1 c_1(\beta_1)}{2\mu\alpha_\mathrm{p}|\xi|} = \frac{\beta_1^2 \bar{c}_1 + \beta\beta_1\bar{c}_2}{2\mu\alpha_\mathrm{p}|\xi|^2} \tag{12}$$

On the other hand, we have (cf. (8))

$$c_3(\beta_1) = \left((\lambda + 2\mu)\frac{\mathrm{d}\bar{u}_3}{\mathrm{d}z} + |\xi|\lambda\bar{u}_1 \right)\Big|_{z=0} = \left(-\frac{(\lambda+2\mu)\alpha_\mathrm{p}^2}{|\xi|} + \lambda|\xi| \right) r(\beta_1)$$

hence

$$r(\beta_1) = \frac{c_3(\beta_1)}{[-(\lambda+2\mu)\alpha_\mathrm{p}^2]/|\xi| + \lambda|\xi|} \tag{13}$$

From the first equation of (10), we deduce that

$$\tilde{u}_1(\beta_1, 0) = \frac{\beta_1 r(\beta_1)}{|\xi|} = \frac{\beta_1 c_3(\beta_1)}{-(\lambda+2\mu)\alpha_\mathrm{p}^2 + \lambda|\xi|^2}$$

and, since $c_3(\beta_1) = \bar{c}_3(\beta_1)$, we get

$$\tilde{u}_1(\beta_1, 0) = \frac{\beta_1 \bar{c}_3(\beta_1)}{-(\lambda+2\mu)\alpha_\mathrm{p}^2 + \lambda|\xi|^2} \tag{14}$$

Suppose now that $\bar{c}_3(\beta_1)$ is not identically zero, then from (12) and (14) it follows that:

$$\frac{\beta_1\bar{c}_1(\beta_1) + \beta\bar{c}_2(\beta_1)}{\bar{c}_3(\beta_1)} = \frac{2\mu\alpha_\mathrm{p}|\xi|^2}{-(\lambda+2\mu)\alpha_\mathrm{p}^2 + \lambda|\xi|^2} \tag{15}$$

The left member of the equality (15) is meromorphic on $\mathbb{C}$ but the right one has branching points at $\pm\mathrm{i}\sqrt{\beta^2 - \omega^2/c_\mathrm{p}^2}$ since $\alpha_\mathrm{p} = \sqrt{\beta_1^2 + \beta^2 - \omega^2/c_\mathrm{p}^2}$ (Figure 3). As this is impossible, $\bar{c}_3(\beta_1) = 0$ on $\mathbb{C}$ and then $c_3(\beta_1) = 0$ since $c_3(\beta_1) = \bar{c}_3(\beta_1)$. From (11) and (13), we deduce that $c_1(\beta_1) = 0$. From (8), the function $\bar{u}_3(\beta_1, z)$ vanishes if $\beta_1^2 \in]0, \omega^2/c_\mathrm{s}^2 - \beta^2[$. Since the function $\beta^1 \to \bar{u}_3(\beta_1, z)$ is not necessarily analytic on $\mathbb{R}$, we cannot conclude that $\bar{u}_3(\beta_1, z)$ is a vanishing function.

But in all cases, we have proved that $(\bar{u}_1, \bar{u}_3)$ is an eigenfunction associated to ω^2 for the operator $(A_{\mathrm{P-SV}}, D(A_{\mathrm{P-SV}}))$, where

$$D(A_{\mathrm{P-SV}}) = \{u = (u_1, u_3) \in (H^1(R_-))^2 / A_{\mathrm{P-SV}}(u) \in (L^2(R_-))^2 \text{ and } \sigma_{1,3}^{\beta_1,\beta}(u) = 0\}$$

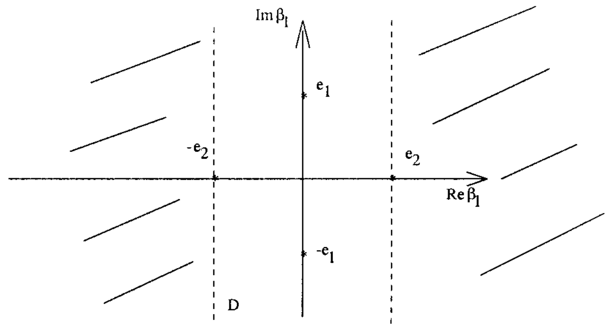

Figure 3. The branching points $\pm e_1$ and the domaine of definition of the equation (15) i.e. $D=\{\beta_1\in\mathbb{C}/(Re\beta_1)^2<\omega^2/c_s^2-\beta^2\}$, where $e_1=\sqrt{\beta^2-\omega^2/c_p^2}$ and $e_2=\sqrt{\omega^2/c_s^2-\beta^2}$.

This operator is self-adjoint, positive and having $[|\xi|^2c_s^2,+\infty[$ as essential spectrum. Its point spectrum is reduced to $|\xi|^2c_R^2$ (see Reference [15]). Then if $\beta_1^2\neq\omega^2/c_R^2-\beta^2$, we get $\omega^2\neq|\xi|^2c_R^2$ and $\bar{u}_1(\beta_1,z)=\bar{u}_3(\beta_1,z)=0$. This implies that $\bar{u}_1(\beta_1,z)=\bar{u}_3(\beta_1,z)=0$ in $L^2(\mathbb{R}^2_-)$ with respect to β_1 and z. As we proved $\bar{u}_2(\beta_1,z)=0$ in $L^2(\mathbb{R}^2_-)$, we deduce that $\bar{u}(\beta_1,z)=0$ in $L^2(\mathbb{R}^2_-)$ and we conclude $u=0$ since $u=\mathscr{F}_x^{-1}\mathscr{L}^{-1}\mathscr{R}^{-1}\bar{u}$.

2.2. *Part b: the general case*

In this part, we consider the general situation of the theorem.

Now let $\omega^2>\beta^2c_p^2$ and $v\in\mathscr{D}(A_\beta)$ such that $A_\beta v=\omega^2 v$. Let $\phi\in C^\infty(\mathbb{R}^2)$ such that $\phi(x,z)=0$ if $x^2+z^2<R$ and $\phi(x,z)=1$ if $x^2+z^2>R+1$. We denote by

$$u(x,z)=\begin{cases}\phi v(x,z) & \text{if } (x,z)\in\mathbb{R}^2_-\cap\Omega\\ 0 & \text{if } (x,z)\in\mathbb{R}^2_-\backslash\Omega\end{cases}$$

Hence, the function u verifies $A_\beta u-\omega^2u=H(x,z)$ a.e. in $\mathbb{R}^2_-$, where H is in $(H^1(\mathbb{R}^2_-))^3$ and satisfy $H(x,z)=0$ if $x^2+z^2>R+1$. Since $u|_{\Omega_R}=0$ we can replace the coefficients λ, μ and ρ by λ_∞, μ_∞ and ρ_∞ in Ω_R.

We denote by $\bar{u}(\beta_1,z)=(\bar{u}_1,\bar{u}_2,\bar{u}_3)=\mathscr{R}\mathscr{L}\mathscr{F}_x(u)$. As the coefficients are constant in $\mathbb{R}^2_-$, the function u is in $(H^2(\mathbb{R}^2_-))^3$ and the lemma can be applied. Similar to the case (a) and using the same notations we verify that the functions $(\bar{u}_1,\bar{u}_3)$ and $(\bar{u}_2)$ satisfy, respectively, the following problems:

$$\begin{cases}\bar{u}_{1,3}\in(H^2(\mathbb{R}_-,L^2(\mathbb{R})))^2\\ A_{\mathrm{P-SV}}(\bar{u}_{1,3})=\omega^2\bar{u}_{1,3}+h_{1,3}\ \text{in}\ (L_2(\mathbb{R}_-))^2\ \text{for almost every}\ \beta_1\in\mathbb{R}\\ \sigma_{1,3}^{\beta_1,\beta}(\bar{u}_{1,3})|_{z=0}=c_{1,3}(\beta_1)\ \text{in}\ (L^2_{1/2}(\mathbb{R}))^2\end{cases}\qquad(16)$$

and

$$\begin{cases}\bar{u}_2 \in (H^2(\mathbb{R}_-, L^2(\mathbb{R})) \\ A_{\mathrm{SH}}(\bar{u}_2) = \omega^2 \bar{u}_2 + h_2 \text{ in } L_2(\mathbb{R}_-) \text{ for almost every } \beta_1 \in \mathbb{R} \\ \sigma_2^{\beta_1,\beta}(\bar{u}_2)|_{z=0} = c_2(\beta_1) \text{ in } L^2_{1/2}(\mathbb{R})\end{cases} \tag{17}$$

where

$$c(\beta_1) = (c_1(\beta_1), c_2(\beta_1), c_3(\beta_1)) = \mathscr{R}\mathscr{L}\mathscr{F}_x(\sigma^\beta(u)\tilde{v}|_{z=0})$$

$$c_{1,3}(\beta_1) = (c_1(\beta_1), c_3(\beta_1))$$

$$h(\beta_1, z) = (h_1(\beta_1, z), h_2(\beta_1, z), h_3(\beta_1, z)) = \mathscr{R}\mathscr{L}\mathscr{F}_x H(x, z)$$

and

$$h_{1,3}(\beta_1, z) = (h_1(\beta_1, z), h_3(\beta_1, z))$$

Since $H(x,z) \in (H^1(\mathbb{R}^2_-))^3$, for almost every $\beta_1 \in \mathbb{R}$, the function $z \to h(\beta_1, z)$ is in $(H^1(\mathbb{R}_-))^3$ and hence in $(C(\overline{\mathbb{R}_-}))^3$. The functions H and $\sigma^\beta(u)\tilde{v}|_{z=0}$ having compact supports, we deduce that the functions $\beta_1 \to c(\beta_1)$ and $\beta_1 \to h(\beta_1, z)$ are analytic with respect to β_1 on $\mathbb{R}$. Thanks to the fact that $H(x,z) = 0$ if $x^2 + z^2 > R + 1$ we have $h(\beta_1, z) = 0$ if $z < -(R+1)$.

We introduce the following two Cauchy problems:

$$\begin{cases}A_{\mathrm{P-SV}}(\psi_{1,3}) = \omega^2 \psi_{1,3} + h_{1,3} \text{ in }]-(R+1), 0[\\ \sigma_{1,3}^{\beta_1,\beta}(\psi_{1,3})(z) = \psi_{1,3}(z) = 0 \text{ if } z = -(R+1)\end{cases} \tag{18}$$

and

$$\begin{cases}A_{\mathrm{SH}}(\psi_2) = \omega^2 \psi_2 + h_2 \text{ in }]-(R+1), 0[\\ \sigma_2^{\beta_1,\beta}(\psi_2)(z) = \psi_2(z) = 0 \text{ if } z = -(R+1)\end{cases} \tag{19}$$

where $\psi_{1,3} = (\psi_1, \psi_3)$.

We extend these two solutions to $]-\infty, -(R+1)[$ by zero. Then the resulting functions (denoted by $\psi_{1,3}, \psi_2$) satisfy, respectively, the first equations of (18) and (19) on $\mathbb{R}_-$.

We claim that each of these two problems has a unique solution and it is analytic with respect to β_1. To see this, for the first problem for example, we write (18) like:

$$\begin{cases}-\frac{\mathrm{d}}{\mathrm{d}z}(A_1 \frac{\mathrm{d}\psi_{1,3}}{\mathrm{d}z} + A_2 \psi_{1,3}) + A_3 \frac{\mathrm{d}}{\mathrm{d}z}(\psi_{1,3}) + A_4 \psi_{1,3} = \rho_\infty \omega^2 \psi_{1,3} + \rho_\infty h_{1,3} \text{ in } \mathbb{R}_- \\ (A_1 \frac{\mathrm{d}\psi_{1,3}}{\mathrm{d}z} + A_2 \psi_{1,3})(z) = \psi_{1,3}(z) = 0 \text{ if } z = -(R+1)\end{cases} \tag{20}$$

where

$$A_1 = \begin{pmatrix}\mu & 0 \\ 0 & \lambda + 2\mu\end{pmatrix}, \quad A_2 = A_3^t = \begin{pmatrix}0 & -|\xi|\mu \\ |\xi|\lambda & 0\end{pmatrix}, \quad A_4 = \begin{pmatrix}|\xi|^2(\lambda + 2\mu) & 0 \\ 0 & |\xi|^2 \mu\end{pmatrix}$$

and $\sigma_{1,3}^{\beta_1,\beta}(\psi_{1,3})(z) = A_1 \mathrm{d}\psi_{1,3}/\mathrm{d}z + A_2\psi_{1,3}$. Hence, the vector

$$Z = \left(\psi_{1,3}, A_1 \frac{\mathrm{d}\psi_{1,3}}{\mathrm{d}z} + A_2\psi_{1,3}\right)$$

satisfy the following Cauchy problem of order one:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}z}(Z) = AZ + \rho_\infty B h_{1,3} \text{ in } \mathbb{R}_- \\ Z(z) = 0 \text{ if } z = -(R+1) \end{cases} \tag{21}$$

where

$$A = \begin{pmatrix} -A_1^{-1}A_2 & A_1^{-1} \\ -A_3A_1^{-1}A_2 + A_4 - \rho_\infty\omega^2 & A_3A_1^{-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Problem (21) has a unique solution, hence also problem (18). Since the coefficients are constant and the function $z \to h_{1,3}(\beta_1, z)$ is continuous, this solution belongs to $C^1(\mathbb{R}_-)$. Furthermore, vanishing for $z \leqslant -(L+1)$, it is square integrable. The coefficients of the 4×4 matrix A are analytic with respect to β_1 on $\mathbb{R}$, hence the solution $Z(\beta_1, z)$ is also analytic with respect to β_1 for all z in $]-\infty, 0]$ (see for example Reference [16]). Hence, $Z(\beta_1, 0)$ is analytic.

Then, for two problems (18) and (19), the functions $\beta_1 \to \sigma_2^{\beta_1,\beta}(\psi_2)$ and $\beta_1 \to \sigma_{1,3}^{\beta_1,\beta}(\psi_{1,3})$ are analytic for all z in $]-\infty, 0]$.

Denote now by $E_2 = \psi_2 - \bar{u}_2$ and $E_{1,3} = \psi_{1,3} - \bar{u}_{1,3}$. These functions verify the following relations:

$$\begin{cases} A_{\mathrm{P-SV}}(E_{1,3}) = \omega^2 E_{1,3} \text{ in } \mathbb{R}_- \\ \sigma_{1,3}^{\beta_1,\beta}(E_{1,3})(0) = \sigma_{1,3}^{\beta_1,\beta}(\psi_{1,3})(0) - c_{1,3}(\beta_1) \end{cases} \tag{22}$$

and

$$\begin{cases} A_{\mathrm{SH}}(E_2) = \omega^2 E_2 \text{ in } \mathbb{R}_- \\ \sigma_2^{\beta_1,\beta}(E_2)(0) = \sigma_2^{\beta_1}(\psi_2)(0) - c_2(\beta_1) \end{cases} \tag{23}$$

Arguing as in the first and the second case of Section 2.1.2, we deduce from (22) and (23) that $E_1(\beta_1, z) = E_2(\beta_1, z) = E_3(\beta_1, z) = 0$ in $\mathbb{R}_-$. This can de done by replacing $c_2(\beta_1)$ and $c_{1,3}(\beta_1)$ by $\sigma_2^{\beta_1}(\psi_2)(0) - c_2(\beta_1)$ and $\sigma_{1,3}^{\beta_1,\beta}(\psi_{1,3})(0) - c_{1,3}(\beta_1)$, respectively, in problems (P – SV) and (SH).

In particular, we obtain $\sigma_{1,3}^{\beta_1,\beta}(E_{1,3})(0) = 0$ and $\sigma_2^{\beta_1,\beta}(E_2)(0) = 0$. Since the functions $\beta_1 \to \sigma_2^{\beta_1,\beta}(E_2)(0)$ and $\beta_1 \to \sigma_{1,3}^{\beta_1,\beta}(E_{1,3})(0)$ are analytic, we deduce that $\sigma_2^{\beta_1,\beta}(E_2)(0) = 0$ and $\sigma_{1,3}^{\beta_1,\beta}(E_{1,3})(0) = 0$ for all $\beta_1 \in \mathbb{R}$.

This implies that E_2 and $E_{1,3}$ verify the following eigenvalue problems:

$$\begin{cases} A_{\mathrm{P-SV}}(E_{1,3}) = \omega^2 E_{1,3} \text{ in } \mathbb{R}_- \\ \sigma_{1,3}^{\beta_1,\beta}(E_{1,3})(0) = 0 \end{cases} \tag{24}$$

$$\begin{cases} A_{\mathrm{SH}}(E_2) = \omega^2 E_2 \text{ in } \mathbb{R}_- \\ \sigma_2^{\beta_1,\beta}(E_2)(0) = 0 \end{cases} \tag{25}$$

The problem given by (24) has only one solution: the Rayleigh wave (cf. Reference [15]), and the one given by (25) gives $E_2 = 0$. This implies that $E_2 = 0$ and $E_{1,3} = 0$. Since $\psi_1(\beta_1, z) = \psi_2(\beta_1, z) = \psi_3(\beta_1, z) = 0$ if $z < -(R+1)$, we conclude that $\bar{u}_1(\beta_1, z) = \bar{u}_2(\beta_1, z) = \bar{u}_3(\beta_1, z) = 0$ if $z < -(R+1)$. Finally, we deduce that $v(x,z) = 0$ if $z < -(R+1)$.

2.3. *Third step: $v = 0$ in Ω*

In Sections 2.1 and 2.2, we have proved that v vanishes in a part of Ω. In this section, we prove that v vanishes in the whole domaine Ω.

We give the result for two situations. In the first, the coefficients are constant everywhere in Ω. The existence of embedded eigenvalues was a question of Reference [1]. In the second one, where the coefficients are not constant, we refer to References [10–12] to use a unique continuation principle given for elasticity system. This principle requires the C^2 regularity of the coefficients. We will see that the $W^{2,\infty}$ regularity is sufficient.

2.3.1. First situation: Although our system is elliptic with constant coefficients, we cannot conclude directly that v vanishes in all Ω, because, in general, the solution of an elliptic system with analytic coefficients and data is not necessarily analytically regular [17, 18]. In our case, we can proceed by adopting the following change of variable: $F = \mathrm{div}^\beta(v)$.

Since the coefficients are constant, expression (1) becomes:

$$A_\beta v = \frac{1}{\rho} \begin{cases} -\mu\Delta v_1 + \mu\beta^2 v_1 - (\lambda+\mu)[\frac{\partial^2 v_1}{\partial x^2} + \beta\frac{\partial v_2}{\partial x} + \frac{\partial^2 v_3}{\partial x \partial z}] \\ -\mu\Delta v_2 + (\lambda+2\mu)\beta^2 v_2 + \beta(\lambda+\mu)[\frac{\partial v_1}{\partial x} + \frac{\partial v_3}{\partial z}] \\ \mu\Delta v_3 + \mu\beta^2 v_3 - (\lambda+\mu)[\frac{\partial^2 v_1}{\partial x \partial z} + \beta\frac{\partial v_2}{\partial z} + \frac{\partial^2 v_3}{\partial z^2}] \end{cases} \tag{26}$$

Since v satisfies the equation $A_\beta v = \omega^2 v$, we deduce that

$$(\lambda+2\mu)(\Delta F - \beta^2 F) = -\rho\omega^2 F \quad \text{on } \Omega$$

Now, the function F is analytic in Ω. Hence, since $v(x,z) = 0$ in $\mathbb{R}^2_-$ for part (a) and $v(x,z) = 0$ if $z \leqslant -(R+1)$ for part (b), we obtain the same result for $F(x,z)$ and therefore $F(x,z) = 0$ in Ω. From (26), we deduce that

$$-\mu\Delta v_i + \mu\beta^2 v_i = \rho\omega^2 v_i, \quad i = 1,2,3, \text{ in } \Omega$$

This implies that the functions v_i, $i = 1,2,3$, are analytic in Ω. We conclude that $v = 0$ in Ω.

2.3.2. Second situation: In this section, we consider the situation where the coefficients are not constant.

In Section 2.1 (part (a)), we have seen that $v|_{\mathbb{R}^2_-}=0$, but in Section 2.2 (part (b)), we have $v|_{\{(x,z)\in\Omega/z<-(R+1)\}}=0$. Since in $\Omega\backslash\Omega_{\rm R}$, the coefficients are constant then, as in Section 2.3.1, we conclude that $v|_{\Omega\backslash\Omega_{\rm R}}=0$ for part (b).

Now for the two cases (a) and (b), we are going to prove that $v(x,z)=0$ in Ω.

We denote by $\mathscr{A}_\beta$ the differential expression of the operator $(A_\beta,\mathscr{D}(A_\beta))$. Let $\Omega_{\rm R}^{i_0}$ a part of $\Omega_{\rm R}$ whose boundary $\partial\Omega_{\rm R}^{i_0}$ has a common part Γ^{i_0} with $\Omega\backslash\Omega_{\rm R}$. We extend the coefficients λ,μ,ρ to $\Omega\backslash\Omega_{\rm R}$ such that the new coefficients become $W^{2,\infty}(B_{i_0})$ where $B_{i_0}=\Omega_{\rm R}^{i_0}\cup(\Omega\backslash\Omega_{\rm R})\cup\Gamma^{i_0}$.

Since $v|_{\Omega\backslash\Omega_{\rm R}}=0$ then v verifies the following properties:

$$\mathscr{A}_\beta v=\omega^2 v \text{ in the distribution sense in } B_{i_0} \text{ and } v_{\Omega\backslash\Omega_{\rm R}}=0$$

Noting by $v_4=\operatorname{div}^\beta(v)$, we see $v_4\in H^1(\Omega)$ since $v\in H^2(\Omega)$. Arguing as in References [11, Theorem 11.6, p. 218;12] we deduce from $\mathscr{A}_\beta v=\omega^2 v$ in B_{i_0} that

$$\Delta v_i=L_i(v_1,v_2,v_3,v_4),\ \ i=1,2,3,4,\ \text{ in the distribution sense in } B_{i_0} \tag{27}$$

where L_i is a differential expression of order one with coefficients in $L^\infty(B_{i_0})$. This system, which is weakly coupled, has the unique continuation property [10, Lemma 2]: if $(v_1,v_2,v_3,v_4)\in(H^1_{\rm loc}(B_{i_0}))^4$ satisfies (27) and vanishes in a neighbourhood of a point, it vanishes everywhere. As $v|_{\Omega\backslash\Omega_{\rm R}}=0$, we conclude that $v=0$ in B_{i_0}, hence in $\Omega_{\rm R}^{i_0}$.

Now, we take the part of $\Omega_{\rm R}$, which we denote by $\Omega_0^{i_1}$, whose boundary has a common part, Γ^{i_1}, with $\partial\Omega_{\rm R}^{i_0}$. Let $B_{i_1}=\Omega_{\rm R}^{i_0}\cup\Omega_{\rm R}^{i_1}\cup\Gamma^{i_1}$. The coefficients in $\Omega_{\rm R}^{i_1}$ are $W^{2,\infty}(\Omega_{\rm R}^{i_1})$, we extend them to $\Omega_{\rm R}^{i_0}$ such that the new coefficients belong to $W^{2,\infty}(B_{i_1})$. Since $v_{|\Omega_{\rm R}^{i_0}}=0$, then v verifies: $\mathscr{A}_\beta v=\omega^2 v$ in the distribution sense in B_{i_1} and $v_{|\Omega_{\rm R}^{i_0}}=0$. As for B_{i_0}, we deduce that $v=0$ in B_{i_1} hence in $\Omega_{\rm R}^{i_1}$.

Making the same thing to the other parts $\Omega_{\rm R}^{i}$, $i\in I$ of $\Omega_{\rm R}$, we deduce that $v=0$ in $\Omega_{\rm R}$. Finally, $v=0$ in Ω.

Remarks

(1) For the density ρ, the $W^{1,\infty}(\Omega_{\rm R})$ regularity is sufficient to give a unique continuation principle.

(2) If $\omega^2\leqslant\beta^2c_{\rm s}^2$, we cannot do the same developments as above. In this case, we have $\omega^2<|\xi|^2c_{\rm s}^2$ for all $\beta_1\in\mathbb{R}$, hence the solutions belonging to $L^2(\mathbb{R}_-)$ of the problem (P$-$SV) are sums of S- and P-waves, in other words there is coupling. When $\omega^2>\beta^2c_{\rm s}^2$, we have seen that, for $\beta_1^2\in]0,\omega^2/c_{\rm s}^2-\beta^2[$, the solutions of the problem (P $-$ SV) are only P-waves, which is crucial. Precisely, we prove that $V=0$ and deduce (6) and (7).

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