The No Response Test for the Reconstruction of Polyhedral Objects in Electromagnetics

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Abstract

We develop a *No Response Test* for the reconstruction of a polyhedral obstacle from two or few time-harmonic electromagnetic incident waves in electromagnetics. The basic idea of the test is to probe some region in space with waves which are small on some test domain and, thus, do not generate a response when the scatterer is inside of this test domain. We will prove that the No Response Test checks analytic continuability of a time-harmonic field from the far field pattern into the domain $B_e := \mathbb{R}^3 \setminus \overline{B}$ for a non-vibrating test domain B.

We show that two incident waves, defined by one incident direction and two appropriately chosen directions of polarization, are enough to recover the convex hull of polyhedrals. Based on this uniqueness result, we build up the No Response Test and we prove convergence in the sense that it fully reconstructs a convex polyhedral scatterer D or the convex hull of an arbitrary polyhedral scatterer.

Further, we will describe the algorithmical realization of the No Response Test and show the feasibility of the method by reconstruction of convex polyhedral objects in three dimensions. This is the first formulation of the No Response Test for electromagnetics.

Key words: Electromagnetic Waves, Maxwell Equations, Inverse Scattering, Object Reconstruction, Sampling Method, No Response Test

1 Introduction

Using electromagnetic waves for probing and investigation of unknown regions in space is widely employed in the natural sciences, ranging from optics and microscopy via X-Ray science to radar and electromagnetic tomography. An introduction into the mathematical theory of inverse problems for acoustic and electromagnetic waves can be found in (Colton and Kress, 1998). A survey about several more recent methods is given by (Potthast, 2006) and a comparative study of some of these methods can be found in (Nakamura et al., 2006) and (Honda et al., 2008).

Our goal here is to formulate and analyse the *No Response Test* first suggested in acoustics by (Luke et al., 2003) for object identification in electromagnetics. In particular, we provide a convergence analysis for the reconstruction of the convex hull of polyhedral perfectly conducting object in three dimensions from the far field pattern of two incident time-harmonic electromagnetic waves.

Let D be a polyhedral domain in $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. We consider the following electromagnetic scattering problem. The propagation of time-harmonic electromagnetic fields in a homogeneous media is governed by the *Maxwell equations*

$$\text{curl } E - i\kappa H = 0, \tag{1}$$
$$\text{curl } H + i\kappa E = 0, \tag{2}$$

in $\mathbb{R}^3 \setminus \overline{D}$ where κ is the real positive *wave number*. At the boundary of the scatterers the total field E satisfies the *perfect conductor boundary condition*

$$\nu \times E = 0 \text{ on } \partial D, \tag{3}$$

where ν is the unit normal vector to ∂D oriented to the exterior of D. We look for solutions of the form $E = E^i + E^s$, $H = H^i + H^s$ of (1), (2) and (3) where the *scattered field* (E^s, H^s) is assumed to satisfy the Silver-Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0, \tag{4}$$

$r = |x|$ and the limit is uniform with respect to all the directions $\theta := \frac{x}{|x|}$, while the incident field (E^i, H^i) is given by

$$\begin{aligned} E^i(x, d, p) &= \frac{i}{\kappa}\text{curl curl } pe^{i\kappa x\cdot d} = i\kappa(d \times p) \times de^{i\kappa x\cdot d}, \\ H^i(x, d, p) &= \text{curl } pe^{i\kappa x\cdot d} = i\kappa d \times pe^{i\kappa x\cdot d}, \end{aligned} \tag{5}$$

where $d \in \mathbb{R}^3$ is the direction of incidence and $p \in \mathbb{R}^3$ is the polarization.

It is proven, for instance, by Cakoni, Colton and Monk (Cakoni et al., 2004) that a solution to this problem exists and it is unique. In addition, from the classical theory as presented for example in (Colton and Kress, 1998), the scattered field satisfies the following asymptotic property,

$$
\begin{aligned}
E^s(x,d,p) &= \frac{e^{i\kappa r}}{r}(E^\infty(\theta,d,p)+O(r^{-1})), \quad r\to\infty,\\
H^s(x,d,p) &= \frac{e^{i\kappa r}}{r}(H^\infty(\theta,d,p)+O(r^{-1})), \quad r\to\infty,
\end{aligned}
\tag{6}
$$

where $(E^\infty(\cdot,d,p),H^\infty(\cdot,d,p))$ defined on the unit sphere $\mathbb{S}^2$ is called the far field pattern associated to the incident field $(E^i(\cdot,d,p),H^i(\cdot,d,p))$.

We will study and solve the following shape reconstruction problem for polyhedral domains.

Definition 1.1 (Shape reconstruction problem) *Given $E^\infty(\cdot,d,p)$ on $\mathbb{S}^2$ with N directions, $N\geq 1$ of incidence $d_i, i=1,...,N$ and two linearely independent polarizations $p_{ij}, j=1,2$, for each d_i, for the scattering problem (1) - (4) reconstruct the obstacle D.*

In this paper, we show that two incident waves defined by one ($N=1$) direction of incidence d coupled with two linearly independent directions of polarizations $p_j, j=1,2$, satisfying $p_j\perp d$ are enough to recover the convex hull of the polyhedrals. Based on this (uniqueness) result we build up the indicator function of the No Response Test to provide a reconstructive proceedure to reconstruct the convex hull of the polyhedrals.

The rest of the paper is organized as follows. In section 2, we start by explaining the idea of the No Response Test, then we state the main theoretical result and give its proof. In section 3, we develop the actual realization of the method and provide some numerical tests.

2 The No Response Test in Electromagnetics

2.1 The Idea of the No Response Test

We consider scattering of incident plane waves with direction of incidence d and with polarization p_j for $j=1,2$. We assume that we have

$$
p_j\perp d, \ \ j=1,2 \text{ and } p_1 \text{ and } p_2 \text{ are not co-linear }. \tag{7}
$$

For every $g\in L^2(\mathbb{S}^2)$, we set $v_g(x) := \int_{\mathbb{S}^2} e^{i\kappa\theta\cdot x}g(\theta)ds(\theta)$ to be the scalar Herglotz wave corresponding the density g.

Then we define

$$I(B) = \lim_{\epsilon \to 0} \Big\{ \sum_{j=1}^{2} \Big| \int_{\mathbb{S}^2} E^\infty(-\theta, d, p_j) g(\theta)\, ds(\theta) \Big| : \ |v_g|_{C^2(\overline{B})} \leq \epsilon \Big\} \tag{8}$$

for any nonvibrating domain B, i.e. B is in the set

$$\mathcal{B} := \left\{ B : \begin{array}{l} \text{A solution } E \text{ to the homogeneous interior Maxwell} \\ \text{problem for } B \text{ with } \nu \times E = 0 \text{ on } \partial B \text{ is zero.} \end{array} \right\} \tag{9}$$

The idea of the No Response Test is to test if the unknown obstacle D is included in some $B \in \mathcal{B}$ by computing $I(B)$.

We will show that $I(B)$ is bounded if the electromagnetic field can be analytically extended into the exterior B_e of B. If it cannot be analytically extended into B_e, we will show that then $I(B)$ will not be bounded.

For convex sets B and convex polyhedral domains D, if $D \not\subset B$ then there will always be some corner z_0 which is not contained in $\overline{B}$. We show that given two incident fields E^i with linearly independent polarizations, one of them will not be analytically extensible into z_0. This will be used to reconstruct the convex hull of D by testing many convex domains B via the behaviour of $I(B)$.

Note that the algorithm of the No Response Test, i.e. the calculation of $I(B)$, does not need any conditions on the scatterer B. The test will work for a wide range of domains D and test analytic continuability. Here, for the case of a convex polyhedral domain and the choice of two incident waves with different polarization we show convergence, i.e. we prove that for exact data the test will reconstruct the full domain D.

2.2 Convergence of the No Response Test.

Our goal is to prove the following reconstruction of the convex hull of a polyhedral domain D.

THEOREM 2.1 (NO RESPONSE CHARACTERIZATION) *The convex hull of D is characterized by*

$$CH(D) = \bigcap_{B \in \mathcal{B}, I(B)=0} B. \tag{10}$$

Further, as a consequence of this theorem we immediately obtain the following uniqueness result.

COROLLARY 2.2 *The convex hull of a polyhedral domain in* $\mathbb{R}^3$ *is uniquely determined by the scattered field for one* ($N = 1$) *directions of incidence and* $M = 2$ *polarizations.*

Before we carry out the proof of Theorem 2.1 we need several preparations.

DEFINITION 2.3 (EXTERIOR CONVEX VERTICES) *We call a vertex of* ∂D *an* exterior convex vertex *if it is in the boundary* $\partial CH(D)$ *of the convex hull* $CH(D)$ *of* D.

REMARK 2.4 *Let* z_0 *be an exterior convex vertex. Then we can continue at least one of the faces of* ∂D *containing* z_0 *to the infinity without crossing* ∂D*, again. The exterior convex vertices characterize the convex hull of* D.

LEMMA 2.5 (EXTENSIBILITY) *Assume that for some positive real number* ρ*, the set of vectors*

$$\left\{ \sup_{|h|=1} \rho^\mu \frac{(h \cdot \nabla_z)^\mu E^s(z, d, p)}{\mu!}, \mu \in \mathbb{Z}_+ \right\} \tag{11}$$

is uniformly bounded in a compact set V*, where here the boundedness is understood componentwise. Then* $E^s(z, d, p)$ *is analytically extensible into an open neighbourhood* $V_\rho = \{x : d(x, V) < \rho\}$ *of* V.

Proof of Lemma 2.5. The basic result can be found in (Honda et al., 2008) or (Potthast, 2007). The authors use (11) as a bound for the Taylor coefficients of the function and construct an analytic extension into the open neighbourhood of V by multi-dimensional Taylor series. □

LEMMA 2.6 *Consider the scattered fields* $E^s(\cdot, d, p_j)$ *for* $j = 1, 2$ *in a neigbourhood of an exterior convex vertex. Then there exists at least one pair* (d, p_j) *such that* $E^s(z, d, p_j)$ *is not analytically extendable into an open neighbourhood of the point* z_0.

Proof of Lemma 2.6. By definition of the exterior convex vertex, there exists at least one face around z_0 which can be extended to infinity without crossing again ∂D. On this face, $E := E + E^s$ satisfies $\nu \times E = 0$. Assume that both $E^s(\cdot, d, p_1)$, and the $E(\cdot, d, p_2)$ are analytically extendible into a neighbourhood of z_0. Then by analyticity we would get $\nu \times E(\cdot, d, p_j) = 0, j = 1, 2$, on an infinite part of the plane Λ which is defined by its normal vector ν and $z_0 \in \Lambda$. Recall $E = E^i + E^s$ and note that E^s tends to zero at infinity then we have $\nu \times E^i(x, d, p_j) \to 0, j = 1, 2$, for $|x| \to \infty$ on Λ. Using explicitly $E^i(x, d, p_j) =$

$i\kappa(d \times p_j) \times d e^{i\kappa x\cdot d}$ we obtain

$$\lim_{|x|\to\infty} \nu \times ((d \times p_j) \times d e^{i\kappa x\cdot d}) = 0, \quad \text{for } x \in \Lambda.$$

This implies that

$$\nu \times ((d \times p_j) \times d) = 0.$$

Since p_j is chosen orthogonal to d, then $(d \times p_j) \times d = p_j$ and hence $\nu \times p_j = 0$.

Having two polarization directions p_1 and p_2 orthogonal to d, then we get $\nu \times p_j = 0, j = 1, 2$, which means that ν is co-linear to both p_1 and p_2. But this contradicts the assumption that p_1 and p_2 are linearly independent. Hence one of the scattered fields $E^s(\cdot, d, p_j)$ $j = 1$ or $j = 2$ is not analytically extendable into an open neighbourhood of the point z_0. □

Proof of Theorem 2.1. Let $B \subset \mathbb{R}^3$ be any convex non-vibrating domain for the Maxwell equation, i.e. let the interior homogeneous boundary value problem in B with boundary condition $\nu \times E = 0$ on ∂B be uniquely solvable. The proof of Theorem 2.1 is based on justifying the following two propoerties.

(A) If $D \subset B$ then $I(B) = 0$.

(B) If $D \not\subset B$ then $I(B) = \infty$.

Since B is taken to be convex, then $D \subset B$ is equivalent to $CH(D) \subset B$. Clearly, a convex set D is given by the intersection of all convex domains B for which $D \subset B$. From (A) and (B) we conclude that $D \subset B$ if and only if $I(B) = 0$. Hence, proving (A) and (B) gives the proof of formula (10) and Theorem 2.1.

We will now prove the statements (A) and (B). As first step, we provide a useful reformulation of the indicator function of the No Response Test. It is based on the identity

$$E^\infty(\theta, d, p) =$$

$$\frac{i\kappa}{4\pi}\theta \times \int_{\partial D} \Big\{\nu(y) \times E^s(y, d, p) + [\nu(y) \times H^s(y, d, p)] \times \theta\Big\} e^{-i\kappa\theta\cdot y} ds(y) \quad (12)$$

given by using the Stratton-Chu formula in $\mathbb{R}^3 \setminus \overline{D}$ for $E^s(\cdot, d, p)$, $H^s(\cdot, d, p)$ and $\Phi(\cdot, y)$, where $\Phi(x, y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$, and their asymptotic behavior at infinity (see (Colton and Kress, 1998), Theorem 6.8). Let $g \in L^2(\mathbb{S}^2)$, then

$$\int_{\mathbb{S}^2} E^\infty(-\theta, d, p) g(\theta) ds(\theta) = \frac{1}{4\pi} \int_{\partial D} \Big\{(\nu(y) \times E^s(y, d, p)) \times \nabla_y v_g(y) \quad (13)$$

$$- \frac{1}{i\kappa}\Big([\nu(y) \times H^s(y,d,p)] \times \nabla_y\Big) \times \nabla_y v_g(y)\Big\}\, ds(y),$$

where we use standard notation from vector analysis, i.e. the operator ∇_y operatos on the vector function on its right-hand side.

(A) Assume that $D \subset B$. Suppose that $|v_g|_{C^2(\overline{B})} \leq \epsilon$. From (13), we have for $p = p_j, j = 1, 2$,

$$|\int_{\mathbb{S}^2} E^\infty(-\theta, d, p) g(\theta) ds(\theta)| \leq C\epsilon$$

with some constant C. This implies that $I(B) = 0$.

(B) Assume that $D \not\subset B$. In this case, we can find at least one exterior convex vertex of ∂D which is not in $\overline{B}$ (since otherwise via convexity we would get $D \subset B$). We denote by z_0 one of these points. It has a positive distance $\rho = d(z_0, B)$ to B. We take a sequence of points z_q which are element of $\mathbb{R}^3 \setminus (\overline{D} \cup B)$ and tent to z_0 for $q \to \infty$.

We consider the *multipole fields*

$$\psi_q := \frac{\epsilon}{2\beta(z_q, \mu_q)} (h_q \cdot \nabla_z)^{\mu_q} \Phi(x, z_q) \tag{14}$$

where h_q is a unit vector, μ_q is a multi-integer and

$$\beta(z_q, \mu_q) := \sup_{y \in \overline{B}, ||h||=1} \{|(h \cdot \nabla_z)^{\mu_q} \Phi(x, z_q)|\}.$$

For every q we take $g_n^q \in L^2(\mathbb{S}^2)$ such that $v[g_n^q]$ tends to ψ_q in $C^2(\overline{B} \cup \overline{D})$. From (13), we get:

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} E^\infty(-\theta, d, p) g_n^q(\theta) ds(\theta) = \frac{1}{4\pi} \int_{\partial D} (\nu(y) \times E^s(y, d, p)) \times \nabla_y \psi_q \tag{15}$$

$$-\frac{1}{i\kappa}\Big([\nu(y) \times H^s] \times \nabla_y\Big) \times \nabla_y \psi_q ds(y)$$

Since $\nabla_y \Phi(x, z) = -\nabla_z \Phi(x, z)$, then $\nabla_y \Psi_q(x, z_q) = -\nabla_z \Psi_q(x, z_q)$. Using the identity curl $(af) = \nabla a \times f = -f \times \nabla a$ for constant vectors f and scalar functions a, (15) becomes:

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} E^\infty(-\theta, d, p) g_n^q(\theta) ds(\theta) \tag{16}$$

$$= \frac{1}{4\pi}\Big\{\text{curl} \int_{\partial D} \nu(y) \times E^s(y, d, p) \psi_q(y, z_q) ds(y)$$

$$-\frac{1}{i\kappa}\text{curl curl}\int_{\partial D}[\nu(y)\times H^s(y,d,p)]\psi_q(y,z_q)\Big\}ds(y).$$

Via the Stratton-Chu formula and due to the form of ψ_q, we obtain:

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{S}^2}E^\infty(-\theta,d,p)g_n^q(\theta)ds(\theta) &= \frac{\epsilon}{8\pi\beta(z_q,\mu_q)}(h_q\cdot\nabla_z)^{\mu_p}E^s(z_q,d,p)\\ &+\frac{1}{4\pi}\text{curl}\int_{\partial\Omega_R}\{\nu(y)\times E^s(y,d,p)\psi_q(y,z_q)\\ &-\frac{1}{i\kappa}\text{curl }[\nu(y)\times H^s(y,d,p)]\psi_q(y,z_q)\Big\}\,ds(y)\end{aligned}\tag{17}$$

where Ω_R is a ball of radius R large enough to contain $\overline{D}$. Arguing as in ((Colton and Kress, 1998), Theorem 6.6), we deduce that the integral over Ω_R tends to zero as R tends to infinity. Hence

$$\lim_{n\to\infty}\int_{\mathbb{S}^2}E^\infty(-\theta,d,p)g_n^q(\theta)ds(\theta) = \frac{\epsilon}{2\beta(z_q,\mu_q)}(h_q\cdot\nabla_z)^{\mu_q}E^s(z_q,d,p).\tag{18}$$

A combination of Lemma 2.5 and Lemma 2.6 implies that for $p=p_1$ or $p=p_2$ there exist sequences $(h_q)\subset\mathbb{S}^2$ and $(\mu_q)\subset\mathbb{N}$ such that

$$\lim_{q\to\infty}\rho^{\mu_q}\frac{(h_q\cdot\nabla_z)^{\mu_q}E^s(z_q,d,p)}{\mu_q!}=\infty.\tag{19}$$

As it is shown in (Honda et al., 2008), the quantities β satisfy

$$|\beta(z_q,\mu_q)|\le C\frac{\mu_q!}{\rho^{\mu_q}}.$$

From (18) and (19), we have

$$\lim_{q\to\infty}\lim_{n\to\infty}\sum_{j=1}^{2}|\int_{\mathbb{S}^2}E^\infty(-\theta,d,p_j)g_n^q(\theta)ds(\theta)|=\infty.\tag{20}$$

For $\epsilon>0$ fixed, we can take q,n large enough such that

$$\|v_{g_n^q}\|_{C^2(B)}\le\|v_{g_n^q}-\psi_q\|_{C^2(B)}+\|\psi_q\|_{C^2(B)}\le\epsilon.\tag{21}$$

By the definition of $I(B)$, (20) and (21) imply that $I(B)=\infty$. □

3 The Implementation of the No Response Test

The basic goal of this chapter is to develop the numerical realization of the No Response Test. We will first describe general preparation steps which are uniform for all subsequent realizations of the No Response Test. Then, we will describe an efficient approach to realize the No Response Test numerically.

We consider an electromangetic *Herglotz wave function*

$$V[a](x) := \frac{i}{\kappa}\text{curl curl} \int_{\mathbb{S}^2} e^{i\kappa x\cdot\theta} a(\theta)\, ds(\theta), \;\; H(x) := \frac{1}{i\kappa}\text{curl}\, V[a](x) \qquad (22)$$

for $x \in \mathbb{R}^3$ with density $a \in T(\mathbb{S}^2)$, where $T(\mathbb{S}^2)$ denotes the set of all vector fields $a \in L^2(\mathbb{S}^2)$ with $\nu(\hat{x})\cdot a(\hat{x}) = 0$ for all $\hat{x} \in \mathbb{S}^2$, where ν here is the normal to $\mathbb{S}$. Clearly, it satisfies the Maxwell equations (1) - (2). Further, consider the *magnetic multipole*

$$\Psi(x,z) := \frac{i}{\kappa}\text{curl curl}\, p\Phi(x,z), \;\; H(x,z) := \frac{1}{i\kappa}\text{curl}\, \Psi(x,z) \qquad (23)$$

for $x \in \mathbb{R}^3$ with source point $z \in \mathbb{R}^3$. Now, let B be a non-vibrating domain in $\mathbb{R}^3$ with boundary of class C^2. Then, with the operator $\mathcal{H} : L^2(\mathbb{S}^2) \to L^2(\partial B)$ defined by

$$(\mathcal{H}a)(x) := \frac{i}{\kappa}\text{curl curl} \int_{\mathbb{S}^2} e^{i\kappa x\cdot\theta} a(\theta)\, ds(\theta), \;\; x \in \partial B, \qquad (24)$$

and $z \in \mathbb{R}^3 \setminus \overline{B}$ we will study approximate solutions to the equation

$$\mathcal{H}a = \Psi(\cdot, z) \text{ on } \partial B. \qquad (25)$$

With $\text{curl}_x(\varphi(x)a) = \text{grad}_x\varphi \times a$ when a does not depend on x we obtain

$$(\mathcal{H}a)(x) = i\kappa \int_{\mathbb{S}^2} e^{i\kappa x\cdot\theta} (\theta \times a(\theta)) \times \theta\, ds(\theta), \;\; x \in \partial B, \qquad (26)$$

and for tangential field $a(\theta) \in T(\mathbb{S}^2)$ this reduces to

$$(\mathcal{H}a)(x) = i\kappa \int_{\mathbb{S}^2} e^{i\kappa x\cdot\theta} a(\theta)\, ds(\theta), \;\; x \in \partial B, \qquad (27)$$

First, we note important properties of equation (25).

LEMMA 3.1 *The equation (25) does not have a solution $a \in L^2(\mathbb{S}^2)$.*

Proof. Assume that there is a solution $a \in L^2(\mathbb{S}^2)$ of equation (25). Then both fields $V[a]$ and $\Psi(\cdot, z)$ solve the Maxwell equations in B with identical boundary values. By the well-posedness of the interior Dirichlet problem in B the two fields will coincide in B. Now, since the fields are both analytic in $\mathbb{R}^3 \setminus \{z\}$, they coincide in $\mathbb{R}^3 \setminus \{z\}$. However, the field $V[a]$ is smooth in $\mathbb{R}^3$, but $\Psi(\cdot, z)$ has a singulity in z which is a contradiction. This proves the lemma. $\square$.

We have shown that (25) does not have a solution. However, the operator $\mathcal{H}$ can be seen to have dense range in $L^2(\partial B)$.

LEMMA 3.2 *The operator $\mathcal{H}$ defined by (24) is injective and has dense range as an operator from $T(\mathbb{S}^2)$ into $L^2(\partial B)$.*

Proof. First, we study the injectivity of $\mathcal{H}$. Let $a \in T(\mathbb{S}^2)$ be some density such that $Ha = 0$ on ∂B. Then, we have $V[a] \equiv 0$ in B due to the well-posedness of the interior Dirichlet problem for the Maxwell equations in B. Due to the analyticity of $V[a]$ in $\mathbb{R}^3$ we have $V[a] \equiv 0$ in $\mathbb{R}^3$. Now, we can apply Theorem 3.15 of (Colton and Kress, 1998) to conclude that $a = 0$. This proves injectivity.

To show the denseness of the range of $\mathcal{H}$ we consider the adjoint operator $\mathcal{H}^*$ which due to (26) is given by

$$(\mathcal{H}^*\psi)(\theta) = i\kappa \int_{\partial B} e^{i\kappa y \cdot \theta} \theta \times (\psi(y) \times \theta)\, ds(y), \quad \theta \in \mathbb{S}^2, \tag{28}$$

with $\psi \in L^2(\partial B)$. Assume that $\mathcal{H}^*\psi = 0$. Then according to (6.26) of (Colton and Kress, 1998) the function

$$W[\psi](x) := \operatorname{curl}\operatorname{curl} \int_{\partial B} \Phi(x, y)\psi(y)\, ds(y), \quad x \in \mathbb{R}^3 \tag{29}$$

has farfield $1/4\pi \cdot \mathcal{H}^*\psi = 0$. Based on Rellichs lemma by Theorem 6.9 of (Colton and Kress, 1998) the field $W[a]$ vanishes in $\mathbb{R}^3 \setminus \overline{B}$. We now pass to the tangental values of this field on the boundary via the vector jump relations (compare (2.86) in combination with Theorem 2.17 of (Colton and Kress, 1983)) and obtain

$$N\psi = \nu \times \operatorname{curl}\operatorname{curl} \int_{\partial B} \Phi(x, y)\psi(y)\, ds(y) = 0, \quad x \in \partial B. \tag{30}$$

This first needs to be carried out in an L^2 sense. Then we argue that solutions $\psi \in L^2$ of $N\psi = 0$ are continuous and use the uniqueness of the interior boundary value problem with homoneneous tangential boundary values and the classical jump relations to conclude that $\psi \equiv 0$ on ∂B. This ends the proof. □

As a consequence of the previous result we obtain that the equation (25) has approximate solutions in the sense that for every $\epsilon > 0$ there is $a \in T(\partial D)$ such that

$$\Big|\Big|\mathcal{H}a - \Psi(\cdot, z)\Big|\Big|_{L^2(\partial B)} \leq \epsilon. \tag{31}$$

In fact, the approximate solution to this equation can be calculated via classical Tikhonov regularization

$$a_\alpha := (\alpha I + \mathcal{H}^*\mathcal{H})^{-1}\mathcal{H}^*\Psi(\cdot, z), \tag{32}$$

which is equivalent to minimizing the functional

$$\mu[a] := ||\mathcal{H}a - \Psi(\cdot, z)||^2_{L^2(\partial B)} + \alpha||a||^2_{L^2(\mathbb{S}^2)}. \tag{33}$$

Clearly, the minimum of the functional (33) tends to zero for $\alpha \to 0$ if $\mathcal{H}$ has dense range. Thus, via (32) we obtain stable approximate solutions for equation (25).

It has been shown in (Ben Hassen et al, 2006) that in fact we do not need to solve the full vectorial equation (25), but that it is sufficient to solve the scalar equation

$$Hg = \Phi(\cdot, z) \text{ on } \partial B_\tau \tag{34}$$

with some parameter $\tau > 0$, $B_\tau := \{x \in \mathbb{R}^3 : \ d(x, B) \leq \tau\}$ and

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{i\kappa x\cdot\theta} g(\theta)\, ds(\theta), \quad x \in \mathbb{R}^m. \tag{35}$$

Then, the function $a_z := pg_z$ with g_z solving (34) is a solution to (25). From a algorithmical point of view to solve a scalar equation is clearly much more efficient. With the same arguments as above we can employ Tikhonov regularization for its solution, i.e. we calculate

$$g_{z,\alpha} := (\alpha I + H^*H)^{-1}H^*\Phi(\cdot, z) \text{ on } \partial B \tag{36}$$

for $\alpha > 0$. Also, it has been shown in (Ben Hassen et al, 2006) that by inserting the approximation of $\Phi(\cdot, z)$ into the Stratton-Chu formula we obtain an approximation

$$\int_{\mathbb{S}^2} E^\infty(\hat{x}) g_{z,\alpha}(\hat{x})\; ds(\hat{x}) \to E^s(z), \quad \alpha \to 0, \tag{37}$$

which holds under the condition that the field E^s can be analytically extended into $\mathbb{R}^3 \setminus B$.

We now describe a direct realization of the No Response Test via the functional

$$I(B, d, p, \epsilon) := \sup \Big\{ \Big| \int_{\mathbb{S}^2} E^\infty(-\theta, d, p) g(\theta) \;\; ds(\theta) \Big| : \; g \in \mathcal{G}_\epsilon \Big\} \tag{38}$$

for some nonvibrating domain B where $\mathcal{G}_\epsilon$ is some set of densities with

$$||v_g||_{C^2(\overline{B})} \leq \epsilon. \tag{39}$$

In particular, we will calculate such densities by solving the integral equation (34) and multiplying the solution with the constant c_ϵ which satisfies

$$c_\epsilon \leq \frac{\epsilon}{2||\Phi(\cdot, z)||_{C^2(\overline{B})}}. \tag{40}$$

Algorithm 3.3 (No Response Test) *The No Response Test estimates the functional (8) by calculating $I(B, d, p, \epsilon)$ defined in (38), where for some domain B, a direction of incidence d and $\alpha > 0$ the density g is calculated by (36) for one or several points $z \in \mathbb{R}^3 \setminus \overline{B}$. In a second step we calculate the intersection*

$$D_{rec} := \bigcap_{I(B,d,p,\epsilon) \leq c} B \tag{41}$$

with some suitable constant c.

We complete this work with some numerical reconstructions which prove the feasibility of the method. Figure 1 shows the simulation of the field via integral equation methods. We have tested the code by solving the exterior boundary value problem with a dipole with source point located in the interior of the object as reference field. The error was clearly below 2% even with a modest number of triangles as shown in Figure 1. Reconstructions are demonstrated in Figure 2. We show a visualization calculated via Algorithm 3.3 for different locations and sizes of the polyhedral domain with wave numbers $\kappa = 1$.

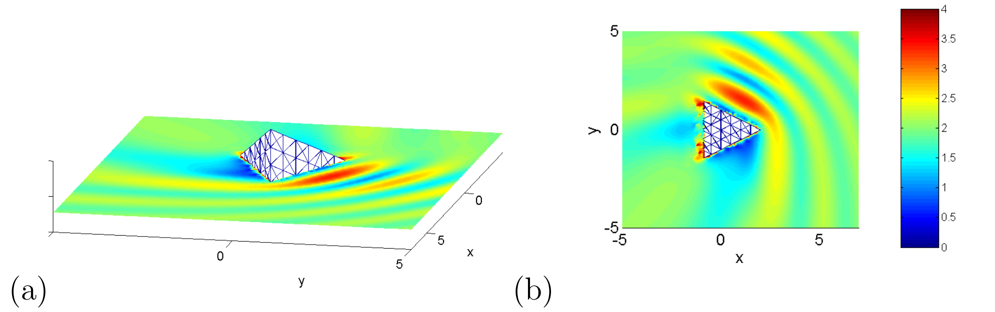

Fig. 1. Modulus of the total electric field for scattering by a polyhedral domain with perfect conductor boundary condition, wave number $\kappa = 2$. We show two different views, (b) from above and (a) looking onto one of the edges. Half of the object is covered by the plane with the field visualization.

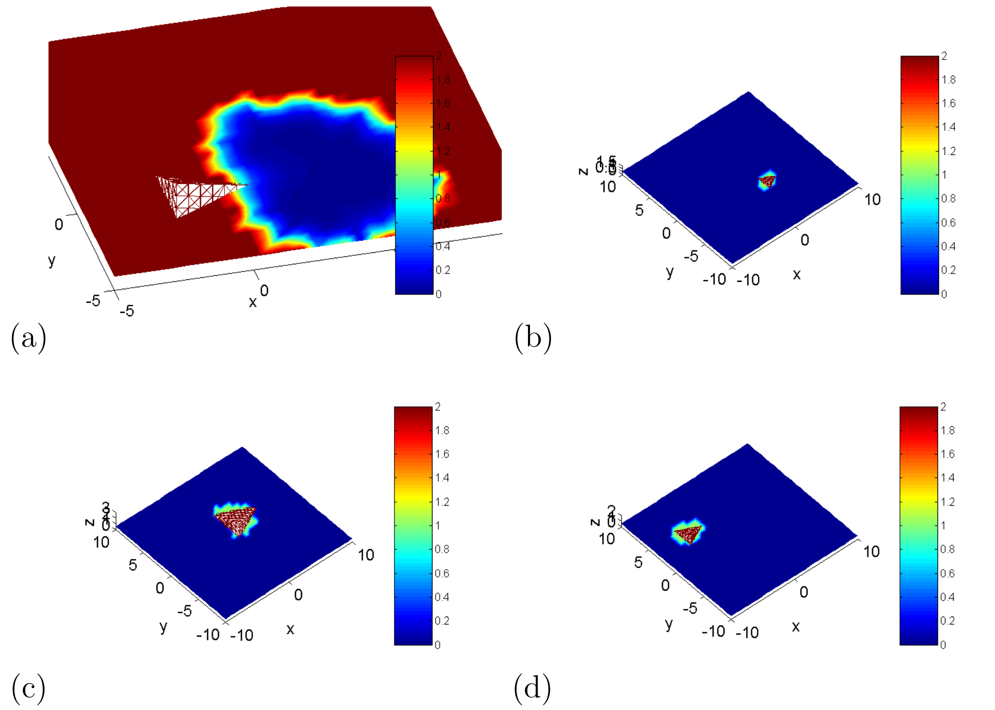

Fig. 2. In (a) we demonstrate the behaviour of the indicator function of the No Response Test for *one* electromagnetic wave only. Here, every image point z corresponds to a test domain $G(z)$ with $z \in \partial G_{\tau}(z)$ and $G(z) \subset \{y \in \mathbb{R}^3 : \; y_1 < z_1\}$. The blue area clearly indicates all such domains for which $D \subset G(z)$, i.e. it indicates a successfull No Response Test for the location of the domain. A second step is then to build the intersections (41). Figure (b) - (d) show reconstructions of some polyhedral domain from the far field pattern of *one* wave via the No Response Test functional with balls as test domains. Here, we show a slice of the mask on a plane intersecting the scatterer. The results here have not been optimized to yield good shape reconstructions, but we worked on a grid with cells of size $h = 0.5$. Clearly, we can easily identify the location and size of the scatterer and prove the feasibility of the ideas described above.

3.1 Conclusions

We have formulated the no response test for domain reconstructions in electromagnetics. In particular, we have shown convergence of the method for reconstruction of a perfectly conducting convex polyhedral domain. An efficient implementation of the scheme has been described, where we can reduce the computational effort by using the close relationship between acoustics and electromagnetics. Finaly, numerical tests have been carried out which prove the feasibility of the method.

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