

The 3D acoustic scattering by complex obstacles. The accuracy issue.

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Abstract. In this paper, we are concerned with the identification of complex obstacles from the scattering data for the 3D acoustic problem. We focus mainly on the question of the accuracy of the reconstruction. Our approach is based on the asymptotic expansion of the indicator functions generated by multipolar sources. We found out that the second term in the expansion is given by the Mean curvature of the surface of the obstacle instead of its Gaussian curvature. This shows how the 3D inverse scattering is more complicated and richer than the 2D one where the curvature, which appears also in the second order term of the corresponding expansion, characterize completely the (strict convexity of the) shape in contrast to the Mean curvature in the 3D Case. We discuss in more details the effects of the geometry and the surface impedance on the reconstruction accuracy and we present extensive numerical examples to support this discussion.

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1. Introduction

Let D be a bounded domain of \mathbb{R}^3 , such that $\mathbb{R}^3 \setminus \overline{D}$ is connected. In addition, we assume that its boundary ∂D is of class C^3 . The propagation of time-harmonic acoustic fields in a homogeneous medium is governed by the *Helmholtz equation*

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (1)$$

where κ is the real positive *wave number*. On the boundary of the scatterers we assume that the total field u satisfies the *impedance boundary condition*

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D. \quad (2)$$

The unit normal ν on ∂D is directed outside of D . We assume that λ is a Lipschitz continuous complex valued function defined on ∂D with a positive real part. The obstacle D is characterized by its shape ∂D and the surface impedance λ distributed on ∂D .

Given an incident field u^i which satisfies $\Delta u^i + \kappa^2 u^i = 0$ we look for a solution $u := u^i + u^s$ of (1) and (2) where the *scattered field* u^s is assumed to satisfy the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - i\kappa u^s \right) = 0, \quad (3)$$

where the limit is uniform with respect to all directions.

Under these assumptions, the exterior impedance problem (1)-(3) is well posed both on classical spaces and on Sobolev spaces (see [6] and [3]). Furthermore, u^s has the asymptotic behaviour of an outgoing spherical wave,

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|} u^\infty(\hat{x}) + O(|x|^{-2}),$$

as $|x| \rightarrow \infty$ uniformly in all directions $\hat{x} = \frac{x}{|x|}$. The function u^∞ is called the far-field pattern of u^s and it is defined on the unit sphere \mathbb{S}^2 .

Taking particular incident fields given by the plane waves, $u^i(x, d) := e^{i\kappa d \cdot x}$ with direction $d \in \mathbb{S}^2$, we define the far-field pattern $u^\infty(\hat{x}, d)$ for $(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2$. The problem we are concerned with is the following

Obstacle reconstruction problem. *Given $u^\infty(\cdot, \cdot)$ on $\mathbb{S}^2 \times \mathbb{S}^2$ for the scattering problem (1)-(3) reconstruct the shape of the obstacle D and the surface impedance λ .*

There are several algorithms proposed to solve this problem. We can divide them into iterative methods and non-iterative methods. We can classify the non-iterative methods into two families (see [19]). The sampling methods are known as the linear sampling of Colton-Kirsh [5], the factorization method of Kirsch [12], the reciprocity gap method of Colton-Haddar [4] and the music algorithms by Devaney [7]. The probing methods are the probe or singular sources method by Ikehata [10]-Potthast [18] (see also [13] and [9] about the equality of these two methods) and the enclosure method by Ikehata [11]. One of the common features of most of these methods is that they

are all based on building indicator functions by using the fundamental solution or a parameter dependent complex geometrical optics solutions of the background to create singularities which allow the detection of the unknown surfaces (or interfaces). So, these methods capture the unknown interface as the surface where the corresponding indicators change drastically the behaviour from bounded values to unbounded values. In addition, in contrast to the iterative methods, these non-iterative methods do not need to know the type of the boundary conditions nor if the obstacle is penetrable or not. However, in a practical level and for numerical purposes, the quality of blowup of the indicator function is the key point. It happens that the (unknown) geometry and the (unknown) distributed material (λ in our case) play a key role in how these indicator functions detect the surface. In recent works ([15] and [22]) we started to study these issues for the bidimensional setting by taking as a pilot method the probing method by Ikehata-Potthast and the linear sampling method by Colton-Kirsch. Our approach is to provide the asymptotic expansion of the indicator functions with respect to the used point sources. We showed, in particular, how the curvature of the surface as well as the surface impedance appear in the second term of the asymptotic behaviour. This explains why the reconstructions of non-convex shapes are less accurate than the ones of convex shapes. In addition, we showed how, even for a very easy shape as a circle, we cannot obtain good reconstructions if the surface impedance is oscillating. Another interesting point is that we can use this surface material to build up interfaces coated in such a way that their reconstruction from exterior measurements is more (or less, if needed) accurate.

The probing methods have been applied to reconstruct 3D obstacles in [2]. The task was to investigate the 'numerical feasibility' of the singular sources method (SSM) to identify impenetrable 3D objects. The crucial ingredient of the algorithm is the singular behaviour of the scattered field $\Phi^s(y, z)$ of an incident point source $\Phi(\cdot, z)$ in a point y on the boundary of the scatterer when the source point z tends to y . Here, Φ denotes the fundamental solution of the Helmholtz equation. We define the *indicator function* of the method as $\Phi^s(z, z)$, which is evaluated in a particular admissibility region. We reconstruct the scattered field $\Phi^s(\cdot, z)$ in the source point z using the point source method ([17], [1]) in a multiwave setting, i.e. when the far-field pattern $u^\infty(\cdot, d)$ is measured for many incident plane waves $u^i(\cdot, d)$. Then, the boundary of the obstacle is determined as the set of points where the indicator function becomes large. In our work, we focus mainly on the 'accuracy issue'. As we said above, this issue has been studied for the 2D case. In this case, it is known that the curvature gives the complete characterization of (the strict convexity of) the surface of the obstacle. For the 3D case, the Gaussian curvature is the one which characterizes the convexity of the shape. We derive the asymptotic expansion of the indicator function and show that the second order term involves the Mean curvature instead of the Gaussian curvature. This shows how the reconstruction issue in the 3D case is more complicated and richer than the 2D case. Indeed, if both the sought shape and the surface impedance are uniform the reconstruction accuracy is high. However, if only one of the values varies, either

the surface impedance oscillates on the uniform surface ∂D or the surface impedance is uniform but defined over a nonuniform surface, we observe a deterioration of the accuracy. We also show that the reconstruction quality improves significantly if the impedance function is uniform with a large imaginary part as compare to the Mean curvature.

For the numerical experiments we generated far field data via solving the direct scattering problem. Using a single-layer potential approach the direct problem is reduced to a boundary integral equation of the second kind. For the surfaces homeomorphic to the unit sphere the boundary integral equation can be solved via a Nyström-type method, introduced by Wienert [23] which is based on spherical harmonics and on the transformation of the boundary surface to a sphere. The method is exponentially convergent for analytic boundaries. For the efficient implementation of the algorithm we used the numerical scheme suggested by Ganesh and Graham [8] which is proved to be equivalent to the Wienert's method.

This paper is organized as follows. In section 2 we present the main theoretical result, that is the asymptotic expansions of the indicator functions. Section 3 is devoted to the proof of the main theorem and auxiliary lemmas. Finally, in section 4 we present the numerical examples supporting the theoretical analysis on the accuracy issue.

2. The main theoretical result

The 'free space fundamental solution' of the Helmholtz equation in 3D is given by

$$\Phi(x, z) := \frac{1}{4\pi} \frac{e^{i\kappa|x-z|}}{|x-z|}, \quad x, z \in \mathbb{R}^3, \quad x \neq z. \quad (4)$$

The incident point-source $\Phi(\cdot, z)$, with source point $z \in \mathbb{R}^3$, is of special interest in this work. We denote with $\Phi_{0,\lambda}^s(\cdot, z)$ the scattered field caused by an incident point source for the scattering problem (1)-(3). We also set $\Phi_{j,\lambda}^s := (\frac{\partial \Phi}{\partial z_j})^s$, $1 \leq j \leq 3$, the scattered field associated with the incident field $\frac{\partial \Phi}{\partial z_j}$.

We start by recalling the following result on the lower and upper estimates of the functions $\Phi_{j,\lambda}^s$, $0 \leq j \leq 3$.

THEOREM 2.1 *There exist constants $\tau > 0$, $c > 0$ and $C > 0$ such that the lower estimate $|\Phi_{0,\lambda}^s(z, z)| \geq \frac{c}{|d(z,D)|}$ holds in the tube $0 < d(z, D) < \tau$ and we have the upper estimate $|\Phi_{0,\lambda}^s(z, z)| \leq \frac{C}{|d(z,D)|}$ for all $z \in B(0, R) \setminus \overline{D}$, where $B(0, R) \supset \overline{D}$ is a ball with fixed radius R around the origin and $d(z, D)$ denotes the Hausdorff distance $d(z, D) := \inf\{|z - y| : y \in D\}$.*

The proof of Theorem 2.1 can be found in [2]. Similar statements can be shown for $j = 1, 2, 3$. This theorem shows that the function $|\Phi_{j,\lambda}^s(z, z)|$, $z \in B(0, R) \setminus \overline{D}$ is bounded in every set $\{z \in B(0, R) \setminus \overline{D} : d(z, D) > \tau > 0\}$ but unbounded when z tends to the boundary of the obstacle, i.e. $\lim_{z \rightarrow \partial D} |\Phi_{j,\lambda}^s(z, z)| = \infty$ holds. Thus the functions

$$I_j(z) := \Phi_{j,\lambda}^s(z, z), \quad 0 \leq j \leq 3 \quad (5)$$

for $z \in B(0, R)$, may serve as *indicator functions* for the reconstruction of the obstacle D .

In the next theorem, we provide the precise behaviour of the indicator functions $\Phi_{j,\lambda}^s(z, z)$, $0 \leq j \leq 3$, for z near ∂D .

THEOREM 2.2 *We have the following asymptotic expansion for the indicator functions (5)*

$$I_0(z) = \frac{1}{8\pi(z-a) \cdot \nu(a)} - \frac{i\lambda(a) + \frac{1}{4}\Delta f_a(a')}{2\pi} \ln |(z-a) \cdot \nu(a)| + O(1), \quad (6)$$

$$I_j(z) = -\frac{\nu_j(a)}{8\pi[(z-a) \cdot \nu(a)]^2} - \nu_j(a) \frac{i\lambda(a) - \frac{9}{16}\Delta f_a(a')}{4\pi(z-a) \cdot \nu(a)} + O(\ln |(z-a) \cdot \nu(a)|), \quad (7)$$

$j = 1, 2, 3$ for every $z := a + h\nu(a)$, $h > 0$ small enough. The quantity $\Delta f_a(a')$ is the Mean curvature of the surface ∂D at the point $a \in \partial D$ where f_a is the local parametrization of ∂D near a and $a' := (a_1, a_2)$.

3. Proof of Theorem 2.2

3.1. The steps of the proof

To prove this theorem, we introduce some functions $\tilde{w}_{\lambda(a)}^j(\cdot, z)$, $0 \leq j \leq 3$, satisfying:

$$\begin{cases} \Delta \tilde{w}_{\lambda(a)}^j(\cdot, z) = 0 & \text{in } \mathbb{R}_+^3, \\ (\partial_{x_3} + i\lambda(a))\tilde{w}_{\lambda(a)}^j(\cdot, z) = -(\partial_{x_3} + i\lambda(a))\psi_j(\cdot, z), & \text{on } \partial\mathbb{R}_+^3 \end{cases} \quad (8)$$

where $\psi_0(x, z) := \Gamma(x, z)$ and $\psi_j(x, z) := \frac{\partial}{\partial z_j} \Gamma(x, z)$, $1 \leq j \leq 3$. Here $\Gamma(x, z) := \frac{1}{4\pi} \frac{1}{|x-z|}$ is the fundamental solution to the Laplace equation in \mathbb{R}^3 .

We also denote by \mathcal{R}_a the rotation satisfying $\mathcal{R}_a(\nu(a)) = (0, 0, 1)$ and \mathcal{T}_a the translation taking the point a to the origin. For $x, z \in \mathbb{R}^3 \setminus \bar{D}$ and $a \in \partial D$, we set

$$w_{\lambda(a)}^j(x, z) := \tilde{w}_{\lambda(a)}^j((\mathcal{T}_a \circ \mathcal{R}_a)(x), (\mathcal{T}_a \circ \mathcal{R}_a)(z)). \quad (9)$$

We state the following propositions. Their proofs will be given in sections 3.2 and 3.3 respectively.

PROPOSITION 3.1 *There exist functions $\tilde{w}_{\lambda(a)}^j$ solutions of (8) having the representations:*

$$\tilde{w}_{\lambda(a)}^0(x, z) := \frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{i(x'-z') \cdot \xi'} e^{-(x_3+z_3)|\xi'|} \frac{|\xi'| + i\lambda(a)}{|\xi'|(|\xi'| - i\lambda(a))} d\xi', \quad (10)$$

$$\tilde{w}_{\lambda(a)}^j(x, z) := \frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{i(x'-z') \cdot \xi'} e^{-(x_3+z_3)|\xi'|} \frac{i\xi_j(|\xi'| + i\lambda(a))}{|\xi'|(|\xi'| - i\lambda(a))} d\xi', \quad j = 1, 2; \quad (11)$$

$$\tilde{w}_{\lambda(a)}^3(x, z) := -\frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{i(x'-z') \cdot \xi'} e^{-(x_3+z_3)|\xi'|} \frac{|\xi'| + i\lambda(a)}{|\xi'| - i\lambda(a)} d\xi', \quad (12)$$

for $x, z \in \mathbb{R}_+^3$ and $a \in \partial D$. In addition, we have the following asymptotics for the functions $\tilde{w}_{\lambda(a)}^j(z, z)$, $z \in \mathbb{R}_+^3$,

$$\tilde{w}_{\lambda(a)}^0(z, z) = \frac{1}{8\pi} \frac{1}{z_3} - \frac{i\lambda(a)}{2\pi} \ln(z_3) + O(1), \quad (13)$$

$$\tilde{w}_{\lambda(a)}^1(z, z) = \tilde{w}_{\lambda(a)}^2(z, z) = 0, \quad (14)$$

$$\tilde{w}_{\lambda(a)}^3(z, z) = -\frac{1}{8\pi} \frac{1}{z_3^2} - \frac{i\lambda(a)}{4\pi z_3} - \frac{(\lambda(a))^2}{\pi} \ln(z_3) + O(1). \quad (15)$$

From (8), (13)-(15) and the explicit form of \mathcal{R}_a , see (23), we deduce that:

$$w_{\lambda(a)}^0(z, z) = \frac{1}{8\pi} \frac{1}{(z-a) \cdot \nu(a)} - \frac{i\lambda(a)}{2\pi} \ln((z-a) \cdot \nu(a)) + O(1), \quad (16)$$

$$\begin{aligned} & w_{\lambda(a)}^j(z, z) \\ &= \nu_j(a) \left[\frac{-1}{8\pi((z-a) \cdot \nu(a))^2} - \frac{i\lambda(a)}{4\pi(z-a) \cdot \nu(a)} - \frac{(\lambda(a))^2}{\pi} \ln((z-a) \cdot \nu(a)) \right] + O(1), \end{aligned} \quad (17)$$

for $j = 1, 2, 3$.

PROPOSITION 3.2 *There exist $\delta(a) > 0$ and $C > 0$ such that*

$$|\Phi_{0,\lambda}^s(z, z) - w_{\lambda(a)}^0(z, z) + \frac{\Delta f_a(a')}{8\pi} \ln((z-a) \cdot \nu(a))| \leq C, \quad (18)$$

$$|\Phi_{j,\lambda}^s(z, z) - w_{\lambda(a)}^j(z, z) - \frac{9}{64\pi} \frac{\Delta f_a(a')}{(z-a) \cdot \nu(a)} \nu_j(a)| \leq C |\ln((z-a) \cdot \nu(a))|, \quad (19)$$

for $a \in \partial D$, $z = a + h\nu(a)$, $0 \leq h < \delta(a)$ and $j = 1, 2, 3$.

The proof of Theorem 2.2 is a combination of (16)-(17) and (18)-(19) by using (5). Therefore, we devote the rest of this section to prove Propositions 3.1 and 3.2.

3.2. Proof of Proposition 3.1.

The explicit form of $\tilde{w}_{\lambda(a)}^0(x, z)$ and its asymptotic expansion were derived in [14, Lemma 3.3]. Following the same approach we find also the representations (11) and (12) for $\tilde{w}_{\lambda(a)}^j$, $1 \leq j \leq 3$.

Due to the antisymmetry of the integrands in (11), we obtain $\tilde{w}_{\lambda(a)}^j(z, z) = 0$, $j = 1, 2$. Finally, to derive the asymptotics of $\tilde{w}_{\lambda(a)}^3(z, z)$, we write:

$$\begin{aligned} \tilde{w}_{\lambda(a)}^3(z, z) &= -\frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-2z_3|\xi'|} \frac{|\xi'| + i\lambda(a)}{|\xi'| - i\lambda(a)} d\xi' \\ &= -\frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-2z_3|\xi'|} d\xi' - \frac{2i\lambda(a)}{8\pi^2} \int_{\mathbb{R}^2} \frac{e^{-2z_3|\xi'|}}{|\xi'| - i\lambda(a)} d\xi'. \end{aligned}$$

Using polar coordinates we obtain (15) after some simple manipulations.

3.3. Proof of Proposition 3.2.

First, we need to summarize some further basic notations.

Let $G_\lambda(\cdot, z) := \varphi_0(\cdot, z) + \Phi_{0,\lambda}^s(\cdot, z)$ be the Green's function of the problem (1)-(3), where $\varphi_0(\cdot, z) := \Phi(\cdot, z)$.

We set also $G_{\lambda(a)}(\cdot, z) := \varphi_0(\cdot, z) + \Phi_{0,\lambda(a)}^s(\cdot, z)$ to be the Green's function of the problem (1)-(3) when the function λ is replaced by the constant $\lambda(a)$.

We have $\frac{\partial}{\partial z_j} G_\lambda(\cdot, z) = \Phi_{j,\lambda}^s(\cdot, z) + \varphi_j(\cdot, z)$ and $\frac{\partial}{\partial z_j} G_{\lambda(a)}(\cdot, z) = \Phi_{j,\lambda(a)}^s(\cdot, z) + \varphi_j(\cdot, z)$,

where $\varphi_j(\cdot, z) := \frac{\partial}{\partial z_j} \Phi(\cdot, z)$ for $1 \leq j \leq 3$.

Then, we define $\Phi_{j,\lambda(a)}^{s,0}$, $0 \leq j \leq 3$, as a solution to

$$\begin{cases} \Delta \Phi_{j,\lambda(a)}^{s,0}(\cdot, z) = 0 & \text{in } \Omega \setminus \overline{D} \\ \Phi_{j,\lambda(a)}^{s,0}(\cdot, z) = 0 & \text{on } \partial\Omega \\ (\partial_\nu + i\lambda(a))\Phi_{j,\lambda(a)}^{s,0}(\cdot, z) = -(\partial_\nu + i\lambda(a))\psi_j(\cdot, z), & \text{on } \partial D \end{cases} \quad (20)$$

with an arbitrary fixed C^3 regular domain $\Omega \supset \overline{D}$, where $z \in \Omega \setminus \overline{D}$. Finally, we introduce $G_{\lambda(a)}^0(\cdot, z) := \psi_0(\cdot, z) + \Phi_{0,\lambda(a)}^{s,0}(\cdot, z)$ which is the Green's function of the problem (20). We have the following lemma which proof can be found in [20] and [21].

LEMMA 3.3 *For every $R > 0$, there exists a positive constant C such that*

1. $|G_\lambda(x, z)| \leq \frac{C}{|x-z|}$, for $x, z \in (\mathbb{R}^3 \setminus D) \cap B(0, R)$,
2. $|\nabla_x G_\lambda(x, z)| \leq \frac{C}{|x-z|^2}$ and $|\nabla_z G_\lambda(x, z)| \leq \frac{C}{|x-z|^2}$, for $x, z \in (\mathbb{R}^3 \setminus D) \cap B(0, R)$.

The rest of the proof of Proposition 3.2 is splitted into the following three lemmas

LEMMA 3.4 *For every $\delta(a) > 0$, $a \in \partial D$, there exists $C > 0$ such that*

$$\begin{aligned} |\Phi_{0,\lambda}^s(x, z) - \Phi_{0,\lambda(a)}^s(x, z)| &\leq C, \\ |\Phi_{j,\lambda}^s(x, z) - \Phi_{j,\lambda(a)}^s(x, z)| &\leq C |\ln(|x-z|)| \quad 1 \leq j \leq 3 \end{aligned}$$

for $x \in \mathbb{R}^3 \setminus D$ and $z \in B(a, \delta(a)) \cap C_{a,\theta}$, where $C_{a,\theta}$ is a cone with center a , angle $\theta \in [0, \frac{\pi}{2}[$ and axis $\nu(a)$ such that $B(a, \delta(a)) \cap C_{a,\theta} \subset \mathbb{R}^3 \setminus D$.

Proof. We set $R_{j,\lambda(a)}(x, z) := \Phi_{j,\lambda}^s(x, z) - \Phi_{j,\lambda(a)}^s(x, z)$, $0 \leq j \leq 3$. Then it satisfies:

$$\begin{cases} (\Delta + \kappa^2)R_{j,\lambda(a)}(\cdot, z) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ (\partial_\nu + i\lambda(a))R_{j,\lambda(a)}(\cdot, z) = -i(\lambda - \lambda(a))(\Phi_{j,\lambda}^s(\cdot, z) + \varphi_j(\cdot, z)) & \text{on } \partial D, \\ R_{j,\lambda(a)}(\cdot, z) \text{ satisfies the Sommerfeld radiation condition.} \end{cases} \quad (21)$$

From (21), we have the representation:

$$R_{j,\lambda(a)}(x, z) = - \int_{\partial D} i(\lambda(y) - \lambda(a))G_{\lambda(a)}(y, x)(\Phi_{j,\lambda}^s + \varphi_j)(y, z)ds(y), \text{ for } x, z \in \mathbb{R}^3 \setminus \overline{D}.$$

The regularity of the surface impedance λ leads to $|\lambda(y) - \lambda(a)| \leq C|y-a|$.

Moreover, there exists $C > 0$ such that $|y-a| \leq C|y-z|$ for $y \in \partial D$ and $z \in C_{a,\theta} \cap B(a, \delta(a))$. This is due to the fact that ∂D and $C_{a,\theta} \cap B(a, \delta(a))$ are separated, i.e. $\partial D \cap C_{a,\theta} \cap B(a, \delta(a)) = \{a\}$.

From Lemma 3.3, $|(\Phi_{0,\lambda}^s + \varphi_0)(x, z)| \leq C|x-z|^{-1}$ and $|(\Phi_{j,\lambda}^s + \varphi_j)(x, z)| \leq C|x-z|^{-2}$. Therefore, for $x \in \mathbb{R}^3 \setminus D$ and $z \in C_{a,\theta} \cap B(0, \delta(a))$, $|R_{0,\lambda(a)}(x, z)| \leq \int_{\partial D} \frac{C}{|y-x|} ds(y) = O(1)$ and $|R_{j,\lambda(a)}(x, z)| \leq C \int_{\partial D} \frac{1}{|y-x||y-z|} ds(y) = O(\ln(|x-z|))$, for $j = 1, 2, 3$. \square

LEMMA 3.5 *There exists $C > 0$ and $\delta(a) > 0$ such that*

$$|\Phi_{j,\lambda(a)}^s(x, z) - \Phi_{j,\lambda(a)}^{s,0}(x, z)| \leq C, \quad 0 \leq j \leq 3,$$

for $x \in \Omega \setminus D$, $a \in \partial D$, $z = a + h\nu(a)$ and $0 < h < \delta(a)$.

Proof. The function $\Phi_{0,\lambda(a)}^s - \Phi_{0,\lambda(a)}^{s,0}$ is a solution of the problem

$$\begin{cases} \Delta(\Phi_{0,\lambda(a)}^s - \Phi_{0,\lambda(a)}^{s,0})(\cdot, z) = -\kappa^2 \Phi_{0,\lambda(a)}^s(\cdot, z) & \text{in } \Omega \setminus \overline{D}, \\ (\partial_\nu + i\lambda(a))(\Phi_{0,\lambda(a)}^s - \Phi_{0,\lambda(a)}^{s,0})(\cdot, z) = 0 & \text{on } \partial D, \\ (\Phi_{0,\lambda(a)}^s - \Phi_{0,\lambda(a)}^{s,0})(\cdot, z) = \Phi_{0,\lambda(a)}^s(\cdot, z) & \text{on } \partial\Omega. \end{cases} \quad (22)$$

According to Lemma 3.6 in [14] and the definition of $G_{\lambda(a)}^0$, we obtain the desired estimate for $\Phi_{0,\lambda(a)}^s - \Phi_{0,\lambda(a)}^{s,0}$. Furthermore, for $j = 1, 2, 3$, we differentiate the equations in (22) with respect to z_j to find the desired estimate for $\Phi_{j,\lambda(a)}^s - \Phi_{j,\lambda(a)}^{s,0}$. \square

LEMMA 3.6 *There exist $C > 0$ and $\delta(a) > 0$ such that*

$$\begin{aligned} |(\Phi_{0,\lambda(a)}^{s,0} - w_{\lambda(a)}^0)(z, z) + \frac{\Delta f_a(a')}{8\pi} \ln((z-a) \cdot \nu(a))| &\leq C, \\ \left| (\Phi_{j,\lambda(a)}^{s,0} - w_{\lambda(a)}^j)(z, z) - \frac{9}{64\pi} \frac{\Delta f_a(a')}{(z-a) \cdot \nu(a)} \nu_j(a) \right| &\leq C |\ln((z-a) \cdot \nu(a))|, \quad j = 1, 2, 3 \end{aligned}$$

for $a \in \partial D$, $z = a + h\nu(a)$ and $0 < h < \delta(a)$.

Proof. It is enough to assume that $a = 0$ and $\nu(a) = e_3 := (0, 0, 1)$. The general case can be deduced as follows. We set

$$\tilde{\Phi}_{j,\lambda(a)}^{s,0}(\tilde{x}, \tilde{z}) = \Phi_{j,\lambda(a)}^{s,0}(x, z)$$

where $\tilde{x} = \mathcal{T}_a \circ \mathcal{R}_a(x)$ and $\tilde{z} = \mathcal{T}_a \circ \mathcal{R}_a(z)$. Then $\tilde{\Phi}_{j,\lambda(a)}^{s,0}$ satisfies the problem

$$\begin{cases} \Delta \tilde{\Phi}_{j,\lambda(a)}^{s,0}(\cdot, \tilde{z}) = 0 & \text{in } \mathcal{T}_a \circ \mathcal{R}_a(\Omega \setminus \overline{D}), \\ \tilde{\Phi}_{j,\lambda(a)}^{s,0}(\cdot, \tilde{z}) = 0 & \text{on } \mathcal{T}_a \circ \mathcal{R}_a(\partial\Omega), \\ \left(\frac{\partial}{\partial \tilde{z}_3} + i\lambda(a) \right) \tilde{\Phi}_{j,\lambda(a)}^{s,0}(\cdot, \tilde{z}) = - \left(\frac{\partial}{\partial \tilde{z}_3} + i\lambda(a) \right) (\mathcal{R}_a^T \nabla_{\tilde{z}} \Gamma)_j(\cdot, \tilde{z}), & \text{on } \mathcal{T}_a \circ \mathcal{R}_a(\partial D) \end{cases}$$

where $(\mathcal{R}_a \nabla_z \Gamma)_j$ is the j th component of $\mathcal{R}_a \nabla_z \Gamma$.

We need then to compute the rotation \mathcal{R}_a transforming $\nu(a)$ to e_3 . Hence \mathcal{R}_a is the rotation matrix defined by the unit vector $A := \frac{\nu(a) \times e_3}{|\nu(a) \times e_3|} = \frac{(\nu_2(a), -\nu_1(a), 0)}{\sqrt{\nu_1^2(a) + \nu_2^2(a)}}$ and the angle θ between the vectors $\nu(a)$ and e_3 . We set $\alpha := \sqrt{\nu_1^2(a) + \nu_2^2(a)}$. This rotation is given by:

$$\mathcal{R}_a = \frac{1}{\alpha^2} \begin{bmatrix} \nu_2^2(a) + \nu_1^2(a)\nu_3(a) & -\nu_1(a)\nu_2(a)(1 - \nu_3(a)) & -\nu_1(a)\alpha^2 \\ -\nu_1(a)\nu_2(a)(1 - \nu_3(a)) & \nu_1^2(a) + \nu_2^2(a)\nu_3(a) & -\nu_2(a)\alpha^2 \\ \nu_1(a)\alpha^2 & \nu_2(a)\alpha^2 & \nu_3(a)\alpha^2 \end{bmatrix}. \quad (23)$$

It satisfies $\mathcal{R}_a^T \mathcal{R}_a = I$ and $\mathcal{R}_a(\nu(a)) = e_3$. Now

$$\mathcal{R}_a^T \nabla_z \Gamma = \begin{pmatrix} \frac{(\nu_2^2(a) + \nu_1^2(a)\nu_3(a))}{\alpha^2} \frac{\partial \Gamma}{\partial z_1} - \frac{\nu_1(a)\nu_2(a)(1 - \nu_3(a))}{\alpha^2} \frac{\partial \Gamma}{\partial z_2} + \nu_1(a) \frac{\partial \Gamma}{\partial z_3} \\ -\frac{\nu_1(a)\nu_2(a)(1 - \nu_3(a))}{\alpha^2} \frac{\partial \Gamma}{\partial z_1} + \frac{(\nu_1^2(a) + \nu_2^2(a)\nu_3(a))}{\alpha^2} \frac{\partial \Gamma}{\partial z_2} + \nu_2(a) \frac{\partial \Gamma}{\partial z_3} \\ -\nu_1(a) \frac{\partial \Gamma}{\partial z_1} - \nu_2(a) \frac{\partial \Gamma}{\partial z_2} + \nu_3(a) \frac{\partial \Gamma}{\partial z_3} \end{pmatrix} \quad (24)$$

Then for general a and $\nu(a)$, the functions $\Phi_{j,\lambda(a)}^{s,0}(z, z)$, for $j = 1, 2, 3$, are deduced from those associated to the case where $a = 0$ and $\nu(a) = e_3$ using the linear combinations given in (24).

Since ∂D is of class C^3 then, there exists a C^3 function f_a defined on the disk of radius r , centered at $O' = (0, 0)$, such that $f_a(O') = \frac{\partial f_a}{\partial x_1}(O') = \frac{\partial f_a}{\partial x_2}(O') = 0$ and

$$D \cap B(0, r) = \{(x_1, x_2, x_3) \in B(0, r); x_3 < f_a(x_1, x_2)\}.$$

In the domain $D \cap B(0, r)$, we introduce the local change of variables $F : x \mapsto \xi$ in the following way

$$\begin{cases} \xi' = x', \\ \xi_3 = x_3 - f_a(x'). \end{cases} \quad (25)$$

3.3.1. First step : An integral representation. Let $B_r^+ := B(0, r) \cap \mathbb{R}_+^3$ and $\partial B_r^+ = S_r \cup S_r^c$ with $S_r := \partial B_r^+ \cap \partial \mathbb{R}_+^3$. We set, $\vartheta_{j,\lambda(a)}^{s,0}(\xi, \eta) := \Phi_{j,\lambda(a)}^{s,0}(F^{-1}(\xi), F^{-1}(\eta))$ and $\tilde{\psi}_j(\xi, \eta) = \psi_j(F^{-1}(\xi), F^{-1}(\eta))$, for $\xi, \eta \in B_r^+$. Then from (20),

$$\begin{cases} \nabla \cdot B \nabla \vartheta_{j,\lambda(a)}^{s,0}(\cdot, \eta) = 0, & \text{in } B_r^+ \\ B \nabla \vartheta_{j,\lambda(a)}^{s,0}(\cdot, \eta) \cdot e_3 + i\lambda(a) |J^T e_3| \vartheta_{j,\lambda(a)}^{s,0}(\cdot, \eta) = \\ \quad -\nabla(\tilde{\psi}_j)(\cdot, \eta) \cdot J^T e_3 - i\lambda(a) |J^T e_3| \tilde{\psi}_j(\cdot, \eta), & \text{on } S_r \end{cases}$$

where $J(\xi) := \frac{\partial \xi}{\partial x}(F^{-1}(\xi))$ and $B(\xi) := J(\xi) J^T(\xi)$.

We set, $\tilde{R}_j(\xi, \eta) := \vartheta_{j,\lambda(a)}^{s,0}(\xi, \eta) - \tilde{w}_{\lambda(a)}^j(\xi, \eta)$, $0 \leq j \leq 3$. Then the function \tilde{R}_j satisfies

$$\begin{cases} \nabla \cdot B \nabla \tilde{R}_j(\cdot, \eta) = \nabla \cdot (I - B) \nabla \tilde{w}_{\lambda(a)}^j(\cdot, \eta), & \text{in } B_r^+ \\ B \nabla \tilde{R}_j(\cdot, \eta) \cdot e_3 + i\lambda(a) |J^T e_3| \tilde{R}_j(\cdot, \eta) = (I - B) \nabla \tilde{w}_{\lambda(a)}^j(\cdot, \eta) \cdot e_3 \\ \quad + i\lambda(a) \left[(1 - |J^T e_3|) \tilde{w}_{\lambda(a)}^j(\cdot, \eta) + \psi_j(\cdot, \eta) - |J^T e_3| \tilde{\psi}_j(\cdot, \eta) \right] \\ \quad + \nabla \psi_j(\cdot, \eta) \cdot e_3 - \nabla \tilde{\psi}_j(\cdot, \eta) J^T e_3, & \text{on } S_r. \end{cases} \quad (26)$$

We denote by G the Green's function satisfying:

$$\begin{cases} \nabla_\xi \cdot B(\xi) \nabla_\xi G(\xi, \eta) = -\delta(\xi, \eta), & \text{in } B_r^+, \\ B(\xi) \nabla_\xi G(\xi, \eta) \cdot \nu(\xi) + i\lambda(a) |J^T(\xi) e_3| G(\xi, \eta) = 0 & \text{on } \partial B_r^+, \end{cases}$$

where ν is the outward unit normal on ∂B_r^+ and $|J^T(\xi) e_3| = \sqrt{1 + |\nabla f_a(\xi')|^2}$.

Integration by parts in (26), using G , generates the following integral representation of $\tilde{R}_j(\xi, \eta)$, $0 \leq j \leq 3$,

$$\begin{aligned} \tilde{R}_j(\xi, \eta) &= \int_{B_r^+} (I - B(z)) \nabla_z G(z, \xi) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) dz \\ &+ i\lambda(a) \left\{ \int_{S_r} [\psi_j(z, \eta) - |J^T(z) e_3| \tilde{\psi}_j(z, \eta)] G(z, \xi) ds(z) \right. \\ &+ \left. \int_{S_r} [1 - |J^T(z) e_3|] \tilde{w}_{\lambda(a)}^j(z, \eta) G(z, \xi) ds(z) \right\} \\ &- \int_{S_r} [\nabla_z \psi_j(z, \eta) \cdot e_3 - \nabla_z \tilde{\psi}_j(z, \eta) \cdot J^T(z) e_3] G(z, \xi) ds(z) + O(1) \end{aligned} \quad (27)$$

for $\xi, \eta \in B_r^+$. The term $O(1)$ represents the total of the surface integrals over S_r^c .

3.3.2. Second step : The estimates for the local coordinates. We have the following lemma which proof is given in the appendix.

LEMMA 3.7 *The following estimates are valid for $\eta \in C_{F(a),\theta}$,*

$$\begin{aligned}\tilde{R}_0(\eta, \eta) &= -\frac{1}{8\pi} \Delta f_a(0') \ln(\eta_3) + O(1), \\ \tilde{R}_j(\eta, \eta) &= O(1), \text{ for } j = 1, 2, \\ \tilde{R}_3(\eta, \eta) &= \frac{9}{64\pi\eta_3} \Delta f_a + O(\ln(\eta_3)).\end{aligned}$$

3.3.3. Third step : The estimates in the original coordinates. So far, we have shown that for $\eta \in C_{F(a),\theta}$,

$$\Phi_{0,\lambda(a)}^{s,0}(F^{-1}(\eta), F^{-1}(\eta)) = \tilde{\omega}_{\lambda(a)}^0(\eta, \eta) - \frac{1}{8\pi} \Delta f_a(0') \ln(\eta_3) + O(1).$$

Hence, from (13),

$$\Phi_{0,\lambda(a)}^{s,0}(F^{-1}(\eta), F^{-1}(\eta)) = \frac{1}{8\pi\eta_3} - \frac{i\lambda(a) + \frac{1}{4\pi} \Delta f_a(0')}{2\pi} \ln(\eta_3) + O(1).$$

Since $\eta = (0, 0, \eta_3)$ then $F^{-1}(\eta) = \eta$. We deduce that for η_3 small enough,

$$\Phi_{0,\lambda(a)}^{s,0}(\eta, \eta) = \frac{1}{8\pi\eta_3} - \frac{i\lambda(a) + \frac{1}{4\pi} \Delta f_a(0')}{2\pi} \ln(\eta_3) + O(1).$$

Similarly, for η_3 small enough, we also have

$$\begin{aligned}\Phi_{1,\lambda(a)}^{s,0}(\eta, \eta) &= O(1) = \Phi_{2,\lambda(a)}^{s,0}(\eta, \eta), \\ \Phi_{3,\lambda(a)}^{s,0}(\eta, \eta) &= -\frac{1}{8\pi\eta_3^2} - \frac{i\lambda(a) - \frac{9}{16\pi} \Delta f_a(0')}{4\pi\eta_3} + O(1).\end{aligned}$$

Going back to the original coordinates and using (24), we finally obtain

$$\Phi_{0,\lambda(a)}^{s,0}(z, z) = \frac{1}{8\pi(z-a) \cdot \nu(a)} - \frac{i\lambda(a) + \frac{1}{4\pi} \Delta f_a(0')}{2\pi} \ln((z-a) \cdot \nu(a)) + O(1),$$

$$\Phi_{j,\lambda(a)}^{s,0}(z, z) = \nu_j(a) \left[-\frac{1}{8\pi((z-a) \cdot \nu(a))^2} - \frac{i\lambda(a) - \frac{9}{16\pi} \Delta f_a(0')}{4\pi(z-a) \cdot \nu(a)} \right] + O(1),$$

for $z = a + h\nu(a)$, h small enough. □

4. Numerical experiments

In the last section we illustrate the feasibility of the proposed method for the obstacle reconstruction and show a dependence of the reconstruction quality on the impedance function and the curvature.

To generate far field data we numerically solve the direct problem via a Galerkin method for the single-layer potential approach, see [8]. We assume that the surface ∂D is C^3 -smooth, homeomorphic to the unit sphere \mathbb{S}^2 , i.e. $\partial D = \{z(x) : x \in \mathbb{S}^2\}$.

We consider four types of impedance functions, that is, a positive constant function comparable with the size of the Mean curvatures

$$\lambda(\theta, \phi) = 1, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad (28)$$

a constant with a large imaginary part

$$\lambda(\theta, \phi) = 0.5 + 10i, \quad (29)$$

a smooth function

$$\lambda(\theta, \phi) = 5 + 4z_3(\theta, \phi), \quad (30)$$

and an oscillating impedance function along z_1 -axis

$$\lambda(\theta, \phi) = 5 + 4 \cos(10 z_1(\theta, \phi)). \quad (31)$$

The scatterers are given by a ball of radius 1 centered at the origin and which is parametrized in spherical coordinates by

$$z(\theta, \phi) = p(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad (32)$$

by an ellipsoid with the parametrization

$$z(\theta, \phi) = \begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 1 \end{pmatrix} p(\theta, \phi), \quad (33)$$

by a cushion-shaped surface

$$z(\theta, \phi) = 0.75 \sqrt{0.8 + 0.5(\cos 2\phi - 1)(\cos 4\theta - 1)} p(\theta, \phi), \quad (34)$$

and a bean-like surface

$$z(\theta, \phi) = \begin{pmatrix} 0.8 \sqrt{(1 - 0.1 \cos(\pi \cos \theta))} \sin \theta \cos \phi \\ 0.8 \sqrt{(1 - 0.4 \cos(\pi \cos \theta))} \sin \theta \sin \phi + 0.3 \cos(\pi \cos \theta) \\ \cos \theta \end{pmatrix}. \quad (35)$$

In Table 1 we present the accuracy of the solution method for the direct problem with the impedance functions and scatterers defined above for an incident wave as a singular source $\Phi(\cdot, x^*)$, where $x^* = (0, 0.1, 0)$.

Table 1. Relative error for the far field.

λ	n	$\ u_\infty - \Phi_\infty(\cdot, x^*)\ _\infty / \ \Phi_\infty(\cdot, x^*)\ _\infty$			
		ball	ellipsoid	cushion	bean
(28)	10	0.0000	0.0000	0.0155	0.0258
	15	–	–	0.0024	0.0048
	20	–	–	0.0007	0.0009
(29)	10	0.0000	0.0000	0.0161	0.0079
	15	–	–	0.0026	0.0006
	20	–	–	0.0008	0.0001
(30)	10	0.0000	0.0000	0.0181	0.0131
	15	–	–	0.0027	0.0013
	20	–	–	0.0009	0.0002
(31)	10	0.0000	0.0000	0.0174	0.0110
	15	–	–	0.0026	0.0011
	20	–	–	0.0009	0.0002

From Table 1 we conclude that choosing $n = 15$ in a Galerkin method, which corresponds to $2(n + 1)^2$ unknowns in the discretized integral equation, we obtain an accurate synthetic data.

In order to find approximations to the indicator functions (5) we proceed analogously to [2].

(i) We set fixed reference configurations

$$\mathcal{G}^{(k)}(0) = 1.5B(0, 1) + 1.55n_k, \quad k = 1, \dots, 26,$$

such that $0 \notin \mathcal{G}^{(k)}(0)$ with n_k unit vectors. Then via a Galerkin approach with $n = 15$ we solve the integral equations

$$\begin{aligned} (\alpha \mathbb{I}_1 + H^* H) g_{0,0}^{\alpha,k} &= H^* \Phi(\cdot, 0), \\ (\alpha \mathbb{I}_1 + H^* H) g_{j,0}^{\alpha,k} &= H^* \frac{\partial \Phi}{\partial x_j}(\cdot, 0), \quad j = 1, 2, 3 \end{aligned}$$

for densities $g_{j,0}^{\alpha,k} \in L^2(\mathbb{S}^2)$, where α is a regularization parameter, \mathbb{I}_1 corresponds to the Sobolev $H^1(\mathbb{S}^2)$ penalty term and H is the Herglotz wave operator defined from $L^2(\mathbb{S}^2)$ to $L^2(\partial \mathcal{G}^{(k)}(0))$ by

$$(Hg)(y) := \int_{\mathbb{S}^2} e^{i\kappa y \cdot d} g(d) ds(d), \quad y \in \partial \mathcal{G}^{(k)}(0)$$

(ii) Having generated a set of sampling points

$$\mathcal{S} := \{z = (-2 + hk, -2 + hl, -2 + hm), \quad h = 0.0404, \quad k, l, m = 1, \dots, 100\}, \quad (36)$$

due to [15, Lemma 3.1 and Theorem 3.2] we can calculate the densities $g_{j,z}^{\alpha,k}$ for the reference domains $\mathcal{G}^{(k)}(z) := z + \mathcal{G}^{(k)}(0)$, via

$$g_{j,z}^{\alpha,k}(d) = e^{-i\kappa z \cdot d} g_{j,0}^{\alpha,k}(d), \quad d \in \mathbb{S}^2.$$

(iii) The approximate indicator functions are found by

$$|\tilde{I}_j^\alpha(z)| = \min_k |\tilde{I}_j^{\alpha,k}(z)|, \quad \text{for } j = 0, 1, 2, 3,$$

where $\tilde{I}_j^{\alpha,k}(z) := 4\pi \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} u^\infty(-\hat{x}, d) g_{j,z}^{\alpha,k}(d) ds(d) g_{0,z}^{\alpha,k}(\hat{x}) ds(\hat{x})$, with the synthetic far field data $u^\infty(-\hat{x}, d)$ given at 512 points for 512 incident directions.

(iv) Steps 2 and 3 are repeated with a different regularization parameter β and the admissibility regions are computed in the following way

$$\begin{aligned} \mathcal{E}_{0,c_1} &:= \{z \in \mathcal{S} : |\tilde{I}_0^\alpha(z) - \tilde{I}_0^\beta(z)| < c_1\}, \\ \mathcal{E}_{123,c_2} &:= \{z \in \mathcal{S} : \max_{j \in \{1,2,3\}} |\tilde{I}_j^\alpha(z) - \tilde{I}_j^\beta(z)| < c_2\}. \end{aligned}$$

(v) Chosen some suitable cut-off constant $C_1 > 0$, the boundary ∂D is identified via the criterion

$$z \in D \quad \text{if} \quad |\tilde{I}_0^\alpha(z)| > C_1 \quad \text{for all } z \in \mathcal{E}_{0,c_1}, \quad (37)$$

or via the criterion

$$z \in D \quad \text{if} \quad \sqrt{\sum_{j=1}^3 |\tilde{I}_j^\alpha(z)|^2} > C_2 \quad \text{for all } z \in \mathcal{E}_{123,c_2}. \quad (38)$$

To calibrate the method we consider a reconstruction of a ball (32) with the constant impedance (28). The regularization parameters are set to $\alpha = 1e - 14$, $\beta = 1e - 15$ and admissibility region parameters were chosen as $c_1 = 0.05$, $c_2 = 0.1$. The cut-off constants are $C_1 = 0.25$ and $C_2 = 0.45$. These parameters are kept fixed for all examples.

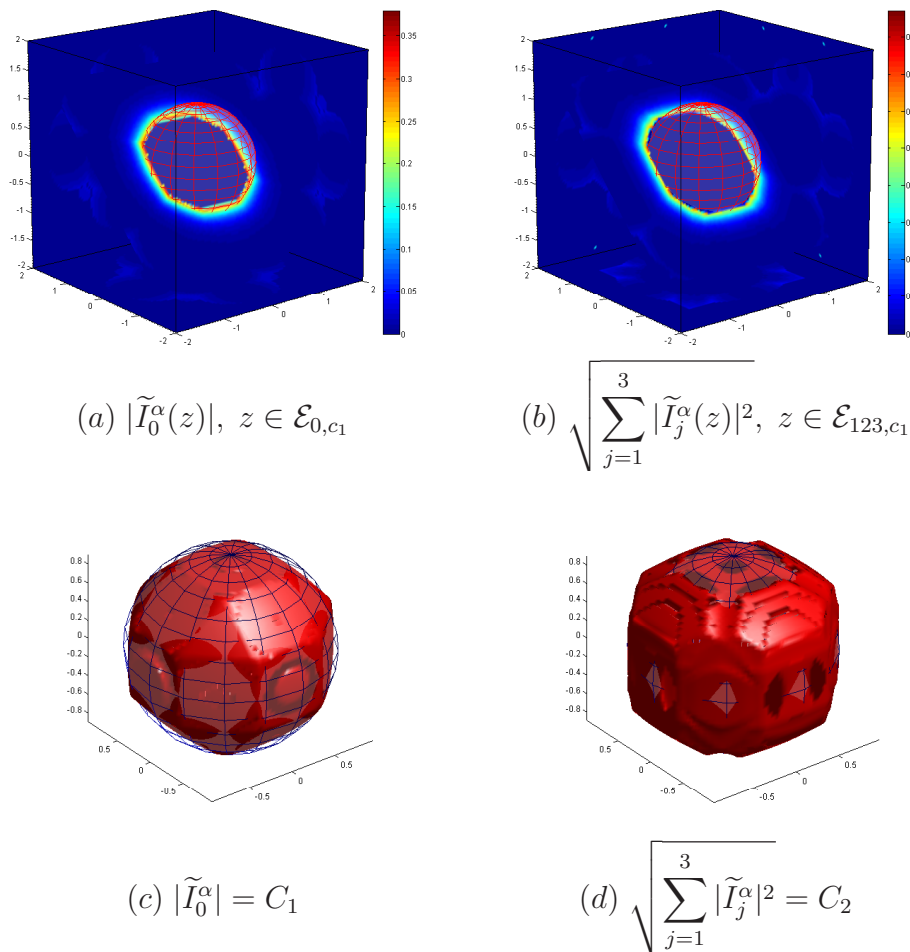


Figure 1. Reconstruction of a ball with the constant impedance (28)

The Figures 1(a) and 1(b) illustrate the absolute value of the indicator functions for the points in the admissibility regions. The reconstructions of the obstacle D are accurate since the local mean curvature $\Delta f_a(a')$ for $a \in \partial D$ is a constant for a ball as well as the impedance function. From Figure 1 we notice that the criterion (37) provides us with a better reconstruction than (38). The same behaviour was observed also for the other shapes and impedance functions. Therefore for the rest of examples only criterion (37) will be applied.

As the second test for the reconstruction accuracy depending on the impedance we choose the constant function (29) with a large imaginary part. The unit ball is reconstructed accurately both from the absolute value and the real part of the indicator function, see Figures 2(a), and 2(b). The imaginary part of the indicator function is less than or equal to 0.07 what is significantly smaller than the values of real part and does not influence the reconstruction from the absolute value. This numerical behaviour agrees with the derived asymptotic expansion (6) of the indicator function.

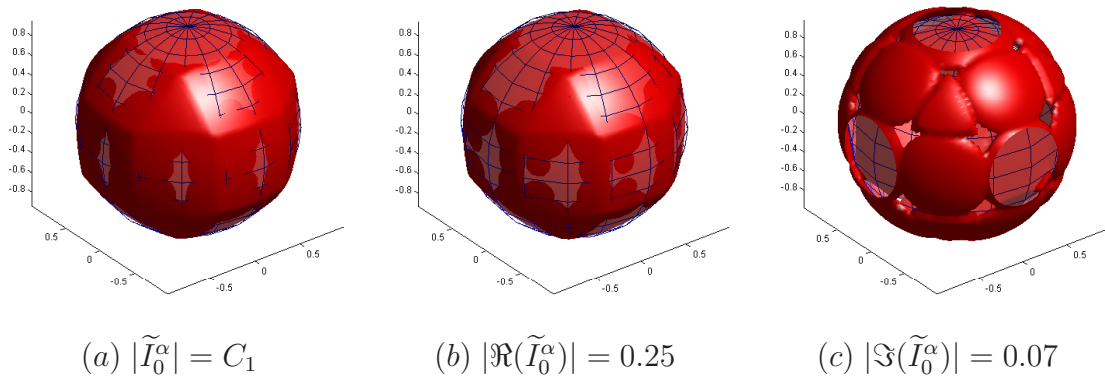


Figure 2. Reconstruction of a ball with the impedance (29)

For the next test we choose the impedance function (30) which takes values from the interval $[1, 9]$ and varies only along z_3 -axis. In Figure 3(a) one can see that the reconstruction is affected by the behaviour of the impedance function. A part of the ball where the impedance function changes insignificantly is accurately reconstructed. However both for the north and south pole where the impedance function takes either its maximum or minimum value, the reconstruction accuracy decreases.

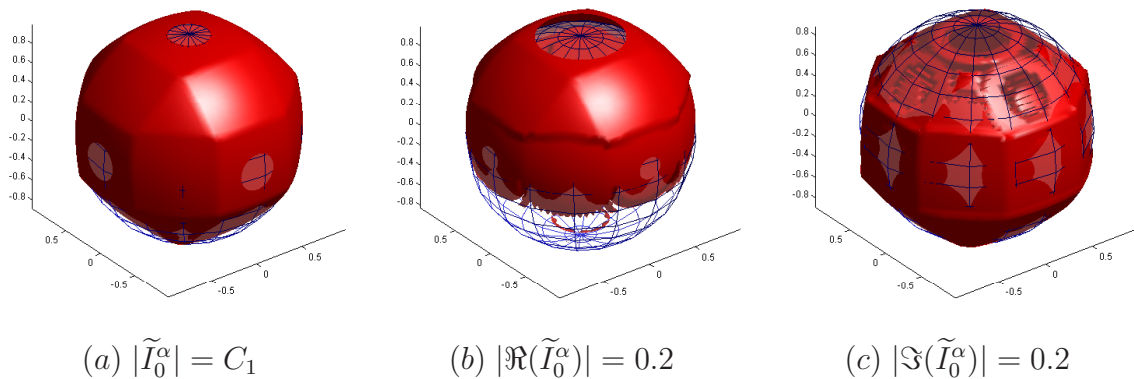


Figure 3. Reconstruction of a ball with the impedance (30)

To investigate this in more details we present the reconstruction obtained from the real and imaginary parts of the indicator function \tilde{I}_0^α . Taking the real part we obtain a better reconstruction of the boundary part where the impedance is small whereas the imaginary part of \tilde{I}_0^α provides us with a more accurate reconstruction of the boundary region where the impedance function grows. The dependence of the indicator function on the impedance can be explained via the term $O(1)$ in (6). We note here that the step length h of the chosen grid (36) is relatively big, e.g. $h = 0.0404$.

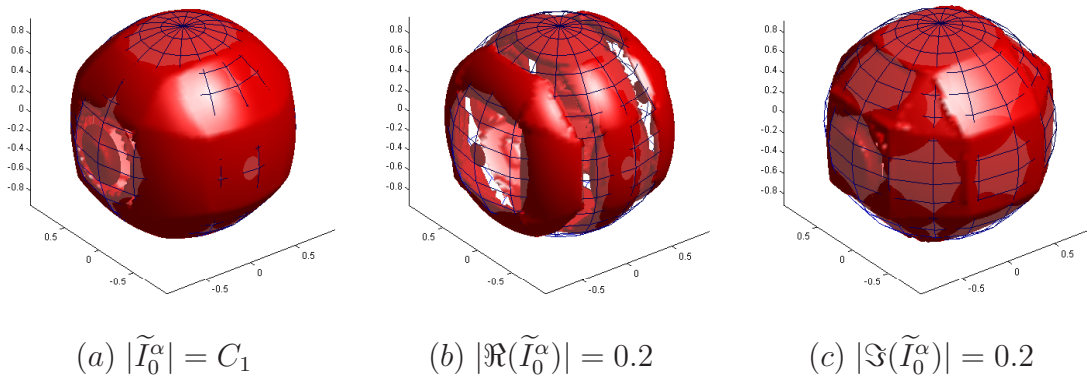


Figure 4. Reconstruction of a ball with the impedance (31)

In Figure 4 we observe that the reconstruction accuracy decreases along z_1 -axis where the impedance function (31) is highly oscillating. Hence, the same characteristic behaviours of the algorithm is remarked as in the previous case. Since the absolute value of the indicator function gives the best reconstruction we do not pursue further reconstruction from the real and imaginary parts of the indicator function.

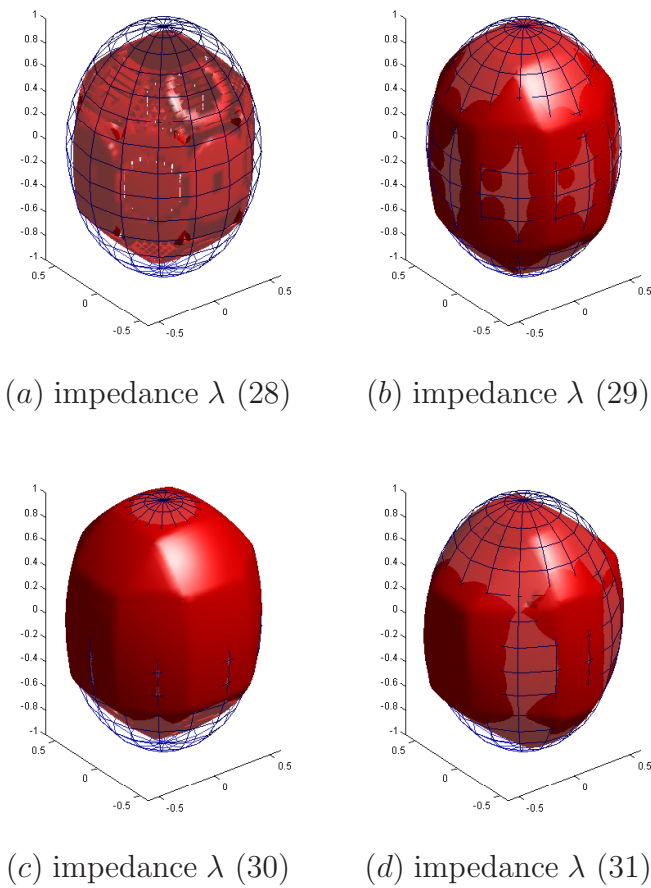


Figure 5. Reconstruction of the ellipsoid (33)

The next obstacle which we try to identify is the ellipsoid (33). Due to the fact that the Mean curvature attains the biggest values at the north and south pole the reconstruction accuracy deteriorates near the poles, see Figure 5(a), whereas the main part of the object is precisely identified. In Figure 5(c) we illustrate the reconstruction of the ellipsoid with the impedance given by (30). Both the impedance function and the Mean curvature of the surface reach their maximum values at the north pole of the ellipsoid. Therefore they eliminate each others effect on the reconstruction on this surface patch. On the contrary, near the south pole the small impedance value and the negative Mean curvature amplify their influence on the reconstruction. In Figure 5(d) the impedance and the Mean curvature affect the reconstruction in different directions, that is, along z_1 - and z_3 -axis, respectively. The best reconstruction, Figure 5(b), we obtain in the case of a uniform impedance (29) with the large imaginary part.

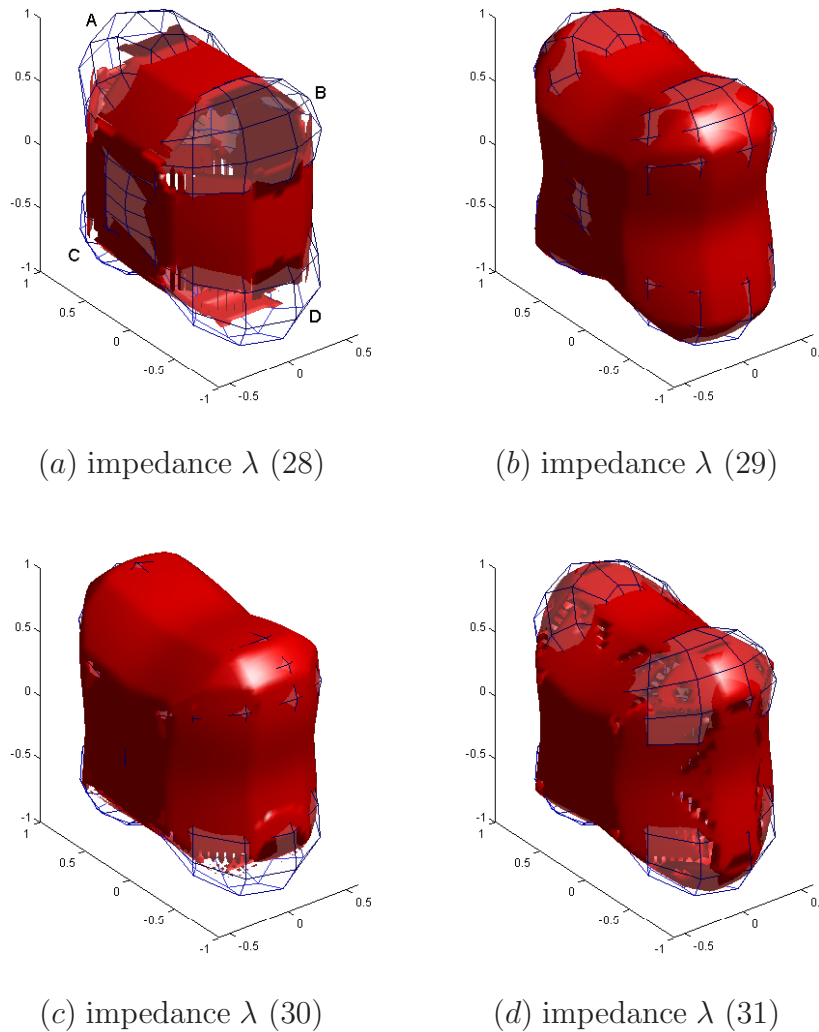


Figure 6. Reconstruction of the cushion (34)

The following obstacle is a cushion-shaped and it has 4 surface patches A, B, C, D,

see Figure 6, where the Mean curvature attains its extrema. These patches are difficult to identify in the case when the positive constant impedance is defined over the surface ∂D and which is comparable in the size with the Mean curvature, see Figure 6(a). The situation changes significantly if the impedance has a constant and large imaginary part, see Figure 6(b). For the impedance function (30), in Figure 6(c) we observe the same behaviour as in Figure 5(c). Even for the highly oscillating along z_1 -axis impedance function (31) we obtain a better reconstruction than for a constant impedance (28) due to the fact that in this case the contribution of the Mean curvature of the surface is balanced by the impedance.

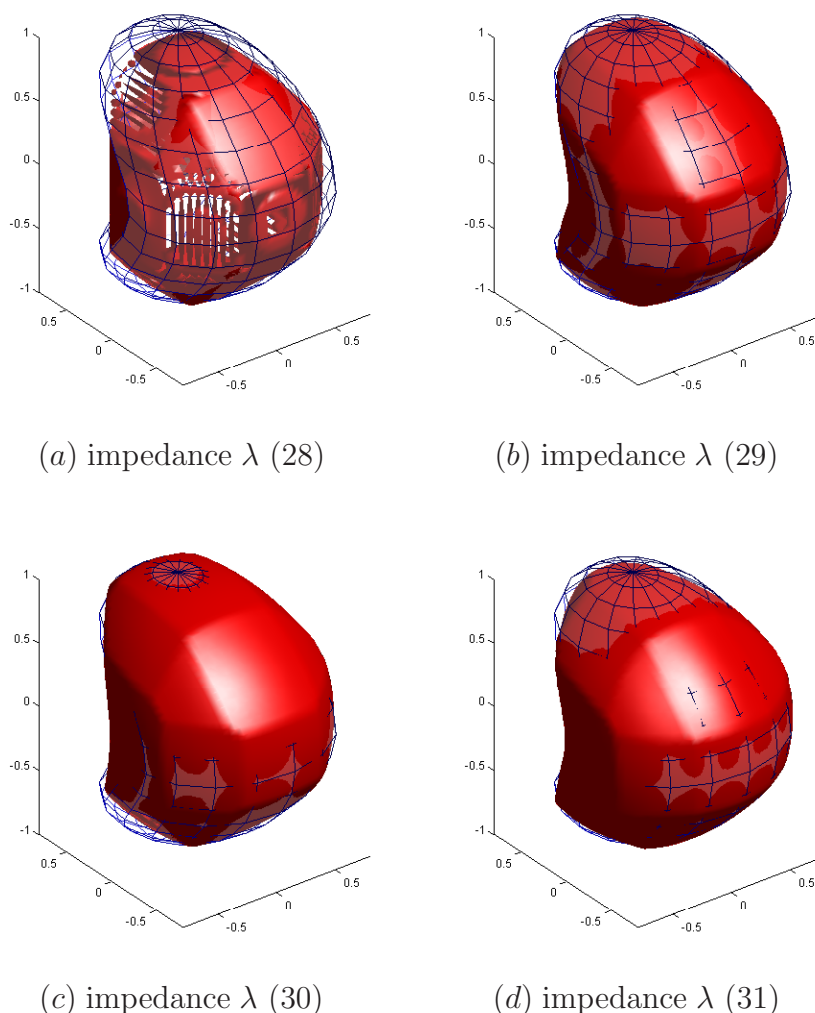


Figure 7. Reconstruction of the bean (35)

In the final example we consider a non-symmetric obstacle with four types of the impedance function. In Figures 7 and 8 we can see pictures illustrating the reconstructions of the convex and the concave side of the obstacle, respectively. Our conclusions about the dependence of the quality of reconstruction on the impedance function and the Mean curvature of ∂D are anew confirmed by the found reconstructions of this bean-shaped scatterer.

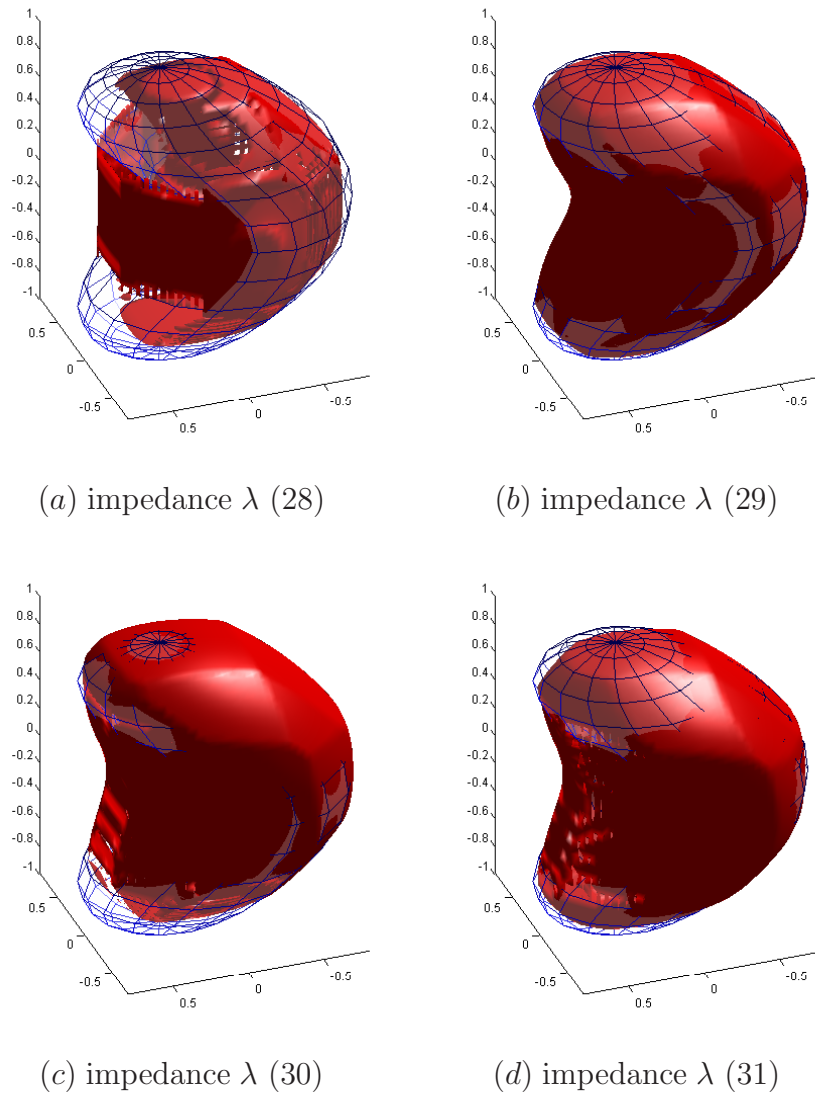


Figure 8. Reconstruction of the bean (35)

Summarizing, theoretically derived asymptotic behaviour (6) of the indicator function I_0 agrees with the numerically illustrated accuracy of reconstructions which depends on the impedance function and the Mean curvature of the sought surface.

Acknowledgments

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Appendix. Proof of Lemma 3.7

LEMMA APPENDIX A.1 *Let $z, \eta \in B_r^+$, then the functions $\tilde{w}_{\lambda(a)}^j$ defined in Proposition 3.1 satisfy*

$$\begin{aligned}\tilde{w}_{\lambda(a)}^0(z, \eta) &= \psi_0(z, \eta^*) + O(|\ln(z_3 + \eta_3)|), \\ \tilde{w}_{\lambda(a)}^j(z, \eta) &= \psi_j(z, \eta^*) + O((z_3 + \eta_3)^{-1}), \quad j = 1, 2, 3, \\ \nabla \tilde{w}_{\lambda(a)}^0(z, \eta) &= \nabla \psi_0(z, \eta^*) + O((z_3 + \eta_3)^{-1}), \\ \nabla \tilde{w}_{\lambda(a)}^j(z, \eta) &= \nabla \psi_j(z, \eta^*) + O((z_3 + \eta_3)^{-2}), \quad j = 1, 2, 3,\end{aligned}$$

where $\eta^* = (\eta_1, \eta_2, -\eta_3)$.

Proof. Let us give a justification for $j = 0$. Similar arguments work for $j = 1, 2, 3$. From the integral representation of $\tilde{w}_{\lambda(a)}^0$, we have:

$$\tilde{w}_{\lambda(a)}^0(z, \eta) = \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(z' - \eta') \cdot \xi'} e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'|} + \frac{2i\lambda(a)}{8\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(z' - \eta') \cdot \xi'} e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'| (|\xi'| - i\lambda(a))}.$$

The first term is nothing but $\Gamma(z, \eta^*) = \psi_0(z, \eta^*)$, see the proof of Lemma 3.3 in [14].

The second term can be estimated in the following way. We have

$$\left| \frac{2i\lambda(a)}{8\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(z' - \eta') \cdot \xi'} e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'| (|\xi'| - i\lambda(a))} \right| \leq O \left(\int_{\mathbb{R}^2} \frac{e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'| (|\xi'| - i\lambda(a))} \right).$$

In addition,

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'| (|\xi'| - i\lambda(a))} &= \int_{\mathbb{R}^2} \frac{e^{-(z_3 + \eta_3)|\xi'|} (|\xi'| + i\lambda(a)) d\xi'}{|\xi'| (|\xi'| + \Im\lambda(a))^2 + (\Re\lambda(a))^2} \\ &= \frac{4\pi}{3} \int_0^\infty \frac{e^{-(z_3 + \eta_3)r} (r + i\lambda(a)) dr}{(r + \Im\lambda(a))^2 + (\Re\lambda(a))^2}.\end{aligned}$$

We write the integral term of the right hand side as

$$\begin{aligned}& e^{(z_3 + \eta_3)\Im\lambda(a)} \int_{\Im\lambda(a)}^\infty \frac{e^{-(z_3 + \eta_3)r} (r + i\lambda(a) - \Im\lambda(a)) dr}{r^2 + (\Re\lambda(a))^2} \\ &= e^{(z_3 + \eta_3)\Im\lambda(a)} \int_{\Im\lambda(a)}^\infty \frac{e^{-(z_3 + \eta_3)r} r dr}{r^2 + (\Re\lambda(a))^2} + O(1).\end{aligned}$$

But since $\Re\lambda(a)$ is positive then this last integral is finite (even if $\Im\lambda(a) \leq 0$) and we have the estimate:

$$\int_{\Im\lambda(a)}^\infty \frac{e^{-(z_3 + \eta_3)r} r dr}{r^2 + (\Re\lambda(a))^2} = -e^{-(z_3 + \eta_3)\Im\lambda(a)} \ln(z_3 + \eta_3) + O(1).$$

Hence

$$\int_{\mathbb{R}^2} \frac{e^{i(z' - \eta') \cdot \xi'} e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'| (|\xi'| - i\lambda(a))} = O(\ln(z_3 + \eta_3)).$$

The absolute value of the gradient of $\int_{\mathbb{R}^2} \frac{e^{i(z' - \eta') \cdot \xi'} e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{|\xi'| (|\xi'| - i\lambda(a))}$ can be estimated as

$$\sqrt{3} \int_{\mathbb{R}^2} \frac{e^{-(z_3 + \eta_3)|\xi'|} d\xi'}{||\xi'| - i\lambda(a)|} = O \left(\int_0^\infty e^{-(z_3 + \eta_3)r} dr \right) = O((z_3 + \eta_3)^{-1}).$$

□

LEMMA APPENDIX A.2 Let $x \in B_r^+$ and $z \in C_{F(a),\theta}$ small enough, then for $|x - z| \rightarrow 0$

$$G(x, z) = \Gamma(x, z) + \Gamma(x, z^*) + O(\ln(|x - z|)),$$

$$\text{and } \frac{\partial}{\partial x_j} G(x, z) = \frac{\partial}{\partial x_j} \Gamma(x, z) + \frac{\partial}{\partial x_j} \Gamma(x, z^*) + O\left(\frac{1}{|x - z|}\right) + O\left(\frac{|z'|}{|x - z|}\right), \quad j = 1, 2, 3.$$

REMARK APPENDIX A.3 It is well known that we have the following rough estimates of G , see for instance [20, 21]:

$$|G(x, z)| \leq c|x - z| \quad \text{and} \quad |\nabla G(x, z)| \leq c|x - z|^{-2}, \quad x, z \in B_r^+.$$

The goal of this lemma is to give a more precise and explicit behaviour of G and its derivatives. The proof of the two dimensional version of this lemma is given in [16]. The proof for the 3D case is the same taking into account the different types of singularities between the 2D and the 3D cases.

Proof. We set $\Gamma_F(x, z) := \Gamma(F^{-1}(x), F^{-1}(z)) + \Gamma(F^{-1}(x), F^{-1}(z^*))$.

It is clear that $\nabla_x \cdot B(x) \nabla_x \Gamma_F(x, z) = -\delta(x, z) - \delta(x, z^*)$ in B_r^+ . Hence $G - \Gamma_F$ satisfies

$$\begin{cases} \nabla_x \cdot B(x) \nabla_x (G - \Gamma_F)(x, z) = 0, & \text{in } B_r^+ \\ B(x) \nabla_x (G - \Gamma_F)(x, z) \cdot \nu(x) + i\lambda(a) |J^T(x) e_3| (G - \Gamma_F)(x, z) = \\ -B(x) \nabla_x \Gamma_F(x, z) \cdot \nu(x) + i\lambda(a) |J^T(x) e_3| \Gamma_F(x, z) & \text{on } \partial B_r^+. \end{cases}$$

An integration by parts gives:

$$\begin{aligned} (G - \Gamma_F)(x, z) = & - \int_{\partial B_r^+} B(t) \nabla_t \Gamma_F(t, z) \cdot \nu(t) G(t, x) ds(t) \\ & + i\lambda(a) \int_{\partial B_r^+} |J^T(t) e_3| \Gamma_F(t, z) G(t, x) ds(t) \end{aligned}$$

which we write as

$$\begin{aligned} (G - \Gamma_F)(x, z) = & \int_{S_r} (I - B(t)) \nabla_t \Gamma_F(t, z) \cdot \nu(t) G(t, x) ds(t) \\ & + \int_{S_r^c} (I - B(t)) \nabla_t \Gamma_F(t, z) \cdot \nu(t) G(t, x) ds(t) \\ & - \int_{S_r} \frac{\partial}{\partial t_3} \Gamma_F(t, z) G(t, x) ds(t) - \int_{S_r^c} \nabla_t \Gamma_F(t, z) \cdot \nu(t) G(t, x) ds(t) \\ & + i\lambda(a) \left[\int_{S_r} |J^T(t) e_3| \Gamma_F(t, z) G(t, x) ds(t) + \int_{S_r^c} |J^T(t) e_3| \Gamma_F(t, z) G(t, x) ds(t) \right]. \end{aligned} \tag{A.1}$$

Since we are interested in $z \in C_{F(a),\theta}$, then the integrals over S_r^c and their derivatives are bounded with respect to x and z . Hence, we will consider only the integrals over S_r .

Recalling that $\Gamma(x, z) := \frac{1}{4\pi|x-z|}$ and that F is given by (25), then

$$\begin{aligned} \left(\frac{\partial}{\partial x_j} \Gamma\right)(F^{-1}(x), F^{-1}(z)) &= -\frac{1}{4\pi} \frac{x_j - z_j}{|F^{-1}(x) - F^{-1}(z)|^3}, \quad j = 1, 2, \\ \left(\frac{\partial}{\partial x_3} \Gamma\right)(F^{-1}(x), F^{-1}(z)) &= -\frac{1}{4\pi} \frac{x_3 - z_3 - f_a(x') + f_a(z')}{|F^{-1}(x) - F^{-1}(z)|^3}. \end{aligned}$$

Expanding $f_a(x')$ to the first order near the point z' , we obtain

$$f_a(x') - f_a(z') = \nabla f_a(z') \cdot (x' - z') + O(|x' - z'|^2).$$

Hence, we can write

$$\begin{aligned} |F^{-1}(x) - F^{-1}(z)|^2 &= |x - z|^2 \left\{ 1 - 2 \frac{\nabla f_a(z') \cdot (x' - z')(x_3 - z_3)}{|x - z|^2} \right. \\ &\quad \left. + |\nabla f_a(z')|^2 \frac{|x' - z'|^2}{|x - z|^2} + O(|x - z|) \right\}. \end{aligned}$$

Taking z small enough and since $\nabla f_a(0') = 0'$ then $\nabla f_a(z') = O(|z'|)$. Therefore,

$$1 - 2 \frac{\nabla f_a(z') \cdot (x' - z')(x_3 - z_3)}{|x - z|^2} + |\nabla f_a(z')|^2 \frac{|x' - z'|^2}{|x - z|^2} = 1 + O(|z'|).$$

Taking, in addition, $|x - z|$ small enough, yields

$$|F^{-1}(x) - F^{-1}(z)|^{-2} = \frac{1 + O(|z'|) + O(|x - z|)}{|x - z|^2},$$

which implies that,

$$\Gamma(F^{-1}(x), F^{-1}(z)) = \Gamma(x, z) + O\left(\frac{|z'|}{|x - z|}\right) + O(1), \quad (\text{A.2})$$

$$\left(\frac{\partial}{\partial x_j} \Gamma\right)(F^{-1}(x), F^{-1}(z)) = -\frac{1}{4\pi} \frac{x_j - z_j}{|x - z|^3} + \frac{O(|z'|)}{|x - z|^2} + O(1), \quad j = 1, 2, 3. \quad (\text{A.3})$$

From the properties of J , we have for $j = 1, 2, 3$,

$$\frac{\partial}{\partial x_j} (\Gamma(F^{-1}(x), F^{-1}(z))) = -\frac{1}{4\pi} \frac{x_j - z_j}{|x - z|^3} + \frac{O(|z'|)}{|x - z|^2} + O(1). \quad (\text{A.4})$$

The identity (A.4) and the chain rule applied to Γ_F imply that $\frac{\partial}{\partial t_3} \Gamma_F(t, z) = \frac{O(z')}{|t-z|^2} + O(1)$ on S_r . Hence, we obtain from (A.1),

$$|(G - \Gamma_F)(x, z)| \leq c_1 \int_{S_r} \frac{|t'| dt'}{|t - z|^2 |t - x|} + c_2 |z'| \int_{S_r} \frac{dt'}{|t - z|^2 |t - x|} + c_3 \int_{S_r} \frac{dt'}{|t - z| |t - x|},$$

with positive constants c_1, c_2 and c_3 . Then

$$|(G - \Gamma_F)(x, z)| = O(\ln(|x - z|)), \text{ for } z \in C_{F(a), \theta}.$$

In addition, differentiating in (A.1), for $j = 1, 2, 3$, we have the estimate:

$$\begin{aligned} & \left| \frac{\partial}{\partial x_j} (G - \Gamma_F)(x, z) \right| \\ & \leq c_1 \int_{S_r} \frac{|t'| dt'}{|t - z|^2 |t - x|^2} + c_2 |z'| \int_{S_r} \frac{dt'}{|t - z|^2 |t - x|^2} + c_3 \int_{S_r} \frac{dt'}{|t - z| |t - x|^2}, \end{aligned}$$

which implies that, for $j = 1, 2, 3$,

$$\left| \frac{\partial}{\partial x_j} (G - \Gamma_F)(x, z) \right| = O\left(\frac{1}{|x - z|}\right) + O\left(\frac{|z'|}{|x - z|}\right), \text{ for } z \in C_{F(a), \theta}. \quad (\text{A.5})$$

Finally from (A.2), (A.4) and (A.5), we obtain the estimates of Lemma Appendix A.2. \square

I. Estimate of the body integral term in (27).

We start by estimating the term $\int_{B_r^+} (I - B(z)) \nabla_z G(z, \xi) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) dz$ in (27), for $\xi = \eta$, $\eta \in C_{F(a), \theta}$. We use the explicit form of B , i.e.

$$B(z) := \begin{bmatrix} 1 & 0 & -\frac{\partial}{\partial z_1} f_a(z') \\ 0 & 1 & -\frac{\partial}{\partial z_2} f_a(z') \\ -\frac{\partial}{\partial z_1} f_a(z') & -\frac{\partial}{\partial z_2} f_a(z') & |\nabla f_a|^2(z') + 1 \end{bmatrix}$$

From $\frac{\partial}{\partial z_j} f_a(z') = \nabla(\frac{\partial}{\partial z_j} f_a)(0') \cdot z' + O(|z'|^2)$, we obtain

$$I - B(z) = E_1 \nabla(\frac{\partial}{\partial z_1} f_a)(0') \cdot z' + E_2 \nabla(\frac{\partial}{\partial z_2} f_a)(0') \cdot z' + O(|z'|^2), \quad (\text{A.6})$$

where $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Using (A.6), we get

$$\begin{aligned} & \int_{B_r^+} (I - B(z)) \nabla_z G(z, \xi) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) dz \\ &= \int_{B_r^+} E_1 \nabla_z G(z, \xi) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) \nabla(\frac{\partial}{\partial z_1} f_a)(0') \cdot z' dz \\ &+ \int_{B_r^+} E_2 \nabla_z G(z, \xi) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) \nabla(\frac{\partial}{\partial z_2} f_a)(0') \cdot z' dz \\ &+ O\left(\int_{B_r^+} |z'|^2 |\nabla_z G(z, \xi)| |\nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta)| dz \right). \end{aligned} \quad (\text{A.7})$$

Due to Lemmas Appendix A.1 and Appendix A.2,

$$\int_{B_r^+} |z|^2 |\nabla_z G(z, \xi)| |\nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta)| dz = O(1).$$

We write the first integral on the right hand side of (A.7) as $K^j = K_1^j + K_2^j$, $0 \leq j \leq 3$, where

$$\begin{aligned} K_1^j(\xi, \eta) &:= \int_{B_r^+} \frac{\partial G}{\partial z_3}(z, \xi) \frac{\partial \tilde{w}_{\lambda(a)}^j}{\partial z_1}(z, \eta) \nabla(\frac{\partial f_a}{\partial z_1})(0') \cdot z' dz, \\ K_2^j(\xi, \eta) &:= \int_{B_r^+} \frac{\partial G}{\partial z_1}(z, \xi) \frac{\partial \tilde{w}_{\lambda(a)}^j}{\partial z_3}(z, \eta) \nabla(\frac{\partial f_a}{\partial z_1})(0') \cdot z' dz. \end{aligned}$$

To estimate K_1^j , $0 \leq j \leq 3$, we apply Lemmas Appendix A.1 and Appendix A.2. Thus, we can replace $\nabla \tilde{w}_{\lambda(a)}^j(z, \eta)$ by $\nabla \psi_j(z, \eta^*)$ and $G(z, \xi)$ by $\Gamma(z, \xi) + \Gamma(z, \xi^*)$ since the remaining terms are bounded. Therefore,

$$\begin{aligned} & K_1^j(\xi, \eta) \\ &= \int_{B_r^+} \frac{\partial \psi_j}{\partial z_1}(z, \eta^*) \left[\frac{\partial \Gamma}{\partial z_3}(z, \xi) + \frac{\partial \Gamma}{\partial z_3}(z, \xi^*) \right] \left[\frac{\partial^2}{\partial z_1^2} f_a(0') z_1 + \frac{\partial^2}{\partial z_1 \partial z_2} f_a(0') z_2 \right] dz + O(1). \end{aligned} \quad (\text{A.8})$$

I.1. Estimate for $j = 0$. In this case,

$$\begin{aligned} K_1^0(\xi, \eta) &= \frac{1}{(4\pi)^2} \int_{B_r^+} \left[\frac{z_3 - \xi_3}{|z - \xi|^3} \frac{z_1 - \eta_1}{|z - \eta^*|^3} + \frac{z_3 + \xi_3}{|z - \xi^*|^3} \frac{z_1 - \eta_1}{|z - \eta^*|^3} \right] z_1 dz \frac{\partial^2}{\partial z_1^2} f_a(0') \\ &\quad + \frac{1}{(4\pi)^2} \int_{B_r^+} \left[\frac{z_3 - \xi_3}{|z - \xi|^3} \frac{z_1 - \eta_1}{|z - \eta^*|^3} + \frac{z_3 + \xi_3}{|z - \xi^*|^3} \frac{z_1 - \eta_1}{|z - \eta^*|^3} \right] z_2 dz \frac{\partial^2}{\partial z_1 \partial z_2} f_a(0') + O(1) \end{aligned}$$

Since B_r^+ is symmetric with respect to z_1 and z_2 , then the second term in K_1^0 is bounded for $\xi = \eta \in C_{F(a), \theta}$.

To estimate the first integral in K_1^0 , we apply the inequality $|x^3 - y^3| \leq |x - y|(x + y)^2$, $x, y > 0$. For $x := |z - \eta^*|$ and $y := |z - \xi|$, it reads $|z - \eta^*|^3 = |z - \xi|^3 + O(|\eta^* - \xi|)(|z - \eta^*| + |z - \xi|)^2$, and

$$\begin{aligned} \int_{B_r^+} \frac{z_3 - \xi_3}{|z - \xi|^3} \frac{z_1 - \eta_1}{|z - \eta^*|^3} z_1 dz &= \int_{B_r^+} \frac{(z_3 - \xi_3)(z_1 - \eta_1)}{|z - \eta^*|^6} z_1 dz \\ &\quad + O(|\eta^* - \xi|) \int_{B_r^+} \frac{(z_3 - \xi_3)(z_1 - \eta_1)(|z - \eta^*| + |z - \xi|)^2}{|z - \xi|^3 |z - \eta^*|^6} z_1 dz. \end{aligned}$$

We find that

$$\int_{B_r^+} \frac{(z_3 - \xi_3)(z_1 - \eta_1)(|z - \eta^*| + |z - \xi|)^2}{|z - \xi|^3 |z - \eta^*|^6} z_1 dz = O(\eta_3^{-1}) + O\left(\frac{\eta_1}{\eta_3}\right).$$

Hence, we obtain

$$K_1^0(\eta, \eta) = \frac{2}{(4\pi)^2} \int_{B_r^+} \frac{z_3(z_1 - \eta_1)}{|z - \eta^*|^6} z_1 dz \frac{\partial^2}{\partial z_1^2} f_a(0') + O(1),$$

which we can write as

$$\begin{aligned} K_1^0(\eta, \eta) &= \frac{2}{(4\pi)^2} \frac{\partial^2}{\partial z_1^2} f_a(0') \left(\int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)^2}{|z - \eta^*|^6} dz + \eta_1 \int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)}{|z - \eta^*|^6} dz \right. \\ &\quad \left. - \eta_3 \int_{B_r^+} \frac{(z_1 - \eta_1)^2}{|z - \eta^*|^6} dz - \eta_1 \eta_3 \int_{B_r^+} \frac{z_1 - \eta_1}{|z - \eta^*|^6} dz \right) + O(1), \end{aligned}$$

and since

$$|\eta_1 \int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)}{|z - \eta^*|^6} dz| \leq |\eta_1| \int_{B_r^+} \frac{1}{|z - \eta^*|^4} dz \leq C \frac{|\eta_1|}{d(\eta, B_r^+)} = C \frac{|\eta_1|}{|\eta_3|} = O(1),$$

$\eta_3 \int_{B_r^+} \frac{(z_1 - \eta_1)^2}{|z - \eta^*|^6} dz = O(1)$ and $\int_{B_r^+} \frac{z_1 - \eta_1}{|z - \eta^*|^6} dz = O(1)$ for $\eta \in C_{F(a), \theta}$, then

$$K_1^0(\eta, \eta) = \frac{2}{(4\pi)^2} \int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)^2}{|z - \eta^*|^6} dz \frac{\partial^2}{\partial z_1^2} f_a(0') + O(1).$$

Similarly as K_1^0 , we can estimate K_2^0 as

$$K_2^0(\eta, \eta) = \frac{2}{(4\pi)^2} \int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)^2}{|z - \eta^*|^6} dz \frac{\partial^2}{\partial z_1^2} f_a(0') + O(1).$$

Gathering K_1^0 and K_2^0 , we obtain

$$K^0(\eta, \eta) = \frac{1}{4\pi^2} \int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)^2}{|z - \eta^*|^6} dz \frac{\partial^2}{\partial z_1^2} f_a(0') + O(1), \quad \eta \in C_{F(a), \theta}.$$

The same arguments as that used in the computation of K^0 can be applied to estimate the second integral in (A.7) to get

$$\begin{aligned} & \int_{B_r^+} E_2 \nabla_z G(z, \xi) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) \nabla \left(\frac{\partial}{\partial z_2} f_a \right) (0') \cdot z' dz \\ &= \frac{1}{4\pi^2} \int_{B_r^+} \frac{(z_3 + \eta_3)(z_2 - \eta_2)^2}{|z - \eta^*|^6} dz \frac{\partial^2}{\partial z_2^2} f_a(0') + O(1) \end{aligned}$$

for $\eta \in C_{F(a), \theta}$. Finally, since,

$$\int_{B_r^+} \frac{(z_3 + \eta_3)(z_j - \eta_j)^2}{|z - \eta^*|^6} dz = -\frac{\pi}{4} \ln \eta_3 + O(1), \quad j = 1, 2.$$

we show that for $\eta \in C_{F(a), \theta}$

$$\int_{B_r^+} (I - B(z)) \nabla_z G(z, \eta) \cdot \nabla_z \tilde{w}_{\lambda(a)}^0(z, \eta) dz = -\frac{1}{16\pi} \ln \eta_3 \Delta f_a(0') + O(1) \quad (\text{A.9})$$

I.2. The cases $j = 1, 2$. Since $\psi_j(z, \eta^*) = -\frac{(z_j - \eta_j)}{4\pi|z - \eta^*|^3}$, then due to the symmetry properties of B_r^+ and those, with respect to z_j , of the integrand in (A.8), we deduce that $K_1^j(\eta, \eta) = O(\ln(\eta_3))$ and

$$\int_{B_r^+} (I - B(z)) \nabla_z G(z, \eta) \cdot \nabla_z \tilde{w}_{\lambda(a)}^j(z, \eta) dz = O(\ln(\eta_3)). \quad (\text{A.10})$$

I.3: The case $j = 3$.

In this case $\psi_3(z, \eta) = -\frac{(z_3 - \eta_3)}{4\pi|z - \eta|^3}$ and $\frac{\partial \psi_3}{\partial z_1}(z, \eta^*) = \frac{3}{4\pi} \frac{(z_3 + \eta_3)(z_1 - \eta_1)}{|z - \eta^*|^5}$. thus,

$$\begin{aligned} K_1^3(\eta, \eta) &= \\ & \frac{-3}{16\pi^2} \int_{B_r^+} \frac{(z_3 + \eta_3)(z_1 - \eta_1)}{|z - \eta^*|^5} \left(\frac{z_3 + \eta_3}{|z - \eta^*|^3} + \frac{z_3 - \eta_3}{|z - \eta|^3} \right) \left(\frac{\partial^2}{\partial z_1^2} f_a(0') z_1 + \frac{\partial^2}{\partial z_1 \partial z_2} f_a(0') z_2 \right) dz \end{aligned}$$

The term involving $\frac{\partial^2}{\partial z_1 \partial z_2} f_a(0')$ is bounded due to symmetry argument. We need to estimate the term involving $\frac{\partial^2}{\partial z_1^2} f_a(0')$. After some computations, we get

$$\begin{aligned} & \int_0^r (z_3 + \eta_3)^2 \int_{-r}^r \int_{-r}^r \frac{z_1^2}{|z - \eta^*|^8} dz_1 dz_2 dz_3 = \frac{\pi}{12\eta_3} + O(\ln(\eta_3)), \\ & \int_{B_r^+} (z_3 - \eta_3)(z_3 + \eta_3) \frac{z_1^2}{|z - \eta^*|^5 |z - \eta|^3} dz = O(\ln(\eta_3)). \end{aligned}$$

Therefore,

$$K_1^3(\eta, \eta) = -\frac{1}{64\pi\eta_3} + O(\ln(\eta_3)). \quad (\text{A.11})$$

We deal now with

$$K_2^3(\eta, \eta) = \int_{B_r^+} \left(\frac{\partial}{\partial z_1} \Gamma(z, \eta) + \frac{\partial}{\partial z_1} \Gamma(z, \eta^*) \right) \frac{\partial}{\partial z_3} \psi_3(z, \eta^*) \frac{\partial^2}{\partial z_1^2} f_a(0') z_1 dz + O(1).$$

Since,

$$\frac{\partial}{\partial z_3} \psi_3(z, \eta^*) = -\frac{1}{4\pi} \frac{(z_1 - \eta_1)^2 + (z_2 - \eta_2)^2 - 2(z_3 + \eta_3)^2}{|z - \eta^*|^5} \quad (\text{A.12})$$

then $K_2^3(\eta, \eta) = \frac{1}{16\pi^2} \frac{\partial^2}{\partial z_1^2} f_a(0') (K_{21}^3 + K_{22}^3 - 2K_{23}^3) + O(1)$ where

$$\begin{aligned} K_{21}^3 &= \int_{B_r^+} \frac{z_1^4}{|z - \eta^*|^8} dz = \frac{\pi}{8\eta_3} + O(\ln(\eta_3)), \\ K_{22}^3 &= \int_{B_r^+} \frac{z_1^2 z_2^2}{|z - \eta^*|^8} dz = \frac{\pi}{24\eta_3} + O(\ln(\eta_3)), \\ K_{23}^3 &= \int_{B_r^+} \frac{z_1^2 (z_3 + \eta_3)^2}{|z - \eta^*|^8} dz = \frac{\pi}{12\eta_3} + O(\ln(\eta_3)). \end{aligned}$$

Hence, $K_2^3(\eta, \eta) = O(\ln(\eta_3))$.

Therefore, thanks to (A.11), $K^3(\eta, \eta) = -\frac{1}{64\pi\eta_3} \frac{\partial^2}{\partial z_1^2} f_a(0') + O(\ln(\eta_3))$ and

$$\int_{B_r^+} (I - B(z)) \nabla_z G(z, \eta) \cdot \nabla_z \tilde{w}_{\lambda(a)}^3(z, \eta) dz = \frac{-1}{64\pi\eta_3} \Delta f_a(0') + O(\ln(\eta_3)). \quad (\text{A.13})$$

II. Estimates of the three surface integrals in (27)

II.1 Let us first consider the surface integral $\int_{S_r} [\psi_j(z, \eta) - |J^T(z)e_3| \tilde{\psi}_j(z, \eta)] G(z, \eta) ds(z)$.

We separate it into two parts as follows

$$\int_{S_r} [\psi_j(z, \eta) - \tilde{\psi}_j(z, \eta)] G(z, \eta) ds(z) + \int_{S_r} [1 - |J^T(z)e_3|] \tilde{\psi}_j(z, \eta) G(z, \eta) ds(z).$$

II.1.1 We start by estimating the first term $\int_{S_r} [\psi_j(z, \eta) - \tilde{\psi}_j(z, \eta)] G(z, \eta) ds(z)$.

Case $j = 0$. Using (A.2), we obtain for $\eta \in C_{F(a), \theta}$,

$$\int_{S_r} [\Gamma(z, \eta) - \Gamma(F^{-1}(z), F^{-1}(\eta))] G(z, \eta) ds(z) = O(|\eta'|) \int_{S_r} \frac{1}{|z - \eta|^2} ds(z) = O(1). \quad (\text{A.14})$$

Cases $j = 1, 2, 3$. As for the case $j = 0$, employing (A.3), it is easy to see that

$$\int_{S_r} [\psi_j(z, \eta) - \tilde{\psi}_j(z, \eta)] G(z, \eta) ds(z) = O(\ln(\eta_3)). \quad (\text{A.15})$$

II.1.2 Using the estimate $1 - |J^T(z)e_3| = O(|z'|)$ and the expansion (A.2), we get

$$\int_{S_r} [1 - |J^T(z)e_3|] \tilde{\psi}_0(z, \eta) G(z, \eta) ds(z) = O(1), \quad (\text{A.16})$$

for $\eta \in C_{F(a), \theta}$ small enough. Similarly, employing (A.4), we show that

$$\int_{S_r} [1 - |J^T(z)e_3|] \tilde{\psi}_j(z, \eta) G(z, \eta) ds(z) = O(\ln(\eta_3)), \quad j = 1, 2, 3. \quad (\text{A.17})$$

II.2 Due to Lemma Appendix A.1, we can replace $\tilde{w}_{\lambda(a)}^j(z, \eta)$ by $\psi_j(z, \eta^*)$. Therefore,

$$\int_{S_r} [1 - |J^T(z)e_3|] \tilde{w}_{\lambda(a)}^0(z, \eta) G(z, \eta) ds(z) = O(1), \quad (\text{A.18})$$

$$\int_{S_r} [1 - |J^T(z)e_3|] \tilde{w}_{\lambda(a)}^j(z, \eta) G(z, \eta) ds(z) = O(\ln(\eta_3)); \quad j = 1, 2, 3. \quad (\text{A.19})$$

II.3 We deal now with the last term in (27). We divide it as follows:

$$\begin{aligned} & \int_{S_r} [\nabla_z \psi_j(z, \eta) \cdot e_3 - \nabla_z \tilde{\psi}_j(z, \eta) \cdot e_3] G(z, \eta) ds(z) \\ & + \int_{S_r} \nabla_z \tilde{\psi}_j(z, \eta) \cdot (e_3 - J^T(z) e_3) G(z, \eta) ds(z). \end{aligned} \quad (\text{A.20})$$

case $j = 0$. We recall that $\partial_{z_3} \Gamma(F^{-1}(z), F^{-1}(\eta)) = -\frac{1}{4\pi} \frac{z_3 - \eta_3 - f_a(z') + f_a(\eta')}{|F^{-1}(z) - F^{-1}(\eta)|^3}$.

The expansion

$$\begin{aligned} |F^{-1}(z) - F^{-1}(\eta)|^2 &= |z - \eta|^2 \times \\ & \times \left(1 - 2 \frac{(z_3 - \eta_3) \nabla f_a(\eta') \cdot (z' - \eta')}{|z - \eta|^2} - \frac{\nabla^2 f_a(\eta') (z' - \eta') \cdot (z' - \eta') (z_3 - \eta_3)}{|z - \eta|^2} + O(|\eta'|^2) \right) \end{aligned} \quad (\text{A.21})$$

yields to

$$\begin{aligned} & \nabla_z(\Gamma)(F^{-1}(z), F^{-1}(\eta)) \cdot e_3 - \nabla_z(\Gamma)(z, \eta) \cdot e_3 \\ &= -\frac{3}{4\pi} \frac{(z_3 - \eta_3)^2 \nabla f_a(\eta') \cdot (z' - \eta')}{|z - \eta|^5} - \frac{3}{8\pi} \frac{(z_3 - \eta_3)^2 \nabla^2 f_a(\eta') (z' - \eta') \cdot (z' - \eta')}{|z - \eta|^5} \\ & + \frac{1}{4\pi} \frac{f_a(z') - f_a(\eta')}{|z - \eta|^3} + \frac{O(|\eta'|^2)}{|z - \eta|^2} + O(1). \end{aligned} \quad (\text{A.22})$$

Since $G(z, \eta) = 2\Gamma(z, \eta)$, for $z, \eta \in S_r$, then

$$\begin{aligned} & \int_{S_r} [\nabla_z \Gamma(z, \eta) \cdot e_3 - \nabla_z \Gamma(F^{-1}(z), F^{-1}(\eta)) \cdot e_3] G(z, \eta) ds(z) \\ &= \frac{3\eta_3^2}{16\pi^2} \int_{S_r} \frac{\nabla^2 f_a(\eta') (z' - \eta') \cdot (z' - \eta')}{|z - \eta|^6} ds(z) + O(1). \end{aligned}$$

We have

$$\int_{S_r} \frac{\nabla^2 f_a(\eta') (z' - \eta') \cdot (z' - \eta')}{|z - \eta|^6} ds(z) = O(\eta_3^{-2}).$$

Therefore,

$$\int_{S_r} [\nabla_z \Gamma(z, \eta) \cdot e_3 - \nabla_z \Gamma(F^{-1}(z), F^{-1}(\eta)) \cdot e_3] G(z, \eta) ds(z) = O(1). \quad (\text{A.23})$$

To estimate the second integral in (A.20), we split it into two terms.

Since $e_3 - J^T(z) e_3 = O(|z'|)$, we deduce from (A.23) that

$$\int_{S_r} [\nabla_z \Gamma(z, \eta) - \nabla_z \Gamma(F^{-1}(z), F^{-1}(\eta))] \cdot (e_3 - J^T(z) e_3) G(z, \eta) ds(z) = O(1). \quad (\text{A.24})$$

Now we write

$$\nabla_z \Gamma(z, \eta) (e_3 - J^T(z) e_3) = \nabla_z \Gamma(z, \eta) \cdot \left(\frac{\partial f_a(z')}{\partial z_1}, \frac{\partial f_a(z')}{\partial z_2}, 0 \right) = -\frac{1}{4\pi} \frac{(\nabla f_a(z') \cdot (z' - \eta'))}{|z - \eta|^3},$$

since $(I - J^T(z)) e_3 = \left(\frac{\partial f_a(z')}{\partial z_1}, \frac{\partial f_a(z')}{\partial z_2}, 0 \right)$. Hence

$$\int_{S_r} \nabla_z \Gamma(z, \eta) (e_3 - J^T(z) e_3) G(z, \eta) ds(z) = -\frac{1}{16\pi^2} \int_{S_r} \frac{\nabla f_a(z') \cdot (z' - \eta')}{|z - \eta|^4} ds(z)$$

$$= -\frac{1}{16\pi^2} \left[\int_{S_r} \frac{\nabla f_a(\eta') \cdot (z' - \eta')}{|z - \eta|^4} ds(z) + \int_{S_r} \frac{\nabla^2 f_a(\eta')(z' - \eta') \cdot (z' - \eta')}{|z - \eta|^4} ds(z) \right] \\ + O\left(\int_{S_r} \frac{|z' - \eta'|^3}{|z - \eta|^4} ds(z) \right).$$

Due to the symmetry of S_r , we have

$$\int_{S_r} \frac{\nabla f_a(\eta') \cdot (z' - \eta')}{|z - \eta|^4} ds(z) = O(1).$$

Using the estimate

$$\int_{S_r} \frac{(z_i - \eta_i)^2}{|z - \eta|^4} ds(z) = -\pi \ln(\eta_3) + O(1), \text{ for } i = 1, 2,$$

we show that

$$\int_{S_r} \nabla_z \Gamma(z, \eta)(e_3 - J^T(z)e_3)G(z, \eta) ds(z) = \frac{1}{16\pi} \Delta f_a(\eta') \ln(\eta_3) + O(1). \quad (\text{A.25})$$

Thus, summing up (A.23)-(A.25), we get

$$\int_{S_r} [\nabla_z \psi_0(z, \eta) \cdot e_3 - \nabla_z \tilde{\psi}_0(z, \eta) \cdot J^T(z)e_3]G(z, \xi) ds(z) = \frac{\Delta f_a(\eta') \ln(\eta_3)}{16\pi} + O(1) \quad (\text{A.26})$$

Cases $j = 1, 2$. We recall that $\partial_{z_3} \psi_j(z, \eta) = \frac{3}{4\pi} \frac{(z_3 - \eta_3)(z_j - \eta_j)}{|z - \eta|^5}$ and

$$\partial_{z_3} \psi_j(F^{-1}(z), F^{-1}(\eta)) = \frac{3}{4\pi} \frac{(z_3 - \eta_3 - f_a(z') + f_a(\eta'))(z_j - \eta_j)}{|F^{-1}(z) - F^{-1}(\eta)|^5}.$$

From (A.21) and a Taylor expansion, we have

$$|F^{-1}(z) - F^{-1}(\eta)|^{-5} \\ = 5 \frac{(z_3 - \eta_3) \nabla f_a(\eta')(z' - \eta')}{|z - \eta|^7} + \frac{5}{2} \frac{\nabla^2 f_a(\eta')(z' - \eta') \cdot (z' - \eta')(z_3 - \eta_3)}{|z - \eta|^7} \quad (\text{A.27}) \\ + |z - \eta|^{-5} + O(|\eta'|^2 |z - \eta|^{-5}) + O(|z - \eta|^{-3})$$

Therefore, using symmetry arguments,

$$\int_{S_r} (\nabla_z \psi_j(z, \eta) - (\nabla_z \psi_j)(F^{-1}(z), F^{-1}(\eta))) \cdot e_3 G(z, \eta) ds(z) = O(\ln(\eta_3)). \quad (\text{A.28})$$

Since $(I - J^T(z))\tilde{\nu} = (\nabla \partial_{z_1} f_a(\eta')(z' - \eta'), \nabla \partial_{z_2} f_a(\eta')(z' - \eta'), 0) + O(|z - \eta|^2)$, we write

$$(I - J^T(z))\tilde{\nu} \cdot (\nabla_z \psi_j)(F^{-1}(z), F^{-1}(\eta)) \\ = \nabla \partial_{z_1} f_a(\eta') \cdot (z' - \eta') \partial_{z_1} \psi_j(F^{-1}(z), F^{-1}(\eta)) \\ + \nabla \partial_{z_2} f_a(\eta') \cdot (z' - \eta') \partial_{z_2} \psi_j(F^{-1}(z), F^{-1}(\eta)) + O(|z - \eta|^2)$$

Again, from symmetry arguments in $(z_j - \eta_j)$, $j = 1, 2$, this implies that

$$\int_{S_r} (I - J^T(z))\tilde{\nu} \cdot (\nabla_z \psi_j)(F^{-1}(z), F^{-1}(\eta))G(z, \eta) ds(z) = O(\ln(\eta_3)). \quad (\text{A.29})$$

Gathering the integrals (A.28) and (A.29), we deduce that for $j = 1, 2$, we have

$$\int_{S_r} [\nabla_z \psi_j(z, \eta) \cdot e_3 - \nabla_z \psi_j(F^{-1}(z), F^{-1}(\eta)) \cdot J^T(z) e_3] G(z, \eta) ds(z) = O(\ln(\eta_3)). \quad (\text{A.30})$$

Case $j = 3$. In this case,

$$\partial z_3 \psi_3(F^{-1}(z), F^{-1}(\eta)) = -\frac{1}{4\pi} \frac{(z_1 - \eta_1)^2 + (z_2 - \eta_2)^2 - 2(z_3 - \eta_3 - f_a(z') + f_a(\eta'))^2}{|F^{-1}(z) - F^{-1}(\eta)|^5}.$$

Thus, using (A.27) gives

$$\begin{aligned} & \partial z_3 \psi_3(F^{-1}(z), F^{-1}(\eta)) - \partial z_3 \psi_3(z, \eta) \\ &= -\frac{5}{4\pi} \frac{((z_1 - \eta_1)^2 + (z_2 - \eta_2)^2 - 2(z_3 - \eta_3)^2)(z_3 - \eta_3) \nabla f_a(\eta')(z' - \eta')}{|z - \eta|^7} \\ & \quad - \frac{5}{8\pi} \frac{((z_1 - \eta_1)^2 + (z_2 - \eta_2)^2 - 2(z_3 - \eta_3)^2) \nabla^2 f_a(\eta')(z' - \eta') \cdot (z' - \eta')(z_3 - \eta_3)}{|z - \eta|^7} \\ & \quad - \frac{1}{\pi} \frac{(z_3 - \eta_3)(f_a(z') - f_a(\eta'))}{|z - \eta|^5} + O(|z - \eta|^{-1}) + O(|\eta'|^2 |z - \eta|^{-3}) \end{aligned}$$

Expanding f_a to the second order around z' ,

$$f_a(x') - f_a(z') = \nabla f_a(z') \cdot (x' - z') + \frac{1}{2} \nabla^2 f_a(z')(x' - z') \cdot (x' - z') + O(|x' - z'|^3),$$

using the symmetry property of some integrands and recalling that $G(z, \eta) = 2\Gamma(z, \eta)$ on S_r gives

$$\begin{aligned} & \int_{S_r} (\nabla_z \psi_3(z, \eta) - (\nabla_z \psi_3)(F^{-1}(z), F^{-1}(\eta))) \cdot e_3 G(z, \eta) ds(z) = \\ & \quad -\frac{5\eta_3}{8\pi^2} \int_{S_r} \frac{[(z_1 - \eta_1)^2 + (z_2 - \eta_2)^2 - 2\eta_3^2][\partial^2 z_1 f_a(\eta')(z_1 - \eta_1)^2 + \partial^2 z_2 f_a(\eta')(z_2 - \eta_2)^2]}{|z - \eta|^8} ds(z) \\ & \quad - \frac{\eta_3}{4\pi^2} \int_{S_r} \frac{\partial^2 z_1 f_a(\eta')(z_1 - \eta_1)^2 + \partial^2 z_2 f_a(\eta')(z_2 - \eta_2)^2}{|z - \eta|^6} ds(z) + O(\ln(\eta_3)). \end{aligned}$$

We write it as,

$$\begin{aligned} & \int_{S_r} (\nabla_z \psi_3(z, \eta) - (\nabla_z \psi_3)(F^{-1}(z), F^{-1}(\eta))) \cdot e_3 G(z, \eta) ds(z) \\ &= -\frac{7\eta_3}{8\pi^2} \int_{S_r} \frac{\partial^2 z_1 f_a(\eta')(z_1 - \eta_1)^2 + \partial^2 z_2 f_a(\eta')(z_2 - \eta_2)^2}{|z - \eta|^6} ds(z) \\ & \quad + \frac{15\eta_3^3}{8\pi^2} \int_{S_r} \frac{\partial^2 z_1 f_a(\eta')(z_1 - \eta_1)^2 + \partial^2 z_2 f_a(\eta')(z_2 - \eta_2)^2}{|z - \eta|^8} ds(z) + O(\ln(\eta_3)) \end{aligned}$$

Since, for $j = 1, 2$,

$$\int_{S_r} \frac{(z_i - \eta_i)^2}{|z - \eta|^6} ds(z) = \frac{\pi}{4\eta_3^2} \quad \text{and} \quad \int_{S_r} \frac{(z_i - \eta_i)^2}{|z - \eta|^8} ds(z) = \frac{\pi}{12\eta_3^4}, \quad (\text{A.31})$$

we deduce that,

$$\begin{aligned} & \int_{S_r} (\nabla_z \psi_3(z, \eta) - (\nabla_z \psi_3)(F^{-1}(z), F^{-1}(\eta))) \cdot e_3 G(z, \eta) ds(z) \\ &= -\frac{1}{16\pi\eta_3} \Delta f_a(0) + O(\ln(\eta_3)). \end{aligned} \quad (\text{A.32})$$

Now, we deal with the last term. Using (A.29),

$$\begin{aligned} & \int_{S_r} (I - J^T(z))e_3 \cdot (\nabla_z \psi_3)(F^{-1}(z), F^{-1}(\eta))G(z, \eta) ds(z) \\ &= \int_{S_r} \partial_{z_1} \psi_3(F^{-1}(z), F^{-1}(\eta)) \nabla(\partial_{z_1} f_a)(\eta') \cdot (z' - \eta') G(z, \eta) ds(z) \\ &+ \int_{S_r} \partial_{z_2} \psi_3(F^{-1}(z), F^{-1}(\eta)) \nabla(\partial_{z_2} f_a)(\eta') \cdot (z' - \eta') G(z, \eta) ds(z) + O(\ln(\eta_3)). \end{aligned} \quad (\text{A.33})$$

Taking the derivative in (A.4), we show that for $j = 1, 2$

$$\partial_{z_j} \psi_3(F^{-1}(z), F^{-1}(\eta)) = \frac{3}{4\pi} \frac{(z_3 - \eta_3)(z_j - \eta_j)}{|z - \eta|^5} + O\left(\frac{|z'| (z_j - \eta_j)}{|z - \eta|^4}\right). \quad (\text{A.34})$$

Thus, combining (A.33), (A.34) and (A.31),

$$\int_{S_r} (I - J^T(z))e_3 \cdot (\nabla_z \psi_3)(F^{-1}(z), F^{-1}(\eta))G(z, \eta) ds(z) = -\frac{3\Delta f_a(\eta')}{32\pi\eta_3} + O(\ln(\eta_3)). \quad (\text{A.35})$$

III. Estimate of \tilde{R}_j , $0 \leq j \leq 3$, given by (27), in the the local coordinates

Combining (A.9), (A.14), (A.16), (A.18) and (A.26) we derive the following asymptotics

$$\tilde{R}_0(\eta, \eta) = \left(-\frac{1}{16\pi} \Delta f_a(0') - \frac{1}{16\pi} \Delta f_a(0'')\right) \ln(\eta_3) + O(1) = -\frac{1}{8\pi} \Delta f_a(0') \ln(\eta_3) + O(1).$$

By (A.10), (A.15), (A.17), (A.19) and (A.30),

$$\tilde{R}_j(\eta, \eta) = O(\ln(\eta_3)), \quad j=1, 2.$$

From (A.13), (A.15), (A.17), (A.20), , (A.19), (A.32) and (A.35)

$$\tilde{R}_3(\eta, \eta) = \left(-\frac{1}{64} + \frac{1}{16} + \frac{3}{32}\right) \frac{\Delta f_a(0')}{\pi\eta_3} + O(\ln(\eta_3)) = \frac{9\Delta f_a(0')}{64\pi\eta_3} + O(\ln(\eta_3)).$$

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