1 Chapter VIII. Newtonian fluids. The Navier-Stokes model.

1.1 Preparations

We start by some preparations. Let ξ be the determinant's function defined on the set of invertible matrices, i.e. for every invertible matrix A, we set $\xi(A) := \det A$.

Lemma 1.1 *The function ξ is differentiable and its differential satisfies:*

$$D\xi(A)[U] = (\det A)tr(UA^{-1})$$

where $D\xi(A)[U]$ means the differential of ξ at the point A in the direction given by the matrix U.

Proof of Lemma 1.1

We know that

$$\det(S - \omega I) = -\omega^3 + i_1(S)\omega^2 - i_2(S)\omega + i_3(S)$$

where, we recall that, $i_1(S) = trS$, $i_2(S) = \frac{1}{2}[(trS)^2 - tr(S^2)]$ and $i_3(S) = \det S$.

If we take $\omega = -1$, then we can write

$$\det(S - \omega I) = 1 + trS + o(S), \text{ for } S \to 0$$

since $i_2(S) = O(S^2)$ and $i_3(S) = O(S^3)$, for $S \to 0$.

Hence if A is invertible then, for every matrix U, we have

$$\det(A+U) = det[(I+UA^{-1})A] = \det A \det(I+UA^{-1}) = \det A[I+tr(UA^{-1})+o(U)), \text{ as } U \to 0$$

and then

$$\det(A+U) = \det A + \det A\, tr(UA^{-1}) + o(U)), \text{ as } U \to 0.$$

Since the mapping, $U \to \det A\, tr(UA^{-1})$ is linear, then it must be the differential of ξ at A in the direction U. □

We recall that $U(x,t) := \varphi(x_R,t) - x_R, \forall (x,t) \in \mathcal{T}$, i.e. $x = x(x_R,t) = \varphi(x_R,t)$. We write $U(x,t) = U(x(x_R,t),t)$, then we have the following properties.

- $\nabla_{x_R} U(x,t) = \nabla_x \cdot \nabla_{x_R}\varphi$.
- Recalling that $v(x,t) := \frac{\partial}{\partial t}\varphi(x_R,t)$, we have

$$\frac{\partial}{\partial t}\nabla_{x_R}\varphi(x_R,t) = \nabla_{x_R}\frac{\partial}{\partial t}\varphi(x_R,t) = \nabla_{x_R}v(x,t) = \nabla_x v \cdot \nabla_{x_R}\varphi.$$

1.2 Isochoric motions

Let $\varphi(x_R, t)$ be a motion. Given any part P of G_R, we know that

$$Vol(P_t) := \int_{P_t} dV, \text{ where } P_t := \varphi(P, t).$$

As we have seen, we can write

$$Vol(P_t) = \int_P \det \nabla\varphi dV$$

thus $\frac{d}{dt}(Vol(P_t)) = \int_P \frac{d}{dt}(\det \nabla\varphi(x_R, t))dV$. However, from the chain rule, we have $\frac{d}{dt}f(g(t)) = Df(g(t))[\frac{d}{dt}g(t)]$, then

$$\frac{d}{dt}(\det \nabla\varphi) = D(\det)(\nabla\varphi)[\frac{d}{dt}\nabla\varphi] = \det(\nabla\varphi)tr((\frac{d}{dt}\nabla\varphi)(\nabla\varphi)^{-1}).$$

Hence

$$\frac{d}{dt}(\det \nabla\varphi) = \det(\nabla\varphi)tr(\nabla_x v).$$

Since $tr(\nabla_x v) = div\ v$, then $\frac{d}{dt}(\det \nabla\varphi) = (\det(\nabla\varphi))div\ v(x, t)$. This gives the relation

$$\frac{d}{dt}Vol(P_t) = \int_P det\nabla\varphi\ div\ v(x, t)dV = \int_{P_t} div\ v(x, t)dV. \tag{1}$$

DEFINITION 1.2 *We say that the motion $\varphi(x_R, t)$ is isochoric, or preserves the volumes, if for every par P_t, we have $\frac{d}{dt}Vol(P_t) = 0$.*

From (1), we obtain the following characterization of isochoric deformations.

THEOREM 1.3 *A deformation is isochoric if and only if $div\ v(x, t) = 0$.*

1.3 Newtonian fluids

As we have seen in the last chapters, the elastic bodies are characterized by the dependency of the Cauchy's stress tensor T on φ via its gradient deformation $\nabla\varphi$, i.e. $T(x_R, t) := \hat{T}(x_R, t, \nabla\varphi(x_R, t))$.

In case of fluids, a measure of the relative motion is furnished by the velocity gradient, i.e.

$$\nabla_x v(x, t) \ \ (= \nabla(\frac{\partial}{\partial t}\varphi)(x, t)).$$

- For fluids, $T(x_R(x, t), t) = \hat{T}(x_R(x, t), t, \nabla_x v(x, t))$, i.e. T depends on the point $(x, t) \in \mathcal{T}$ and on $\nabla_x v$ on it.
- In addition, for fluids, the divergence equation of the Cauchy theorem in chapter VII is stated in the (x, t) variables and not in the (x_R, t) variables. In other words, for the fluids, we work in the Eulerian coordinates instead of the Lagrangian coordinates (as we did it for the elasticity).

In this chapter, we will study a particular type of fluids called the Newtonian fluids.

Definition 1.4 *A Newtonian fluid is a fluid for which the Cauchy's stress tensor T is given by*

$$T := -\pi I + 2\mu_0(\nabla v + (\nabla v)^T).$$

In addition, we assume that the Newtonian fluid is *incompressible* which means that the deformation is isochoric. Hence we assume that

$$div\ v(x,t) = 0.$$

Finally, we assume also that the density ρ and the viscosity μ_0 are constants, i.e. the material is homogeneous.

Hence, in the following sections, we will be concerned with *homogeneous and incompressible Newtonian fluids.*

1.4 The Navier-Stokes model

Let us recall the equation of motion

$$\rho\dot{v} = div\ T + \rho b$$

where $\dot{v}(x,t) = \frac{\partial}{\partial t}v(x,t) + (\nabla v)\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}v(x,t) + (\nabla v)v$.

Since $[div(\nabla v)^T]_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}\sum_{J=1}^{3}\frac{\partial v_j}{\partial x_j} = [\nabla(div\ v)]_i$, then $div(\nabla v)^T = \nabla(div\ v)$. Using this identity, we can write:

$$div\ T = -\nabla\pi_0 + 2\mu_0(div\nabla v + \nabla(div\ v)) = -\nabla\pi_0 + 2\mu_0\Delta v,$$

since $div\ v = 0$ where $\Delta v = div\ \nabla v$ is the laplacian. Thus we get the following equations

$$\begin{cases} \rho[v' + (\nabla v)v] = \mu_0\Delta v - \nabla\pi_0 + \rho b \text{ in } G_t \\ div\ v = 0, \text{ in } G_t. \end{cases} \tag{2}$$

These relations are called the *Navier-Stokes equations.* If we define the new notations $\mu := \frac{\mu_0}{\rho}$ and $\pi := \frac{\pi_0}{\rho}$, we can write the Navier-Stokes equations in the form:

$$\begin{cases} v' + (\nabla v)v = \mu\Delta v - \nabla\pi + b \text{ in } G_t \\ div\ v = 0, \text{ in } G_t. \end{cases} \tag{3}$$

If we neglect the non-linear term $(\nabla v)v$, then we obtain

$$\begin{cases} v' = \mu\Delta v - \nabla\pi + b \text{ in } G_t \\ div\ v = 0, \text{ in } G_t. \end{cases} \tag{4}$$

which are called the *Stokes equations.*

We recall that the unknown for our problem is the couple (v, π).

For these homogeneous and incompressible Newtonian fluids, we add the restriction that the fluid adhere, without 'slipping', to the boundary. This is formulated mathematically by the boundary condition

$$v = 0, \text{ on } \partial G_t = \partial G \times (0, L). \tag{5}$$

We assume, in addition, that the initial velocity is given by v_0, i.e.

$$v(0,x) = v_0(x) \text{ in } (0,L). \tag{6}$$

The object of the next chapter is to study mathematically the initial-boundary value problem (3) – (5) – (6).