

# Unification of the Probe and Singular Sources Methods for the Inverse Boundary Value Problem by the No-Response Test

GEN NAKAMURA<sup>1</sup>, ROLAND POTTHAST<sup>2</sup>,  
AND MOURAD SINI<sup>1</sup>

<sup>1</sup>Department of Mathematics, Hokkaido University, Sapporo, Japan

<sup>2</sup>Institute for Numerical and Applied Mathematics,  
University of Göttingen, Germany & Tomoscience GbR,  
Wolfsburg-Göttingen, Germany

*In this article, we use the no-response test idea, introduced in Luke and Potthast (2003) and Potthast (Preprint) and the inverse obstacle problem, to identify the interface of the discontinuity of the coefficient  $\gamma$  of the equation  $\nabla \cdot \gamma(x)\nabla + c(x)$  with piecewise regular  $\gamma$  and bounded function  $c(x)$ . We use infinitely many Cauchy data as measurement and give a reconstructive method to localize the interface. We will base this multiwave version of the no-response test on two different proofs. The first one contains a pointwise estimate as used by the singular sources method. The second one is built on an energy (or an integral) estimate which is the basis of the probe method. As a conclusion of this, the probe and the singular sources methods are equivalent regarding their convergence and the no-response test can be seen as a unified framework for these methods. As a further contribution, we provide a formula to reconstruct the values of the jump of  $\gamma(x)$ ,  $x \in \partial D$  at the boundary. A second consequence of this formula is that the blow-up rate of the indicator functions of the probe and singular sources methods at the interface is given by the order of the singularity of the fundamental solution.*

**Keywords** Conductivity problem; Dirichlet to Neumann map; Inverse problems.

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## 1. Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n = 2, 3$  with  $C^2$  regular boundary. We assume that  $\Omega$  contains a bounded domain  $D$  with its boundary  $\partial D$  such that  $\Omega \setminus \bar{D}$  is connected. We suppose that  $\partial D$  has the  $C^{1,1}$  regularity. We consider a function  $\gamma$  of

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Address correspondence to Gen Nakamura, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan; E-mail: gnaka@math.sci.hokudai.ac.jp

the form

$$\gamma(x) := 1 + \chi_D A(x),$$

where  $\chi_D$  is the characteristic function of  $D$  and  $A(x)$  is a  $C^1(\overline{D})$  function satisfying  $A(x) > 0$  in  $\overline{D}$ . The function  $\gamma$  is sometimes called the *conductivity*. We denote by

$$L_\gamma := \nabla \cdot \gamma \nabla \quad \text{and} \quad M_\gamma := L_\gamma + c(x),$$

where  $c(x)$  is a bounded measurable function. This function is sometimes called the *index of refraction*. For the inverse boundary value problem,  $D$  is called an *inclusion* and for the inverse scattering problem it is called a *penetrable obstacle*.

Consider  $f \in H^{\frac{1}{2}}(\partial\Omega)$  and let  $u^f$  be the  $H^1(\Omega)$  solution of

$$\begin{cases} M_\gamma u^f = 0 & \text{in } \Omega, \\ u^f = f & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

This problem is well posed by assuming that *zero* is not an eigenvalue for the related operator. By taking all the functions  $f \in H^{\frac{1}{2}}(\partial\Omega)$ , we define the Dirichlet to Neumann map

$$\Lambda : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega), \quad f \mapsto \Lambda(f) := \left. \frac{\partial u^f}{\partial \nu} \right|_{\partial\Omega},$$

where  $\nu$  is the exterior normal of  $\partial\Omega$ .

**Inverse Problem.** We assume that  $\Omega$ ,  $D$ ,  $\gamma(x)$ , and  $c(x)$  have the regularity properties we gave before and that zero is not an eigenvalue for the operator related to the problem (1.1). Let the function  $c(x)$  and the Dirichlet to Neumann map  $\Lambda$  be known. Our task is:

- 1) Reconstruct the interface  $\partial D$  of discontinuity of the coefficient  $\gamma(x)$ ;
- 2) Recover the values of  $A(x)$ ,  $x \in \partial D$ .

This inverse boundary value problem was initiated in Isakov (1988) where uniqueness for identifying the inclusion  $D$  was proven. In a recent article, Alessandrini and Di Cristo (2005), a stability result concerning this problem of localization of the interface of discontinuity is given. Regarding the reconstruction issue, in Ikehata (1998), a method for identifying the inclusion was proposed. The idea of this *probe method* is to define an indicator function depending on some parameter and use its behavior to extract some information on the shape of the discontinuity of the conductivity. The method has been generalized to deal with general scalar equations with mixed boundary conditions and with source term (Daido and Nakamura, 2004) and for anisotropic elastic systems (Ikehata et al., 1999).

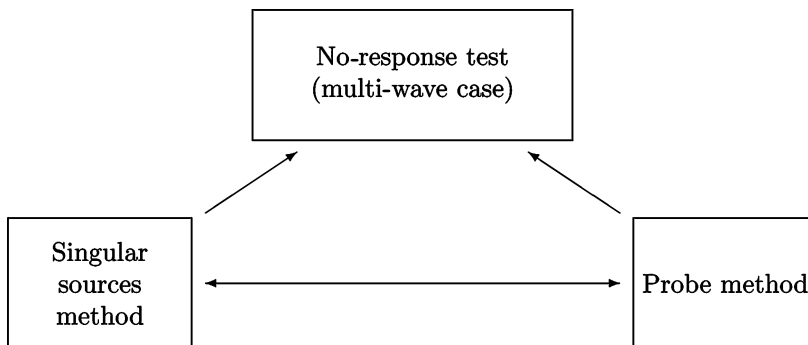
For the inverse obstacle problem, in Luke and Potthast (2003) and Potthast (Preprint) the *no-response test* is proposed to localize an obstacle from finitely or infinitely many measurements, and in Potthast (2001) we find the description of the *singular sources method* for shape reconstruction for the inverse scattering obstacle. These two methods also define indicator functions depending on parameters and

their behaviors are used to detect the obstacle. Convergence of these methods means that they detect the unknown. To distinguish the definitions, the no-response test using infinitely many measurements is called the *multi-wave version* of the no-response test.

The purpose of this article is to use the idea of the no-response test to reconstruct the inclusion from infinitely many measurements and to clarify its relation to the probe and the singular sources methods.

To be precise, we start by stating the near field version of the singular sources method and by recalling the probe method, as stated in Ikehata (1998), and then we show that these two methods are equivalent regarding their convergence by proving that their corresponding indicator functions are equivalent. After stating the no-response test indicator function, we show that its blow-up behavior can be based on the one of the probe method (energy estimate) or on the one of singular sources method (pointwise estimate). In addition, we will see that the testing domains of the probe and singular sources methods are particular testing domains of the no-response test, see Remark 3.1. This means that these two methods are two faces of the no-response test. This is why we say that the no-response test unifies them. The equivalence between the probe and the singular sources methods will be shown in Theorem 2.1 and the convergence of the no-response test will be given in Theorem 2.2. The relation of the methods is visualized in Figure 1. This implies that in any case where the probe method or the singular sources method converges, then the no-response test also converges.

As a further contribution, we derive a formula to reconstruct the values of  $A(x)$ ,  $x \in \partial D$ . A similar formula has been given in Ikehata and Nakamura (2004). In their formula, one needs to compute the integrals of the gradient of the fundamental solution on  $D$ . We also refer to Potthast (2004). We will explain more about our formula after the statement of the main result in the next section, see Theorem 2.2. A consequence of this formula is that the blow-up rate of the indicator function of the probe method and the singular sources method is of the order of the singularity of the fundamental solution. Hence the rate grows with the dimension of the space  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ). This precise blow-up rate of the indicator functions of the probe and



**Figure 1.** The no-response test is a method with a general optimization formulation which unifies the singular sources method (SSM) (pointwise estimate of fundamental solution) and the probe method (PM) (energy estimate of fundamental solution). The SSM and PM are equivalent regarding their convergence. Convergence of the SSM or PM is used to prove the convergence of the no-response test.

the singular sources methods given in this article is new. For the numerical realization of these methods this fact is quite useful. This will be given in Theorem 2.1.

The idea to justify the blow-up in a pointwise sense of the indicator function is to transform its behavior to the one of the Green's function of the equation  $\nabla \cdot \gamma(x)\nabla + c(x)$ . Then we prove that this Green's function is locally (near any point  $a \in \partial D$ ) equivalent, in the  $L^\infty$  norm sense, to the fundamental solution of  $\nabla \cdot (1 + A(a)\chi^-)\nabla$  where  $\chi^-$  is the characteristic function of the negative half-space. The explicit form of this last fundamental solution gives the result. The proof of this equivalence is given by freezing and flattening the coefficient  $\gamma(x)$  near the point  $a$ . To justify these two steps, we combined some recent estimates of the corresponding Green's functions given in Alessandrini and Di Cristo (2005),  $L^p$ ,  $1 < p < \infty$ , and  $L^\infty$  estimates of solutions for scalar divergence form elliptic equations with discontinuous coefficients, see Seftel (1963) and the recent work Li and Vogelius (2000) respectively.

The article is organized as follows. In the next section we describe the no-response test, the probe and the singular sources methods and state the results. In Section 3, we give the proofs of Theorems 2.1 and 2.2 using some auxiliary lemmas, and in Section 4, we give the proofs of these auxiliary lemmas.

## 2. Statement of Results

Let  $M_1$  and  $L_1$  be the operators  $M_\gamma$  and  $L_\gamma$  respectively when  $\gamma(x) = 1$ ,  $x \in \Omega$ , extended by 1 to  $\mathbb{R}^n \setminus \Omega$  and  $c(x)$  extended by 1 to  $\mathbb{R}^n \setminus \Omega$ . Conventionally, by  $\Phi$  we denote the fundamental solution of  $L_1$ . In this article we denote it by  $\Phi'$  and use the notation  $\Phi$  for the fundamental solution of  $M_1$ . This is motivated by the fact that we are using as the known background the medium given by  $(\gamma = 1, c(x))$  and the unknown perturbation is given by  $\chi_D A(x)$ . We denote by  $\mathbb{N}$  the set of positive integers and we will use the notation:

$$\lim_{p, n \rightarrow \infty} := \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty}.$$

In the next subsections, we start by recalling the main idea of the probe method and state the singular sources method for this inverse boundary value problem. Then we explain the idea of the no-response test introduced in Luke and Potthast (2003) and Potthast (Preprint) for the inverse obstacle problem and show how to adapt it to our problem. We finish these subsections by giving the equivalence between the probe and the singular sources methods and by stating the convergence of the no-response test and the formula for detecting the jumps of the conductivity across the interface.

### 2.1. The Probe and Singular Sources Methods

As we said in the introduction, both of the probe and the singular sources methods are based on indicator functions which are used to detect the shapes of the inclusion. Before explaining how these indicator functions are constructed, we need some preparations.

Let  $z_p \in \Omega \setminus \bar{D}$  such that  $z_p$  tends to  $z \in \Omega$  when  $p$  tends to  $\infty$ . We set  $E(z_p)$  any  $C^2$ -regular domain such that  $z_p \in \Omega \setminus \bar{E}(z_p)$ ,  $\Omega \setminus \bar{E}(z_p)$  is connected and  $D \subset \subset E(z_p) \subset \Omega$ . Using the Runge approximation (see Lax, 1956), we can find a

sequence of functions,  $v_n^p \in H^1(\Omega)$  satisfying  $\Delta v_n^p + cv_n^p = 0$  in  $\Omega$ , such that  $\|v_n^p - \Phi(\cdot, z_p)\|_{H^1(E(z_p))}$  tends to zero when  $n$  tends to  $\infty$ . Instead of using the general Runge approximation, we can use the single layer potential operator as in the coming Lemma 3.6 to chose  $v_n^p := \int_{\partial\Omega} \Phi(\cdot, y)\varphi_n^p(y)ds(y) \in H^1(\Omega)$  with a density  $\varphi_n^p \in L^2(\partial\Omega)$ . The advantage of using the single layer potential is that it is a more constructive way.

Indeed, the single layer operator  $S$  defined on  $L^2(\partial\Omega)$  with values in  $L^2(\partial E(z_p))$  by

$$S(\varphi) := \int_{\partial\Omega} \Phi(\cdot, y)\varphi(y)ds(y)$$

is compact since the kernel  $\Phi(\cdot, \cdot)$  is bounded in  $\partial\Omega \times \partial E(z_p)$ . In addition, the condition on  $E(z_p)$  that zero is not a Dirichlet eigenvalue for  $M_1$  implies that this operator is injective with dense range, see Lemma 3.6. These properties justify the use of the regularization methods to construct the sequence  $\varphi_n^p$ , see Colton and Kress (1998), Engle et al. (1996) or Kress (1999).

**Remark 2.1.** We wish to add the following two comments to show how we can handle in practice this eigenvalue condition.

1. If we define the operator  $S$  from  $L^2(\partial\Omega)$  with the values in

$$\overline{\{v|_{\partial E(z_p)}, v \in H^1(E(z_p)) \text{ and } M_1v = 0, \text{ in } E(z_p)\}}^{L^2(\partial E(z_p))},$$

then it has also a dense range, see Lemma 3.6. Remark that for this point, we do not need the eigenvalue condition on  $E(z_p)$ . If in addition we take its restriction on  $[N(S)]^\perp$ , then it becomes injective, compact, and with dense range. The regularization methods can be applied for this operator, see Engle et al. (1996). This means in particular that the regularization methods give a practical realization of the Runge approximation (or the weak unique continuation).

If on  $E(z_p)$ , zero is not a Dirichlet eigenvalue of  $M_1$ , then

$$\overline{\{v|_{\partial E(z_p)}, v \in H^1(E(z_p)) \text{ and } M_1v = 0, \text{ in } E(z_p)\}}^{L^2(\partial E(z_p))} = L^2(\partial E(z_p)),$$

see Weck (2004) and the references there and obviously  $[N(S)]^\perp = L^2(\partial\Omega)$ .

2. Considering a one parameter family of continuous, monotone deformations of  $E(z_p)$  there are at most finitely many of such deformed domains for which zero is a Dirichlet eigenvalue for  $M_1$ . We will not give the details to justify this fact but we want to mention that it can be based on the following two known properties:

- (i) For every domain  $E(z_p)$ , fixed, the sequence of the Dirichlet eigenvalues goes to infinity;
- (ii) By decreasing strictly the domain  $E(z_p)$ , the eigenvalues strictly increase. The proof of this property is based on the weak unique continuation property and the Courant mini-max principle, see Courant and Hilbert (1955, Ch. VI, Sec. 2, Theorem 3) or Chavel (1984) and Uhlenbeck (1976).

This means that we can almost chose the set  $E(z_p)$  satisfying the eigenvalue condition. We want to finish this comment by saying that for any given domain  $B \subset \Omega$ , we can always test whether  $\kappa^2$  is a Dirichlet eigenvalue for  $-\Delta$  by using the Courant min-max principle.

Now we give the indicator functions of the probe and the singular sources method.

*2.1.1. The Indicator Function of the Probe Method.* Here, we assume that zero is not a Dirichlet eigenvalue for  $M_1$  on  $\Omega$ . With this condition, the problem (1.1), with  $\gamma = 1$ , is well posed. Let  $\Lambda_0$  be the corresponding Dirichlet–Neumann map.

The functional of the probe method is defined by

$$\int_{\partial\Omega} (\Lambda - \Lambda_0)f(x) \cdot f(x)ds(x).$$

We take now  $f_n^p := v_n^p|_{\partial\Omega}$  and evaluate  $\int_{\partial\Omega} (\Lambda - \Lambda_0)f_n^p(x) \cdot f_n^p(x)ds(x)$ , then, see Ikehata (1998), for every  $p$  fixed we obtain

$$\lim_{n \rightarrow \infty} \int_{\partial\Omega} (\Lambda - \Lambda_0)f_n^p(x) \cdot f_n^p(x)ds(x) = \int_D A(x)(\nabla w(\cdot, z_p) + \nabla\Phi)(x) \cdot \nabla\Phi(x)dx \quad (2.1)$$

where  $w(\cdot, z_p)$  is the  $H^1$ -solution of

$$\begin{cases} M_\gamma w(\cdot, z_p) = -\nabla \cdot \chi_D A(x) \nabla\Phi(\cdot, z_p) & \text{in } \Omega, \\ w(\cdot, z_p) = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

called the *reflected solution*.

The characterization of  $z$  to be in  $\partial D$  is given by the testing

$$\lim_{p, n \rightarrow \infty} \int_{\partial\Omega} (\Lambda - \Lambda_0)f_n^p(x) \cdot f_n^p(x)ds(x) = \infty.$$

For every  $z_p$ , we set

$$I_{pb}(z_p) := \lim_{n \rightarrow \infty} \int_{\partial\Omega} (\Lambda - \Lambda_0)f_n^p(x) \cdot f_n^p(x)ds(x).$$

Similarly, for any  $z \in \Omega \setminus \bar{D}$  we set

$$I_{pb}(z) := \lim_{n \rightarrow \infty} \int_{\partial\Omega} (\Lambda - \Lambda_0)f_n^z(x) \cdot f_n^z(x)ds(x), \quad (2.3)$$

where  $f_n^z$  is constructed relatively to  $z$  in the same way we constructed  $f_n^p$  relatively to  $z_p$ . We call (2.3) the *indicator function* of the probe method.

*2.1.2. The Indicator Function of the Singular Sources Method.* We define  $u_n^p$  as the solution of

$$\begin{cases} M_\gamma u_n^p = 0 & \text{in } \Omega, \\ u_n^p = v_n^p & \text{on } \partial\Omega. \end{cases}$$

Then  $w_n^p := u_n^p - v_n^p$  satisfies

$$\begin{cases} M_\gamma w_n^p = -\nabla \cdot \chi_D A(x) \nabla v_n^p & \text{in } \Omega, \\ w_n^p = 0 & \text{on } \partial\Omega. \end{cases}$$

Tending  $n$  to  $\infty$ , we deduce that  $w_n^p$  tends to  $w(\cdot, z_p)$  in  $H^1(\Omega)$  which is the solution of (2.2). From the data  $(u_n^p, \frac{\partial u_n^p}{\partial \nu})|_{\partial\Omega}$ , we solve the multidimensional Cauchy problem inside  $\Omega \setminus \bar{D}$  by regularization methods, like the point sources method for example (Potthast, 2001). We compute the values  $u_n^p(z_p)$ , then we compute

$$\lim_{n \rightarrow \infty} (u_n^p(z_p) - v_n^p(z_p)) = w(z_p, z_p). \tag{2.4}$$

The characterization of  $z$  to be in  $\partial D$  is given by the testing

$$\lim_{p \rightarrow \infty} w(z_p, z_p) = \infty.$$

For every  $z \in \Omega \setminus \bar{D}$ , we set

$$I_{ss}(z) := w(z, z). \tag{2.5}$$

We call (2.5) the indicator function of the singular sources method.

Let  $a \in \partial D$  and  $\theta \in [0, 2\pi)$ . We set  $\Omega_{a,\theta}$  to be the cone of vertex  $a$ , angle  $\theta$  and axis  $\nu(a)$ , the interior normal on  $a$  of  $\partial D$ . We take this cone having the opposite direction of  $\nu(a)$ , i.e., the cone is outside  $D$  near the point  $a$ . We have the following theorem which gives as the first point the equivalence of the singular sources method and the probe method regarding their convergence. The second point gives the precise dominant part of (2.3) and (2.5) with respect to the parameter  $z$ . The blow-up rate is of the order of the singularity of the fundamental solution.

**Theorem 2.1.** *The two functionals  $I_{pb}$  and  $I_{ss}$  satisfy the following properties:*

1) *For every subset  $B \subset \Omega$ ,*

$$|I_{pb}(z) + I_{ss}(z)| = O(1), \quad z \in B \setminus \bar{D}. \tag{2.6}$$

2) *Let  $a \in \partial D$  and  $\theta \in [0, \frac{\pi}{2})$ . Then there exists  $\delta = \delta(a) > 0$  such that for every  $\alpha$ ,  $0 < \alpha < 1$ , we have*

$$\begin{cases} I_{ss}(z) = \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|z - z^*|} + O\left(\frac{1}{|z - z^*|^\alpha}\right), & z \in \Omega_{a,\theta} \cap B(a, \delta), \text{ for } n = 3, \\ I_{ss}(z) = -\frac{A(a)}{A(a) + 2} (2\pi)^{-1} \ln|z - z^*| + O(1), & z \in \Omega_{a,\theta} \cap B(a, \delta), \text{ for } n = 2, \end{cases} \tag{2.7}$$

where  $z^*$  is the symmetric to  $z$  with respect to the tangent at  $a$  of  $\partial D$ .

From the parts 1) and 2) we deduce that the probe functional  $I_{pb}$  also satisfies (2.7).

## 2.2. The No-Response Test

By (1.1) and Green's formula, we write

$$\begin{aligned} u^f(x) &= \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \\ &\quad + \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \end{aligned} \quad (2.8)$$

for  $x \in \Omega \setminus \bar{D}$ . Letting  $x \rightarrow \partial\Omega$  in (2.8) and using Green's formula, we obtain

$$\begin{aligned} u^f(x) &= \frac{1}{2} u^f(x) + \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \\ &\quad + \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \end{aligned} \quad (2.9)$$

for  $x \in \partial\Omega$ , where the double layer potential  $\int_{\partial\Omega} u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y)$  is understood as the Cauchy principle value integral. From our Cauchy data on  $\partial\Omega$ , we know the function

$$J^f(x) := \frac{1}{2} u^f(x) - \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in \partial\Omega$$

hence

$$J^f(x) = \frac{1}{2} f(x) - \int_{\partial\Omega} \left\{ \Lambda(f)(y) \Phi(x, y) - f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in \partial\Omega. \quad (2.10)$$

By (2.9) we have

$$J^f(x) = \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in \partial\Omega. \quad (2.11)$$

For  $\varphi \in L^2(\partial\Omega)$  we define the single layer potential  $v[\varphi](y)$  by

$$v[\varphi](y) = \int_{\partial\Omega} \Phi(x, y) \varphi(x) ds(x) \quad y \in \Omega.$$

Multiplying (2.11) by  $\varphi$ , integrating over  $\partial\Omega$  and exchanging the order of integration, we obtain

$$\begin{aligned} &\int_{\partial\Omega} J^f(x) \varphi(x) ds(x) \\ &= \int_{\partial\Omega} \varphi(x) \left\{ \int_{\partial D} \left( \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right) ds(y) \right\} ds(x) \\ &= \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \int_{\partial\Omega} \varphi(x) \Phi(x, y) ds(x) - u^f(y) \int_{\partial\Omega} \varphi(x) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(x) \right\} ds(y). \end{aligned}$$

Hence

$$\int_{\partial\Omega} J^f(x)\varphi(x)ds(x) = \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y)v[\varphi] - u^f(y)\frac{\partial v[\varphi]}{\partial \nu(y)} \right\} ds(x). \tag{2.12}$$

Let now  $B$  be a domain inside  $\Omega$ . Noticing that  $v[\varphi]|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$ , see McLean (2000), we define the functional

$$I_\epsilon(B) := \sup_{\varphi \in \mathbf{M}_\epsilon(B)} \left\{ \left| \int_{\partial\Omega} J^f(x)\varphi(x)ds(x), f := v[\varphi]|_{\partial\Omega} \right| \right\} \tag{2.13}$$

where

$$\mathbf{M}_\epsilon(B) := \{ \varphi \in L^2(\partial\Omega) : \|v[\varphi]\|_{H^1(B)} \leq \epsilon \}. \tag{2.14}$$

Our main *indicator function* is defined by

$$I(B) := \lim_{\epsilon \rightarrow 0^+} I_\epsilon(B). \tag{2.15}$$

Note that it is defined on a set of domains, not in the underlying “physical” space. The limit in (2.15) may exist or may not exist. This is the key observation to detect  $\partial D$ . We will see that this functional has two values. If  $\bar{D} \subset B$  then  $I_\epsilon(B) \leq c\epsilon^2$  with some positive constant  $c$  and hence  $I(B) = 0$ . If  $\bar{D} \not\subset B$  then  $I_\epsilon(B) = \infty$  for every  $\epsilon > 0$  and hence  $I(B) = \infty$ .

Now, using the data given by the Dirichlet to Neumann map we may calculate the functional (2.10) and then indicator function  $I(B)$  defined in (2.15), respectively. In Section 3, we give the proof of the following theorem which gives a reconstructive way how to localize  $\partial D$  and how to reconstruct the values of  $A(x)$ ,  $x \in \partial D$ .

**Theorem 2.2.** 1) *We have the following characterization of  $D$  from the Dirichlet to Neumann map:*

$$\bar{D} = \bigcap_{B \in \mathbf{B}} B,$$

where  $\mathbf{B} := \{B \subset \Omega : I(B) = 0\}$ .

2) *Knowing  $\partial D$ , then for every  $a \in \partial D$ , we reconstruct a sequence  $\varphi_n^p \in L^2(\partial\Omega)$  such that the following formula is valid:*

$$\begin{cases} \frac{A(a)}{A(a) + 2} = \lim_{p, n \rightarrow \infty} (4\pi)|z_p - z_p^*| \int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x), & \text{for } n = 3, \\ \frac{A(a)}{A(a) + 2} = - \lim_{p, n \rightarrow \infty} (2\pi)(\ln|z_p - z_p^*|)^{-1} \int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x), & \text{for } n = 2, \end{cases} \tag{2.16}$$

where  $f_n^p := v[\varphi_n^p]|_{\partial\Omega}$  and  $z_p$  is any sequence of points in  $\Omega \setminus \bar{D}$  tending to  $a$  as  $p$  tends to  $\infty$  and  $z_p^*$  is the point symmetric to  $z_p$  with respect to the plane tangent to  $\partial D$  at the point  $a$ .

Based on Lemma 3.6, given later, the functions  $\varphi_n^p$ , and hence  $f_n^p$ , can be reconstructed using the Tikhonov regularization scheme. Note also that from (2.16)

we do not need to know  $D$  everywhere to reconstruct  $A(a)$ . It is enough to know  $D$  near the point  $a$ . In fact it is enough to know the point  $a$  and the plane tangent to  $\partial D$  at the point  $a$ .

**Remark 2.2.** The conditions on  $\partial D$  and  $A(x)$  can be weakened by considering  $\partial D$  having the  $C^{1,\alpha}$  regularity,  $A(x) \in C^{0,\alpha}(\overline{D})$  where  $0 < \alpha \leq 1$  and  $A(x) \neq 0$  near  $\partial D$ . We take the limit cases to simplify the exposition.

### 3. Proofs of Theorems 2.1 and 2.2

We give the proof for the case  $n = 3$ . The case  $n = 2$  can be treated similarly with the appropriate changes for the behavior of the related Green's functions.

#### 3.1. Proof of Theorem 2.1

*Proof of the point 1).* Let  $z \in \Omega \setminus \overline{D}$ . Considering  $\Phi(\cdot, z)$  and  $w(\cdot, z)$  and using an integration by parts in  $\Omega \setminus \overline{D}$ , we get

$$w(z, z) = \int_{\partial\Omega} \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) ds(x) + \int_{\partial D} \left\{ \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) - w(x, z) \frac{\partial \Phi}{\partial \nu}(x, z) \right\} ds(x) \tag{3.1}$$

We write:

$$\begin{aligned} & \int_{\partial D} \left\{ \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) - w(x, z) \frac{\partial \Phi}{\partial \nu}(x, z) \right\} ds(x) \\ &= \int_{\partial D} \left\{ \Phi(x, z) \frac{\partial(w + \Phi)}{\partial \nu}(x, z) - (w + \Phi)(x, z) \frac{\partial \Phi}{\partial \nu}(x, z) \right\} ds(x). \end{aligned} \tag{3.2}$$

We remark that  $w + \Phi$  satisfies  $\nabla \cdot \gamma \nabla(w + \Phi) + c(w + \Phi) = 0$  in  $D$  and recalling that  $\Delta \Phi + c\Phi = 0$  in  $D$  we deduce that:

$$\begin{aligned} & \int_{\partial D} \left\{ \Phi(x, z) \frac{\partial(w + \Phi)}{\partial \nu}(x, z) - (w + \Phi)(x, z) \frac{\partial \Phi}{\partial \nu}(x, z) \right\} ds(x) \\ &= - \int_{\partial D} \left\{ \Phi(x, z)(1 + A(x)) \frac{\partial(w + \Phi)}{\partial \nu^+}(x, z) - (w + \Phi)(x, z) \frac{\partial \Phi}{\partial \nu^+}(x, z) \right\} ds(x) \\ &= - \int_D A(x) \nabla \Phi(x, z) \cdot \nabla(\Phi + w)(x, z) dx \end{aligned} \tag{3.3}$$

where  $\nu^+$  is the unit normal oriented into  $\Omega \setminus \overline{D}$ . Hence from (3.1) and (3.3) we get

$$w(z, z) - \int_{\partial\Omega} \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) ds(x) = - \int_D A(x) \nabla \Phi(x, z) \cdot \nabla(\Phi + w)(x, z)$$

which means

$$I_{ss}(z) + I_{pb}(z) = \int_{\partial\Omega} \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) ds(x).$$

The following proposition will be proven in Section 4.

**Proposition 3.1.** *The function  $\int_{\partial\Omega} \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) ds(x)$  is bounded with respect to  $z \in B$  for any domain  $B$  such that  $\bar{B} \subset \Omega$ .*

From this proposition we deduce that for  $z \in B \setminus \bar{D}$ , with  $\bar{B} \subset \Omega$ , we have

$$I_{ss}(z) = -I_{pb}(z) + O(1).$$

*Proof of the point 2).* To prove this second point we start by stating some useful lemmas which we will prove in Section 4. We recall that the reflected solution satisfies:

$$\begin{cases} M_\gamma w = -\nabla \cdot \chi_D A \nabla \Phi(\cdot, z_p) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Let us consider the sequence  $w(z_p, z_p)$ . From (3.4) we see that the distribution  $G := w + \Phi$  satisfies

$$\begin{cases} M_\gamma G = -\delta(x - z_p) & \text{in } \Omega, \\ G = \Phi(\cdot, z_p) & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

The purpose of the following lemmas is to localize the dominant part of  $G$  and hence of  $w$  in the pointwise sense.

We set  $\tilde{G}$  the Green's function of  $M_\gamma$  on  $\Omega$  with homogeneous Dirichlet boundary condition.

**Lemma 3.1.** *For every domain  $B, B \subset \subset \Omega$ , the function  $(G - \tilde{G})(x, z)$  is bounded for  $x \in \Omega$  and  $z \in B$ .*

Let us define the Green's function  $G'$  of  $L_\gamma$  on  $\Omega$  with homogenous Dirichlet boundary condition, i.e.,

$$\begin{cases} L_\gamma G' = -\delta(x - z) & \text{in } \Omega, \\ G' = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

**Lemma 3.2.** *The function  $(\tilde{G} - G')(x, z)$  is bounded for  $(x, z) \in \Omega^2$ .*

Let now  $a \in \partial D$ . We set  $L_{\gamma(a)}$  as the expression  $L_\gamma$  with  $\gamma$  replaced by  $\gamma(a) := 1 + \chi_D A(a)$ . We denote by  $G'_{\gamma(a)}$  the Green's function of  $L_{\gamma(a)}$  on  $\Omega$  with homogenous Dirichlet boundary condition. We recall also that  $\Omega_{a,\theta}$  is the cone of vertex  $a$  with axis  $-\nu(a)$  and angle  $\theta \in [0, \frac{\pi}{2})$  where  $\nu(a)$  is the normal to  $\partial D$  on the point  $a$  oriented inside  $D$ .

**Lemma 3.3.** *Let  $\epsilon > 0, \theta \in [0, \frac{\pi}{2})$  and  $B \subset \subset \Omega$  such that  $D \subset B$  be fixed. There exists a constant  $c(\epsilon, B, \theta) > 0$  such that*

$$|(G' - G'_{\gamma(a)})(x, z)| \leq c(\epsilon, B, \theta) [d(z, \partial D)]^{-\frac{\epsilon}{3-\epsilon}}$$

for  $x \in \Omega$  and  $z \in \Omega_{a,\theta} \cap B$ .

Let  $\Phi'_{\gamma(a)}$  be the fundamental solution of  $\nabla \cdot (1 + A(a)\chi^-)\nabla \cdot$ , where  $\chi^-$  is the characteristic function of the negative half-space of  $\mathbb{R}^3$  given by  $\mathbb{R}^3_- := \{x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ . Let  $T$  be the transformation of coordinates which transforms the half-space given by the points which are below the plane tangent to  $\partial D$  on  $a$  to the half-space  $\mathbb{R}^3_-$ . We set  $B(a, \delta)$ , the ball of center at  $a$  with radius  $\delta > 0$ .

**Lemma 3.4.** *Let  $\theta \in [0, \frac{\pi}{2})$  be fixed. For  $\epsilon > 0$  and  $\delta > 0$  small enough, there exists a positive constant  $c(\delta, \theta) > 0$  such that*

$$|(G'_{\gamma(a)} - \Phi'_{\gamma(a)} \circ T)(x, z)| \leq c(\delta, \theta)[d(z, \partial D)]^{-\epsilon}$$

for  $x \in \Omega$  and  $z \in \Omega_{a,\theta} \cap B(a, \delta)$ .

**Lemma 3.5.** *The distribution  $(\Phi - \Phi')(x, z)$  is bounded for  $(x, z) \in \Omega^2$ .*

Now, we have the following formula, see Alessandrini et al. (1995):

$$\Phi'_{\gamma(a)}(x, z) - \frac{(4\pi)^{-1}}{|x - z|} = \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|x - z^*|},$$

with  $(x, z) \in \mathbb{R}^n_+$ , where  $z^*$  is the point symmetric to  $z$  with respect to the plane  $\{z \in \mathbb{R}^3 : z = (z_1, z_2, 0)\}$ .

We write

$$G = (G - \tilde{G}) + (\tilde{G} - G') + (G' - G'_{\gamma(a)}) + G'_{\gamma(a)}$$

and we recall that  $w = G - \Phi$ , then from Lemmas 3.1–3.3, we obtain

$$|w(x, z) - (G'_{\gamma(a)} - \Phi)| = |G - G'_{\gamma(a)}| \leq c[d(z, D)]^{\frac{-\epsilon}{3-\epsilon}} \tag{3.7}$$

for  $x \in \Omega$  and  $z \in \Omega_{a,\theta} \cap B$ .

We write

$$G'_{\gamma(a)} - \Phi = G'_{\gamma(a)} - \Phi'_{\gamma(a)} \circ T + \left[ \Phi'_{\gamma(a)} \circ T - \frac{(4\pi)^{-1}}{|x - z|} + \frac{(4\pi)^{-1}}{|x - z|} - \Phi \right]. \tag{3.8}$$

By Lemma 3.4 the term  $G'_{\gamma(a)} - \Phi'_{\gamma(a)} \circ T$  is bounded by  $c[d(z, D)]^{-\epsilon}$  for  $(x, z) \in \Omega \times (\Omega_{a,\theta} \cap B(a, \delta))$  and by Lemma 3.5,  $\frac{(4\pi)^{-1}}{|x - z|} - \Phi(x, z)$  is also bounded for  $(x, z) \in \Omega^2$ . Recall also that  $T$  is an isometry since it is given by a combination of a translation and a rotation, then using the identity

$$\Phi'_{\gamma(a)} \circ T(x, z) - \frac{(4\pi)^{-1}}{|x - z|} = \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|x - z^*|} \tag{3.9}$$

we deduce from (3.8) and (3.9) that there exists a positive constant  $c$  such that

$$\left| G'_{\gamma(a)} - \Phi - \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|x - z^*|} \right| \leq c[d(z, \partial D)]^{-\epsilon} \tag{3.10}$$

for  $x \in \Omega$  and  $z$  in  $\Omega_{a,\theta} \cap B(a, \delta)$ . Hence combining (3.7) and (3.10) we have the estimate

$$|z - z^*|^\alpha \left[ w(z, z) - \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|z - z^*|} \right] \leq c[d(z, \partial D)]^{-\frac{\epsilon}{3-\epsilon}} |z - z^*|^\alpha$$

which we write as

$$\begin{aligned} &|z - z^*|^\alpha \left[ w(z, z) - \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|z - z^*|} \right] \\ &\leq c[d(z, \partial D)]^{\alpha - \frac{\epsilon}{3-\epsilon}} + c[d(z, \partial D)]^{-\frac{\epsilon}{3-\epsilon}} [d(z^*, \partial D)]^\alpha. \end{aligned}$$

Choosing  $\delta > 0$  small enough such that  $\Omega_{a,\theta} \cap B(a, \delta) \cap \partial D = \{a\}$ , then there exist constants  $c(\delta, \theta) > 0$  and  $C(\delta, \theta) > 0$  such that  $C(\delta, \theta)d(z, a) \geq d(z, \partial D) \geq c(\delta)d(z, a)$  for  $z \in \Omega_{a,\theta} \cap B(a, \delta)$ .

Taking  $\epsilon > 0$  satisfying  $-\frac{\epsilon}{3-\epsilon} + \alpha > 0$ , the second member of the last inequality is bounded. Hence we deduce (2.7).

### 3.2. Proof of Theorem 2.2

We consider the property 1) We will prove that:

- (a) Case one.  $\overline{D} \subset B \Rightarrow I(B) = 0$ , and
- (b) Case two.  $\overline{D} \not\subset B \Rightarrow I(B) = \infty$ .

These two properties imply the desired result.

3.2.1. *Case One.* Let  $D$  be such that  $\overline{D} \subset B$ . Let also  $\varphi \in L^2(\partial\Omega)$  be such that  $\|v[\varphi]\|_{H^1(B)} < \epsilon$  and  $f := v[\varphi]|_{\partial\Omega}$ . Then the function  $U := u^f - v[\varphi] \in H^1(\Omega)$  satisfies

$$\begin{cases} M_\gamma U = -\nabla \cdot \chi_D A(x) \nabla(v[\varphi]) & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Since  $\overline{D} \subset B$ , then the well posedness of the problem (3.11) gives  $\|U\|_{H^1(\Omega)} \leq c_1 \epsilon$  with a positive constant  $c_1$  and then  $\|u^f\|_{H^1(B)} \leq (c_1 + 1)\epsilon$ . From (2.12), this implies that  $I_\epsilon(B) \leq c_2 \epsilon^2$  where  $c_2 > 0$  is a constant. This means that if we have  $\overline{D} \subset B$ , then

$$I(B) = 0.$$

3.2.2. *Case Two.* We suppose that  $\overline{D} \not\subset B$ . To prove that  $I(B) = \infty$ , we will prove that for every  $\epsilon > 0$ ,  $I_\epsilon(B) = \infty$ . Let  $\epsilon > 0$  be fixed. We will construct a sequence of densities  $\varphi$  in  $L^2(\partial\Omega)$  creating the blowup for the functional (2.13). Indeed, we take a point  $a$  in  $\partial D \setminus \overline{B}$  and a sequence  $z_p \in \Omega \setminus (\overline{D} \cup \overline{B})$  such that  $z_p$  tends to  $a$ . We denote by  $E(z_p)$  a  $C^2$ -regular domain containing  $D$  and  $B$  such that zero is not a Dirichlet eigenvalue of  $\Delta + c(x)$  on  $E(z_p)$ ,  $z_p \in \Omega \setminus E(z_p)$  and  $\Omega \setminus \overline{E(z_p)}$  is connected. We start by the following lemma whose proof is given in the Appendix.

**Lemma 3.6.** 1. *The single layer potential  $S : L^2(\partial\Omega) \rightarrow L^2(\partial E(z_p))$  is compact, injective and has a dense range.*

2. If  $f \in \overline{\{v|_{\partial E(z_p)}, v \in H^1(E(z_p)) \text{ and } M_1 v = 0, \text{ in } E(z_p)\}}^{L^2(\partial E(z_p))}$ , then we can find a sequence  $\varphi_n \in L^2(\partial E(z_p))$  such that  $S(\varphi_n)$  tends to  $f$  in  $L^2(\partial E(z_p))$ . This property is valid without the eigenvalue condition on  $E(z_p)$ .

We consider the sequence of functions  $\Phi(\cdot, z_p)$ . By Lemma 3.6, for every  $p \in \mathbb{N}$  and  $\beta \in \mathbb{R}$ , we can find a sequence of functions  $\varphi_n^p(x) \in L^2(\partial\Omega)$  such that  $\|v[\varphi_n^p] - \beta\Phi(\cdot, z_p)\|_{L^2(\partial E(z_p))}$  tends to zero when  $n$  tends to  $\infty$ . We wrote  $\varphi_n^p$  independent of the parameter  $\beta$  since we will apply it for  $\beta$  taken as a function of  $p$ . The parameter  $\beta$  will be chosen to assure that  $\phi_n^p \in \mathbf{M}_e$ . We call this a *normalization* of the sequence  $\phi_n^p$ . For every  $(p, n) \in \mathbb{N}^2$ , we set  $f_n^p := v[\phi_n^p]$ . Then for every  $(p, n) \in \mathbb{N}^2$  fixed, we have:

$$\int_{\partial\Omega} J^{f_n^p}(x)\phi_n^p(x)dx = \int_{\partial D} \left( \frac{\partial u^{f_n^p}}{\partial v}(y)v[\varphi_n^p] - u^{f_n^p}(y)\frac{\partial v[\phi_n^p]}{\partial v(y)} \right) ds(y). \tag{3.12}$$

We set  $w_n^p := u^{f_n^p} - v[\varphi_n^p]$ . Hence  $w_n^p$  satisfies:

$$\begin{cases} M_\gamma w_n^p = -\nabla \cdot \chi_D A(x)\nabla v[\varphi_n^p] & \text{in } \Omega, \\ w_n^p = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

For  $p \in \mathbb{N}$  fixed, we obtain that  $v[\phi_n^p]$  tends to  $\beta\Phi(\cdot, z_p)$  on any subdomain of  $E(z_p)$  with the  $H^1$  norm. Indeed, from the fact that  $\|v[\varphi_n^p] - \beta\Phi(\cdot, z_p)\|_{L^2(\partial E(z_p))}$  tends to zero when  $n$  tends to  $\infty$  and since  $v[\varphi_n^p]$  and  $\beta\Phi(\cdot, z_p)$  satisfy the same equation in  $E(z_p)$ , the regularity of very weak solution for elliptic problems (see, Nečas, 1967) implies that  $v[\phi_n^p]$  tends to  $\beta\Phi(\cdot, z_p)$  in  $L^2(E(z_p))$ . Interior estimates give the convergence in any subdomain of  $E(z_p)$  with the  $H^1$  norm. Hence, the right hand side of (3.13) tends to  $-\beta\nabla \cdot \chi_D A\nabla\Phi(\cdot, z_p)$  in  $H^{-1}(\Omega)$ . From Lax–Milgram lemma we deduce that  $w_n^p$  is bounded in  $H^1(\Omega)$  and tends weakly to some  $w_\beta \in H^1(\Omega)$  which satisfies

$$\begin{cases} M_\gamma w_\beta = -\beta\nabla \cdot \chi_D A(x)\nabla\Phi(\cdot, z_p) & \text{in } \Omega, \\ w_\beta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Since  $D \subset E(z_p)$ , then from 3.12 we get

$$\int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x) = \int_{\partial D} \left\{ v[\varphi_n^p] \left[ \frac{\partial u^{f_n^p}}{\partial v} - \frac{\partial v[\varphi_n^p]}{\partial v} \right] + \left[ v[\varphi_n^p] - u^{f_n^p} \right] \frac{\partial v[\varphi_n^p]}{\partial v} \right\} ds(x)$$

tends to

$$\beta \int_{\partial D} \left\{ \Phi(x, z_p) \frac{\partial w_\beta}{\partial v}(x, z_p) - w_\beta(x, z_p) \frac{\partial \Phi}{\partial v}(x, z_p) \right\} ds(x). \tag{3.15}$$

From (2.2) and (3.14), we remark that  $w_\beta = \beta w$ .

I. Integration in  $\Omega \setminus \bar{D}$ : The pointwise version of the no-response test. Using the Green’s representation formula applied in  $\Omega \setminus \bar{D}$  and (3.15), we deduce that

$\int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x)$  tends to:

$$\beta w_\beta(z_p, z_p) - \beta \int_{\partial\Omega} \Phi(x, z_p) \frac{\partial w_\beta}{\partial \nu}(x, z_p) ds(x)$$

which is equal to

$$\beta^2 \left\{ I_{ss}(z_p) - \int_{\partial\Omega} \Phi(x, z_p) \frac{\partial w}{\partial \nu}(x, z_p) ds(x) \right\}. \tag{3.16}$$

II. Integration in D: The integral version of the no-response test. Arguing as in the point 1) of Theorem 2.1, recalling that  $\beta\Phi + w_\beta = \beta(\Phi + w)$ , we get obviously:

$$\begin{aligned} & \int_{\partial D} \left\{ \Phi(x, z_p) \frac{\partial w_\beta}{\partial \nu}(x, z_p) - w_\beta(x, z_p) \frac{\partial \Phi}{\partial \nu}(x, z_p) \right\} ds(x) \\ &= \int_D A(x) \nabla \Phi(x, z_p) \nabla(\beta\Phi + w_\beta)(x, z_p) dx. \end{aligned} \tag{3.17}$$

Hence from (3.15) and (3.17),  $\int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x)$  tends to

$$-\beta^2 I_{pb}(z_p). \tag{3.18}$$

Now we choose:

$$\beta := \beta(z_p, \epsilon) = \frac{\epsilon}{4} \left[ \max \left( \int_B |\Phi(x, z_p)|^2 dx, \int_B |\nabla_x \Phi(x, z_p)|^2 dx \right) \right]^{-1}.$$

With this choice, we have  $\|\beta\Phi(\cdot, z_p)\|_{H^1(B)} \leq \frac{\epsilon}{2}$ . Since  $\|v[\varphi_n^p] - \beta\Phi(\cdot, z_p)\|_{H^1(E(z_p))}$  tends to zero as  $n$  tends to  $\infty$ , for  $n$  large enough we obtain  $\|v[\varphi_n^p]\|_{H^1(B)} \leq \epsilon$ .

As a conclusion we have a sequence of functions  $\varphi_n^p$  such that for every fixed  $p \in \mathbb{N}$  there is  $N(p, \epsilon) \in \mathbb{N}$  such that for all  $n \geq N(p, \epsilon)$  we have

$$\|v[\varphi_n^p]\|_{H^1(B)} \leq \epsilon \quad \text{i.e., } v[\varphi_n^p] \in \mathbf{M}_\epsilon.$$

This sequence has the property: for  $p$  fixed,  $\int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x)$  tends to, see (3.18),

$$\beta^2 I_{pb}(z_p).$$

We remark that  $\beta := \beta(z_p, \epsilon)$  is bounded with respect to  $z_p$  since  $(z_p)_{p \in \mathbb{N}} \subset \subset \Omega \setminus \bar{B}$ . This remark and the point 2) of Theorem 2.1 imply that

$$\lim_{p, n \rightarrow \infty} \int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x) = \infty.$$

Hence  $I_\epsilon(B) = \infty$ .

We consider now the part 2). We take  $\beta = 1$  and the corresponding sequence  $\varphi_n^p$ , used to approximate the point source  $\Phi(\cdot, z_p)$  in  $L^2(\partial E(z_p))$ , then also from (3.18), we deduce that

$$\lim_{n \rightarrow \infty} \int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x) = I_{pb}(z_p)$$

and the justification of this part 2) is given by (2.7) applied for the probe functional  $I_{pb}(z)$ .  $\square$

**Remark 3.1.** In the last proof, we gave the two integrations by parts, one in  $D$  and the other in  $\Omega \setminus \bar{D}$ , to show how the blowup of the functional of the no-response test can be based either on the probe functional or on the singular sources functional. It is also remarkable that the testing domains  $E(z_p)$  used for both the singular sources and the probe methods are particular testing domains used for the no-response test.

#### 4. Proofs of Lemmas and Proposition 3.1

We start by proving the lemmas and then we prove Proposition 3.1.

*Proof of Lemma 3.1.* We have

$$M_\gamma(G - \tilde{G}) = 0 \quad \text{in } \Omega, \quad G - \tilde{G} = \Phi(\cdot, z) \quad \text{on } \partial\Omega. \tag{4.1}$$

The fundamental solution  $\Phi(\cdot, z)$  is bounded in any  $L^2(\tilde{\Omega})$ ,  $\bar{\Omega} \subset \tilde{\Omega}$ . For  $z \in B \subset\subset \Omega$ , we have  $(\Delta + c(x))\Phi(\cdot, z) = 0$  in  $\tilde{\Omega} \setminus \bar{B}$ . Hence  $\Phi(\cdot, z)$  is bounded in  $H^2_{loc}(\tilde{\Omega} \setminus \bar{B})$  with  $z \in B$ . From (4.1), by Lax-Milgram Lemma, we see that  $(G - \tilde{G})(\cdot, z)$  is bounded in  $L^2(\Omega)$ ,  $z \in B$ . Now we rewrite (4.1) as

$$L_\gamma(G - \tilde{G}) = -c(x)(G - \tilde{G}) \quad \text{in } \Omega, \quad G - \tilde{G} = \Phi(\cdot, z) \quad \text{on } \partial\Omega. \tag{4.2}$$

By Sobolev embedding theorem  $\Phi(\cdot, z)$  is bounded in  $C^\mu(\partial\Omega)$ ,  $0 < \mu < \frac{1}{2}$ , with  $z \in B$ . Finally from (4.2), we deduce that  $(G - \tilde{G})(x, z)$  is bounded in  $\Omega \times B$ , see Li and Vogelius (2000, Corollary 1.3) and the comments given later.  $\square$

*Proof of Lemma 3.2.* We have

$$L_\gamma(\tilde{G} - G') = -c(x)\tilde{G} \quad \text{in } \Omega, \quad \tilde{G} - G' = 0 \quad \text{in } \partial\Omega. \tag{4.3}$$

Using the Green's representation in  $\Omega$ , we write:

$$(\tilde{G} - G')(x, z) = \int_\Omega c(y)\tilde{G}(y, z)G'(y, x)dy. \tag{4.4}$$

We know that both of  $\tilde{G}$  and  $G'$  satisfy

$$|\tilde{G}(x, y)|, \quad |G'(x, y)| \leq C|x - y|^{-1}$$

with some positive constant  $C$ , see Littman et al. (1963). Hence from (4.4), we deduce that  $(\tilde{G} - G')$  is bounded in  $(x, z) \in \Omega^2$ . This ends the proof.  $\square$

*Proof of Lemma 3.3.* Let  $B \subset\subset \Omega$  be a fixed subdomain such that  $D \subset B$ . We set

$$R(x, z) := G'(x, z) - G'_{\gamma(a)}(x, z).$$

Then, for every  $z \in \Omega$  fixed,  $R(x, z)$  satisfies

$$\begin{cases} L_\gamma(R(x, z)) = -\nabla \cdot (\chi_D(\gamma - \gamma(a))\nabla G'_{\gamma(a)}) & \text{in } \Omega, \\ R(x, z) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Since  $\gamma$  is of class  $C^1$ , there exists  $c > 0$  such that

$$|\gamma(x) - \gamma(a)| \leq c|x - a| \quad \text{for } x \in D. \tag{4.6}$$

We recall that  $\Phi'_{\gamma(a)}(x, z)$  is the fundamental solution for the operator  $\nabla \cdot \gamma(a)\nabla$ . It is proven in Alessandrini and Di Cristo (2005, Proposition 3.2), that

$$|\nabla_x \Phi'_{\gamma(a)}(x, z)| \leq c|x - z|^{-2}. \tag{4.7}$$

$(x, z) \in (\mathbb{R}^3)^2$  with  $c > 0$  is a constant. In  $\Omega$ , we have  $\nabla \cdot \gamma(a)\nabla(\Phi'_{\gamma(a)} - G'_{\gamma(a)}) = 0$ . We know that both of  $\Phi'_{\gamma(a)}(\cdot, z)$  and  $G'_{\gamma(a)}(\cdot, z)$  are bounded, in  $L^2(\Omega)$  since  $|\Phi'_{\gamma(a)}(x, y)|, |G'_{\gamma(a)}(x, y)| \leq C|x - y|^{-1}$ . Using interior estimates for divergence form elliptic problems with piecewise regular coefficients (see Li and Vogelius, 2000, Theorem 1.1), we deduce that  $\Phi'_{\gamma(a)} - G'_{\gamma(a)}(x, z)$  is bounded for  $(x, z) \in B \times \Omega$ . From (4.7), we deduce that:

$$|\nabla_x G'_{\gamma(a)}(x, z)| \leq c(B)|x - z|^{-2}. \tag{4.8}$$

$(x, z) \in B \times \Omega$  with  $c(B) > 0$  is a constant depending only on  $B$ .

For  $\theta \in [0, \frac{\pi}{2})$  fixed, there exists a constant  $c(\theta) > 0$  such that for  $x \in D$  and  $z \in \Omega_{a,\theta}$ , we have

$$|x - a| \leq c(\theta)|x - z|. \tag{4.9}$$

Hence, using (4.6), (4.8), and (4.9), we get:

$$|\chi_D(\gamma(x) - \gamma(a))\nabla_x G'_{\gamma(a)}(x, z)| \leq c(B, \theta)|x - z|^{-1} \tag{4.10}$$

for every  $(x, y) \in \Omega \times \Omega_{a,\theta}$ . Then for every  $\epsilon > 0$ , we have the estimate

$$\int_\Omega |\chi_D(\gamma(x) - \gamma(a))\nabla_x G'_{\gamma(a)}(x, z)|^{3-\epsilon} dx \leq c(B, \theta) \int_D |x - z|^{-3+\epsilon} dx \leq c'(B, \theta), \tag{4.11}$$

for  $z \in \Omega_{a,\theta}$ .

We return now to (4.5). Using the  $L^p$  regularity of second order partial differential problems with piecewise regular coefficients, see Seftel (1963), we deduce that

$$\|\nabla R(\cdot, z)\|_{L^{3-\epsilon}(D)}^{3-\epsilon} \leq c''(B, \theta).$$

From (4.5), we write

$$R(x, z) = \int_D (\gamma(y) - \gamma(a))\nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y G'(x, y) dy, \quad (x, z) \in \Omega.$$

We rewrite it as

$$\begin{aligned}
 R(x, z) &= \int_D (\gamma(y) - \gamma(a)) \nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y G'_{\gamma(a)}(x, y) dy \\
 &\quad + \int_D (\gamma - \gamma(a)) \nabla_y G'_{\gamma(a)}(x, z) \cdot \nabla_y R(z, y) dy.
 \end{aligned}
 \tag{4.12}$$

Hence, the first part satisfies

$$\left| \int_D (\gamma(y) - \gamma(a)) \nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y G'_{\gamma(a)}(x, y) dy \right| \leq c \int_D |y - a| |y - z|^{-2} |x - y|^{-2} dy.
 \tag{4.13}$$

For  $x \in \bar{D}$  and  $z \in \Omega_{a,\theta}$ , arguing as in Alessandrini and Di Cristo (2005), p. 11 inequality (4.12), this last integral is bounded by  $\tilde{c} |\ln(|x - z|)|$  with some positive constant  $\tilde{c}$ . In Alessandrini and Di Cristo (2005), the authors took  $z$  on the normal  $\nu(a)$  but their proof still justified also for  $z \in \Omega_{a,\theta}$  since the critical point is the inequality (4.9). Using the inequalities  $|x - z| \leq d(x, \partial D) + d(z, \partial D)$  and  $|\ln(|x - z|)| \leq c|x - z|^{-t}$ , locally for every  $t > 0$ , we deduce that

$$\left| \int_D (\gamma(y) - \gamma(a)) \nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y G'_{\gamma(a)}(x, y) dy \right| \leq c(d(x, \partial D) + d(z, \partial D))^{-t}.
 \tag{4.14}$$

Let us now consider the term  $\int_D (\gamma(y) - \gamma(a)) \nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y R(x, y) dy$ . Since  $\nabla_x R(x, z)$  is bounded in  $(L^{3-\epsilon}(\Omega))^3$ , then by (4.8) and the Holder inequality we have

$$\begin{aligned}
 &\left| \int_D (\gamma(y) - \gamma(a)) \nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y R(x, y) dy \right| \\
 &\leq cc(B) \int_D |y - a| |z - y|^{-2} |\nabla_y R(x, y)| dy \\
 &\leq cc(B) \left[ \int_D [|y - a| |z - y|^{-2}]^{p'} dy \right]^{\frac{1}{p'}} \left[ \int_D |\nabla_y R(x, y)|^{3-\epsilon} dy \right]^{\frac{1}{3-\epsilon}},
 \end{aligned}
 \tag{4.15}$$

where  $p' := p'(\epsilon) = \frac{3-\epsilon}{2-\epsilon} > \frac{3}{2}$ . Using

$$\int_D [|y - a| |z - y|^{-2}]^{p'} dy \leq \tilde{c} \int_D |z - y|^{-2p'} dy \leq c'' \frac{2-\epsilon}{\epsilon} [d(z, \partial D)]^{-\frac{\epsilon}{2-\epsilon}},$$

we estimate

$$\left| \int_D (\gamma(y) - \gamma(a)) \nabla_y G'_{\gamma(a)}(z, y) \cdot \nabla_y R(x, y) dy \right| \leq c(B, \theta, \epsilon) [d(z, \partial D)]^{-\frac{\epsilon}{3-\epsilon}},$$

where  $c(B, \theta, \epsilon)$  depends on  $B, \theta$  and  $\epsilon$ .

This means that for  $x \in \bar{D}$  and  $z \in \Omega_{a,\theta} \cap B$ , we have the estimate

$$|R(x, z)| \leq c(B, \theta, \epsilon) [d(z, \partial D)]^{-\frac{\epsilon}{3-\epsilon}}.
 \tag{4.16}$$

Now, for  $x \in (\Omega \setminus \overline{D})$ ,  $R(x, z)$  satisfies  $\Delta_x R(x, z) = 0$  with the estimate

$$|R(x, z)| \leq c(B, \theta, \epsilon) [d(z, \partial D)]^{-\frac{\epsilon}{3-\epsilon}}$$

for  $x \in \partial D$  and  $R(x, z) = 0$  for  $x \in \partial \Omega$ . Then we have also (4.16) for  $x \in \Omega \setminus \overline{D}$ . All together, we have (4.16) for  $x \in \Omega$  and  $z \in \Omega_{a,\theta} \cap B$ .  $\square$

*Proof of Lemma 3.4.* To prove Lemma 3.4, it is enough to prove that

$$|(G'_{\gamma(a)} \circ T^{-1} - \Phi'_{\gamma(a)})(x, z)| \leq c(\delta, \theta) (d(z, \partial T(D)))^{-t}$$

for  $x \in T(\Omega)$  and  $z \in T(\Omega_{a,\theta} \cap B(a, \delta))$ . The set  $T(\Omega_{a,\theta})$  is the intersection of  $T(\Omega)$  and the cone with vertex at the origin  $O = (0, 0, 0)$ , the axis in the direction  $\eta(O) := (0, 0, 1)$  and angle  $\theta$ .

We proceed in several steps.

**First step.** Arguing as in Alessandrini and Di Cristo (2005, Proposition 3.2), we get the following estimate:

$$|(G'_{\gamma(a)} \circ T^{-1} - \Phi'_{\gamma(a)})(x, z)| \leq c(r) |\ln(|x - z|)| \tag{4.17}$$

for  $x \in \overline{T(D)} \cap B(0, \frac{r}{2})$  and  $z = t\eta(O)$  with  $t$  small enough such that  $z \in B(0, \frac{r}{2})$  with  $r > 0$  depending on  $\partial D$  via its parametrization.

Following their proof, we find that the same result is true by taking

$$x \in \overline{\mathbb{R}^3 \cup T(D)} \cap B(0, r) \quad \text{and} \quad z \in T(\Omega_{a,\theta}) \cap B\left(0, \frac{r}{2}\right),$$

i.e., the inequality (4.12) of Alessandrini and Di Cristo (2005) is valid for those points. This is due to the inequalities  $|x| \leq c(\theta)|x - z|$  and  $|\Psi(x)| \leq c'(\theta)|\Psi(x) - \Psi(z)|$ , where  $c(\theta)$  and  $c'(\theta)$  depend only on  $\theta$ , which are satisfied for these points. The second inequality is a consequence of the first one and the form of the change of variables, we denote here by  $\Psi$ , introduced to flatten  $\partial T(\Omega)$  near the point  $O$ . We omit the details here to avoid complicating more the exposition. The argument behind is that  $\overline{\mathbb{R}^3 \cup T(D)}$  and  $T(\Omega_{a,\theta})$  are separated near the point  $O$  in the sense that  $\overline{\mathbb{R}^3 \cup T(D)} \cap \overline{T(\Omega_{a,\theta})} \cap B(a, r) = \{T(a) = O\}$  for some  $r > 0$  small enough.

Hence as for the proof of Lemma 3.3, we get

$$\begin{aligned} |(G'_{\gamma(a)} \circ T^{-1} - \Phi'_{\gamma(a)})(x, z)| &\leq c(r, \theta) |\ln(|x - z|)| \\ &\leq c(r, \theta) (d(x, \partial T(D)) + d(z, \partial T(D)))^{-t} \end{aligned} \tag{4.18}$$

for every  $t > 0$ ,  $x \in \overline{\mathbb{R}^3 \cup T(D)} \cap B(0, r)$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ .

Remark that  $\overline{\mathbb{R}^3 \cup T(D)} \cap B(0, r) = \overline{\mathbb{R}^3 \cap T(\Omega)} \cap B(0, r)$ , for  $r$  small enough.

**Second step.** Now, we show that (4.18) is also valid for  $x \in \mathbb{R}^3_+ \cap T(\Omega) \cap B(0, r)$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ . To do so we observe that on  $\mathbb{R}^3_+ \cap T(\Omega) \cap B(0, r)$ , we have  $F(x, z) := G'_{\gamma(a)} \circ T^{-1}(x, z) - \Phi'_{\gamma(a)}(x, z)$  satisfies  $\Delta F = 0$  with uniformly bounded boundary conditions. The uniform boundedness of the boundary

conditions is justified by:

- 1) the fact that  $|F(x, z)| \leq c(r, \theta)[d(z, \partial T(D))]^{-t}$ , for  $x$  on the part of the boundary of  $\mathbb{R}^3_+ \cap T(\Omega) \cap B(0, r)$  located on  $\partial\mathbb{R}^3_+$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ , which can be deduced from (4.18), and that
- 2) both of  $G'_{\gamma(a)} \circ T^{-1}(x, z)$  and  $\Phi'_{\gamma(a)}(x, z)$  are bounded for  $x$  in  $\partial B(0, r)$  and  $z \in B(0, \frac{r}{2})$ .

The second point is justified since for  $x \in T(\Omega) \setminus B(0, \frac{2}{3}r)$  and  $z \in B(0, \frac{r}{2})$ , both of  $G'_{\gamma(a)} \circ T^{-1}(x, z)$  and  $\Phi'_{\gamma(a)}(x, z)$  satisfy divergence form elliptic equation with discontinuous coefficient and homogenous second member. Indeed, since  $G'_{\gamma(a)} \circ T^{-1}(\cdot, z)$  and  $\Phi'_{\gamma(a)}(\cdot, z)$  are bounded in  $L^2(T(\Omega))$  and their traces on  $\partial(T(\Omega) \setminus \overline{B(0, \frac{2}{3}r)})$  are bounded in  $C^2(T(\Omega) \setminus \overline{B(0, \frac{2}{3}r)})$  since  $z$  is in  $B(0, \frac{r}{2})$ . Then Theorem 1.2 of Li and Vogelius (2000) implies that they are bounded in the space  $C(\overline{T(\Omega) \setminus (T(\partial D) \cup \overline{B(0, \frac{2}{3}r)})})$ . This implies 2) since  $\partial B(0, r) \subset T(\Omega) \setminus (T(\partial D) \cup \overline{B(0, \frac{2}{3}r)})$ .

We deduce that  $|F(x, z)| \leq c(r, \theta)[d(z, \partial T(D))]^{-t}$  for  $x \in \mathbb{R}^3_+ \cap T(\Omega) \cap B(0, r)$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ .

**Third step.** Taking all together, we showed that  $F(x, z)$  is bounded by  $c(r, \theta)[d(z, \partial T(D))]^{-t}$  for  $x \in T(\Omega) \cap B(0, r)$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ . Since for  $x \in T(\Omega) \setminus B(0, r)$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$  both of  $G'_{\gamma(a)} \circ T^{-1}(x, z)$  and  $\Phi'_{\gamma(a)}(x, z)$  are bounded then we get (4.18) for  $x \in T(\Omega)$  and  $z \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ . To finish the proof we take  $\delta = \frac{r}{2}$ . This ends the proof.  $\square$

*Proof of Lemma 3.5.* The distribution  $\Psi(x, z) := \Phi(x, z) - \frac{4\pi}{|x-z|}$  satisfies

$$\begin{cases} (\Delta_x + c(x))\Psi = -c(x)\frac{(4\pi)^{-1}}{|x-z|} & x \text{ in } \tilde{\Omega}, \\ \Psi(x, z) = \Phi(x, z) - \frac{(4\pi)^{-1}}{|x-z|} & x \text{ on } \partial\tilde{\Omega}, \end{cases} \tag{4.19}$$

where we took  $\tilde{\Omega}$  as a  $C^\infty$  domain such that  $\Omega \subset\subset \tilde{\Omega}$  and (4.19) is well posed, i.e., zero is not a Dirichlet eigenvalue. This last property is possible since the eigenvalues are strictly monotonic with respect to the variation of  $\tilde{\Omega}$ . We have  $\Psi(x, z) \in C^\infty(\partial\tilde{\Omega} \times \Omega)$ , since  $\Phi(x, z)$  and  $\frac{1}{|x-z|}$  are, because  $c(x) = 1$  in  $\tilde{\Omega} \setminus \Omega$ . The function  $c(x)\frac{(4\pi)^{-1}}{|x-z|}$  is bounded in  $L^2(\tilde{\Omega})$  for  $z$  in  $\Omega$ . Hence from the problem (4.19),  $\Psi(x, z)$  is bounded in  $\tilde{\Omega} \times \Omega$ .  $\square$

*Proof of Proposition 3.1.* To show that

$$\int_{\partial\Omega} \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) ds(x)$$

is bounded with respect to  $z$ , we investigate the term  $\frac{\partial w}{\partial \nu}(\cdot, z)$ . Near  $\partial\Omega$  the function  $w$  satisfies  $\Delta w + c(x)w = 0$ . We recall that  $w = G - \Phi$ . Hence  $w(\cdot, z)$  is also bounded in  $L^2(\Omega)$  since  $|G(x, y)|, |\Phi(x, y)| \leq C|x-y|^{-1}$ . By interior estimates we deduce that  $w(\cdot, z)$  is bounded in  $H^1(B)$  for every  $B \subset\subset (\Omega \setminus \overline{D})$ . We take now  $B$  as a corona surrounding  $D$  and denote by  $\partial B$  the exterior part of the boundary

of  $B$ . Then solving the problem  $M_\gamma w = 0$  in  $\Omega_B$ , where  $\Omega_B$  is the domain limited by  $\partial\Omega$  and  $\partial B$ , with Dirichlet condition on  $\partial\Omega \cup \partial B$  bounded in  $H^{\frac{1}{2}}(\partial\Omega \cap \partial B)$ , we deduce that  $\frac{\partial w}{\partial \nu}(\cdot, z)$  is bounded in  $H^{-\frac{1}{2}}(\partial\Omega)$ . Also by interior estimates and the trace theorem, it is easy to see that  $\Phi(\cdot, z)$  is bounded in  $H^{\frac{1}{2}}(\partial\Omega)$  with respect to  $z$  away from  $\partial\Omega$ . This implies that  $\int_{\partial\Omega} \Phi(x, z) \frac{\partial w}{\partial \nu}(x, z) ds(x)$  is bounded with respect to  $z$  away from  $\partial\Omega$ .  $\square$

## 5. Appendix

### 5.1. Justification of the Green and Jumps Formulas for the Equation $\Delta + c(x)$ Where $c(x)$ Is a Bounded Function

In this Appendix, we justify the Green and jump formulas for the equation  $\Delta + c(x)$  where  $c(x)$  is a bounded function. In the case where  $c(x)$  is continuous these results are known, see Isakov (1990). We refer to Mitrea and Taylor (2000) and the references there for related results. We explain one way how to justify them for the case where coefficient  $c(x)$  is bounded. We assume that  $c(x) = 1$  for  $x \in \mathbb{R}^n \setminus \Omega$ . The following argument is true in  $\mathbb{R}^n$ ,  $n = 2, 3$ . We give the details for  $n = 3$ . As in the proof of Lemma 4.19, let us consider the distribution  $\Psi(x, z) := \Phi(x, z) - \frac{(4\pi)^{-1}}{|x-z|}$ . Then  $\Psi(\cdot, z)$  satisfies

$$\begin{cases} (\Delta_x + c(x))\Psi = -c(x) \frac{(4\pi)^{-1}}{|x-z|} & x \text{ in } \tilde{\Omega}, \\ \Psi(x, z) = \Phi(x, z) - \frac{(4\pi)^{-1}}{|x-z|} & x \text{ on } \partial\tilde{\Omega}, \end{cases} \tag{5.1}$$

where  $\Omega \subset\subset \tilde{\Omega}$  and  $\tilde{\Omega}$  is a  $C^\infty$  domain and (5.1) is well posed. For  $z \in \bar{\Omega}$  fixed, the right hand side of (5.1) is in  $L^2(\tilde{\Omega})$  and a direct computation shows that its  $L^2(\tilde{\Omega})$ -norm is continuous with respect to  $z \in \bar{\Omega}$ . Also  $\Psi(x, z)$  is in  $C^\infty(\partial\tilde{\Omega} \times \bar{\Omega})$  since  $\Phi(x, z)$  and  $|x-z|^{-1}$  are. By the well posedness of the problem (5.1), we deduce that  $\Psi$  is continuous with respect to  $z \in \bar{\Omega}$  with values in  $H^2(\tilde{\Omega})$ .

1) To prove the Green's formula, it is enough to prove that

$$\lim_{r \rightarrow 0} \int_{\partial B(z,r)} \frac{\partial u}{\partial \nu}(x) \Phi(x, z) - u(x) \frac{\partial \Phi}{\partial \nu}(x, z) ds(x) = u(z), \quad (r := |x-z|).$$

We recall the notation  $\Phi'(x, z) = \frac{(4\pi)^{-1}}{|x-z|}$ . We write

$$\begin{aligned} & \int_{\partial B(z,r)} \frac{\partial u}{\partial \nu}(x) \Phi(x, z) - u(x) \frac{\partial \Phi}{\partial \nu}(x, z) ds(x) \\ &= - \int_{\partial B(z,r)} u(x) \frac{\partial \Phi'}{\partial \nu}(x, z) ds(x) + \int_{\partial B(z,r)} \frac{\partial u}{\partial \nu}(x) \Phi(x, z) - u(x) \frac{\partial \Psi}{\partial \nu}(x, z) ds(x). \end{aligned} \tag{5.2}$$

Now, since  $u$  satisfies

$$(\Delta + c(x))u(x) = 0 \quad \text{in } \Omega,$$

then  $u \in H^2_{loc}(\Omega)$ , hence  $u(x)$  is continuous in  $\Omega$  by Sobolev embedding theorem in  $\mathbb{R}^3$ . Since  $u$  is continuous and  $\frac{\partial \Phi'}{\partial \nu} |_{\partial B(z,r)} = -\frac{1}{4\pi r^2}$ , then using the mean value theorem we deduce that the first term is tending to  $u(z)$  as  $r$  tends to zero.

Using the Cauchy–Schwartz inequality, the fact that  $\Phi(x, z) = \frac{1}{4\pi r}$  and  $u$  and  $\Psi(\cdot, z)$  are in  $H^2(\Omega)$  for every  $z \in \Omega$ , we get

$$\begin{aligned} & \left| \int_{\partial B(z,r)} u(x) \frac{\partial \Psi}{\partial \nu}(x, z) - \frac{\partial u}{\partial \nu}(x) \Phi(x, z) ds(x) \right| \\ & \leq \int_{\partial B(z,r)} |u(x)|^2 ds(x) \int_{\partial B(z,r)} \left| \frac{\partial \Psi}{\partial \nu}(x, z) \right|^2 ds(x) \\ & \quad + \frac{1}{4\pi r} \int_{\partial B(z,r)} \left| \frac{\partial u}{\partial \nu}(x) \right|^2 ds(x) \text{meas}(\partial B(z, r)). \end{aligned} \tag{5.3}$$

The left-hand side tends to zero as  $r$  tends to zero. This ends the proof of the point 1.

For the case where the dimension  $n > 3$  we need the continuity of the coefficient  $c(x)$ , in which case the solution of  $(\Delta + c(x))u(x) = 0$  in  $\Omega$  is continuous.

2) To justify the jump formula, we also use the decomposition  $\Phi(x, z) = \Psi(x, z) + \Phi'(x, z)$ . Then we have

$$\int_{\partial \Omega} \frac{\partial \Phi}{\partial \nu}(x, z) \varphi(x) ds(x) = \int_{\partial \Omega} \frac{\partial \Phi'}{\partial \nu}(x, z) \varphi(x) ds(x) + \int_{\partial \Omega} \frac{\partial \Psi}{\partial \nu}(x, y) \varphi(x) ds(x) \tag{5.4}$$

where  $z \in \Omega$ . Now, letting  $z$  tend to  $\partial \Omega$ , using the jump formula for the Green’s function  $\Phi'(x, z)$  we get the jump from the first term of the right hand side of (5.4). The second term is continuous with respect to  $z \in \bar{\Omega}$ . This is due to the continuity of  $\Psi(\cdot, z)$  with respect to  $z$  with values in  $H^2(\Omega)$ , the continuity of the trace theorem and an application of the Cauchy–Schwartz inequality. We deduce the desired jump formula for the Green’s function  $\Phi(x, z)$ .

**5.2. Proof of Lemma 3.6**

1. The integral operator  $S$  is compact since its kernel  $\Phi(x, y)$ ,  $x \in \partial \Omega$  and  $y \in \partial E(z_p)$ , is bounded. Let us prove its injectivity. Let  $\varphi$  in  $L^2(\partial \Omega)$  such that  $S(\varphi) = 0$ . Abusing the notation, we denote  $U(x) := S(\varphi)(x)$  for  $x \in \mathbb{R}^3$ . Since zero is not a Dirichlet eigenvalue of  $\Delta + c(x)$  on  $E(z_p)$  and that  $(\Delta + c(x))U(x) = 0$  in  $\Omega$  and hence in  $E(z_p)$ , we deduce that  $U(x) = 0$  for every  $x$  in  $E(z_p)$ . By unique continuation,  $U(x) = 0$  in  $\Omega$ . We have  $(\Delta + 1)U(x) = 0$  in  $\mathbb{R}^3 \setminus \bar{\Omega}$ ,  $U(x)$  is continuous across  $\partial \Omega$  and it satisfies the Sommerfeld Radiation condition, then we get  $U(x) = 0$  in  $\mathbb{R}^3$ . Using the jumps formulas of the normal derivative of the single layer potential we deduce that  $\varphi = 0$ .

We consider now the denseness of the range of  $S$ . The adjoint of  $S$  is given by the operator,  $S^* : L^2(\partial E(z_p)) \rightarrow L^2(\partial \Omega)$  where  $S^*(\varphi)(y) := \int_{\partial E(z_p)} \Phi(x, y) \varphi(y) ds(x)$ . We write

$$L^2(\partial E(z_p)) = \overline{R(S)} \oplus N(S^*),$$

where

$$N(S^*) := \{\varphi \in L^2(\partial E(z_p)) : S^*(\varphi) = 0\}.$$

It is enough to prove that  $N(S^*) = \emptyset$ . For this, we have just to mimic the previous arguments applied for  $S$  in a reverse order starting by the exterior problem on  $\mathbb{R}^3 \setminus \Omega$ , the unique continuation, the use of the fact that zero is not a Dirichlet eigenvalue of  $(\Delta + c(x))$  on  $E(z_p)$  and then we conclude by the jumps formulas across  $\partial E(z_p)$ .

2. We start by taking  $v \in H^1(E(z_p))$  satisfying  $(\Delta + c(x))v = 0$  in  $E(z_p)$  and prove that  $v|_{\partial E(z_p)}$  is in  $N(S^*)^\perp$ . Let  $\varphi \in N(S^*)$ , then  $S^*\varphi = 0$ . We set  $V(x) := S^*\varphi(x)$  for  $x \in \mathbb{R}^3$  and argue as above we get  $V(x) = 0$  in  $\mathbb{R}^3 \setminus \Omega$  and hence in  $\mathbb{R}^3 \setminus E(z_p)$ . Since  $V$  and  $v$  satisfy  $(\Delta + c(x))V = 0 = (\Delta + c(x))v$  in  $E(z_p)$ , an integration by parts give

$$\int_{\partial E(z_p)} \frac{\partial v}{\partial \nu} V \, ds(x) = \int_{\partial E(z_p)} \left( \frac{\partial V}{\partial \nu} \right)_- v \, ds(x),$$

where  $\left( \frac{\partial V}{\partial \nu} \right)_-$  is given by the limit from inside of the derivative. Since  $V(x) = 0$  on  $\partial E(z_p)$ , then  $\int_{\partial E(z_p)} \left( \frac{\partial V}{\partial \nu} \right)_- v \, ds(x) = 0$ . From the jumps formulas and the fact that  $V(x) = 0$  for  $x$  in  $\mathbb{R}^3 \setminus E(z_p)$ , we get

$$\left( \frac{\partial V}{\partial \nu} \right)_- = -\varphi.$$

Hence  $\int_{\partial E(z_p)} \varphi v \, ds(x) = 0$ . We end the proof by an approximation argument.  $\square$

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