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Relative Gröbner bases and dimension polynomials in difference-differential modules

Franz Winkler
RISC, JKU

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Beijing

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and

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1. Introduction

- we introduce the concept of generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ and on difference-differential modules
- we introduce the concept of relative Gröbner basis w.r.t. two different generalized term orders
- based on relative Gröbner bases we present a new algorithmic approach for computing the Hilbert function and Hilbert polynomial (or dimension polynomial) of difference-differential modules

Difference-differential modules

Def.1.1:

- R a commutative Noetherian ring
- $\Delta = \{\delta_1, \dots, \delta_m\}$ a set of derivations on R
 $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ a set of automorphisms of R
s.t. $\alpha \circ \beta = \beta \circ \alpha$ for any $\alpha, \beta \in \Delta \cup \Sigma$
- Then R is called a *difference-differential ring* with the basic set of derivations Δ and the basic set of automorphisms Σ , or shortly a *Δ - Σ -ring*;
- if R is a field, then it is called a *Δ - Σ -field*.

Ex.1.1: Let $R = K[x_1, \dots, x_n]$ for a field K , $\delta_i = \partial/\partial x_i$ and σ_i the automorphism which maps x_i to $x_i - 1$. Then R is a Δ - Σ -ring for $\Delta = \{\delta_1, \dots, \delta_n\}$ and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$.

Def.1.2:

- R a Δ - Σ -ring
- Λ the free commutative semigroup of words over Δ and $\tilde{\Sigma}$ (containing the elements of Σ and their inverses)
- then a *difference-differential operator* (or shortly a *Δ - Σ -operator*) over R is

$$\sum_{\lambda \in \Lambda} a_{\lambda} \lambda, \quad (1.2)$$

where $a_{\lambda} \in R$ and only finitely many coefficients a_{λ} are different from 0

fundamental relations:

$$\begin{aligned}
\sum_{\lambda \in \Lambda} a_\lambda \lambda + \sum_{\lambda \in \Lambda} b_\lambda \lambda &= \sum_{\lambda \in \Lambda} (a_\lambda + b_\lambda) \lambda, \\
a(\sum_{\lambda \in \Lambda} a_\lambda \lambda) &= \sum_{\lambda \in \Lambda} (aa_\lambda) \lambda, \\
(\sum_{\lambda \in \Lambda} a_\lambda \lambda) \mu &= \sum_{\lambda \in \Lambda} a_\lambda (\lambda \mu), \\
\delta a &= a\delta + \delta(a), \quad \sigma a = \sigma(a)\sigma,
\end{aligned} \tag{1.3}$$

for $a_\lambda, b_\lambda \in R$, $\lambda, \mu \in \Lambda$, $a \in R$, $\delta \in \Delta$, $\sigma \in \tilde{\Sigma}$

Def.1.3: The (non-commutative) ring of all Δ - Σ -operators over a Δ - Σ -ring R is called the *ring of difference-differential operators* (or shortly the *ring of Δ - Σ -operators*) over R . It will be denoted by D .

A left D -module M is called a *difference-differential module* (or a Δ - Σ -module).

- for $\Sigma = \emptyset$: $D = R[\delta_1, \dots, \delta_m]$, the ring of differential operators
- if $R = K[x_1, \dots, x_m]$ over a field K and $\delta_i = \partial/\partial x_i$ for $1 \leq i \leq m$, then D is the Weyl algebra $\mathcal{A}_m(K)$.

In the ring of Δ - Σ -operators the terms are of the form

$$\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \sigma_1^{l_1} \dots \sigma_m^{l_n}$$

and the exponents of σ_i are in \mathbb{Z} . The notion of term order, as commonly used in Gröbner basis theory, is no longer valid. We need to generalize the concept of term order.

2. Generalized term order

Def.2.1: Let $\mathbb{Z}^n = \bigcup_{j=1}^k \mathbb{Z}_j^n$, where

- (i) $(0, \dots, 0) \in \mathbb{Z}_j^n$, and \mathbb{Z}_j^n does not contain any pair of invertible elements $c = (c_1, \dots, c_n) \neq 0$ and $-c = (-c_1, \dots, -c_n)$,
- (ii) \mathbb{Z}_j^n is isomorphic to \mathbb{N}^n as a semigroup,
- (iii) the group generated by \mathbb{Z}_j^n is \mathbb{Z}^n .

Then $\{\mathbb{Z}_j^n \mid j = 1, \dots, k\}$ is called an *orthant decomposition* of \mathbb{Z}^n and \mathbb{Z}_j^n is called the *j -th orthant* of the decomposition.

Ex.2.1: Let $\mathbb{Z}_1^n, \dots, \mathbb{Z}_{2^n}^n$ be all distinct Cartesian products of n sets, each of which is either \mathbb{Z}_+ or \mathbb{Z}_- . Then this is an orthant decomposition of \mathbb{Z}^n . The set of generators of \mathbb{Z}_j^n as a semigroup is

$$\{(c_1, 0, \dots, 0), (0, c_2, 0, \dots, 0), \dots, (0, \dots, 0, c_n)\},$$

where c_j is either 1 or -1 , $j = 1, \dots, n$.

This is the *canonical orthant decomposition* of \mathbb{Z}^n .

Ex.2.2: The decomposition $\mathbb{Z}^2 = \mathbb{Z}_0^2 \cup \mathbb{Z}_1^2 \cup \mathbb{Z}_2^2$, where

$$\mathbb{Z}_0^2 = \{(a, b) | a \geq 0, b \geq 0, a, b \in \mathbb{Z}\},$$

$$\mathbb{Z}_1^2 = \{(a, b) | a \leq 0, b \geq a, a, b \in \mathbb{Z}\},$$

$$\mathbb{Z}_2^2 = \{(a, b) | b \leq 0, a \geq b, a, b \in \mathbb{Z}\},$$

is an orthant decomposition of \mathbb{Z}^2 .

Def. 2.4: Let $\{\mathbb{Z}_j^n \mid j = 1, \dots, k\}$ be an orthant decomposition of \mathbb{Z}^n . Let $E = \{e_1, \dots, e_q\}$ be a set of q distinct elements. A total order \prec on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is called a *generalized term order* on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ w.r.t. the decomposition, if the following conditions hold:

- (i) $(0, \dots, 0, e_i)$ is the smallest element in $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$ for any $e_i \in E$,
- (ii) if $(a, e_i) \prec (b, e_j)$, then $(a + c, e_i) \prec (b + c, e_j)$ for any c similar to b .

Example 2.3: Given the canonical orthant decomposition of \mathbb{Z}^n , an order " \prec " in $E = \{e_1, \dots, e_q\}$, for two elements

$$\begin{aligned} a &= (k_1, \dots, k_m, l_1, \dots, l_n, e_i) \text{ and} \\ b &= (r_1, \dots, r_m, s_1, \dots, s_n, e_j) \end{aligned}$$

of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ define:

$$|a|_1 = \sum_{j=1}^m k_j, \quad |a|_2 = \sum_{j=1}^n |l_j|.$$

$$(a, e_i) \prec (b, e_j) : \iff$$

$$\begin{aligned} & (|a|_1, |a|_2, e_i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n) \\ & < (|b|_1, |b|_2, e_j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, s_1, \dots, s_n) \end{aligned}$$

in lexicographic order.

Then " \prec " is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$.

□

Lemma 2.1: Let $\{\mathbb{Z}_j^n \mid j = 1, \dots, k\}$ be an orthant decomposition of \mathbb{Z}^n and " \prec " be a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ with respect to the orthant decomposition.

- (a) Every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n$ is finite. In particular, every subset of $\mathbb{N}^m \times \mathbb{Z}^n$ contains a smallest element.
- (b) Every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is finite. In particular, every subset of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ contains a smallest element.

3. Relative Gröbner bases

In the following let $\{\mathbb{Z}_j^n, j = 1, \dots, k\}$ be a fixed orthant decomposition of \mathbb{Z}^n .

Definition 3.1: The subset Λ_j of Λ ,

$$\Lambda_j = \{\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \mid (l_1, \dots, l_n) \in \mathbb{Z}_j^n\},$$

is called j -th **orthant** of Λ .

Let F be a finitely generated free D -module and ΛE be the set of terms of F . Then

$$\Lambda_j E = \{\lambda e_i \mid \lambda \in \Lambda_j, e_i \in E\}$$

is called j -th **orthant** of ΛE . \square

Lemma: Let F be a finitely generated (with generators in E) free D -module and $f \in F \setminus \{0\}$. Let $\lambda \in \Lambda$ and $a \in R$, " \prec " be a generalized term order on $\Lambda E \subseteq D$. Then

- (i) $\lambda a = a'\lambda + \xi$, where $a' = \alpha(a)$ for some $\alpha \in \Sigma^*$ (if $a \neq 0$ then $a' \neq 0$), $\xi \in D$ with $lt(\xi) \prec \lambda$ and all terms of ξ are in the same orthant as λ .
- (ii) If $\lambda \in \Lambda$, then $lt(\lambda f) = \max_{\prec} \{\lambda u_i\}$ where u_i are terms of f , and $lt(\lambda f) = \lambda u$ for a unique term u of f .
- (iii) If $lt(f) \in \Lambda_j E$ then $lt(\lambda f) = \lambda lt(f) \in \Lambda_j E$ holds for any $\lambda \in \Lambda_j$.
- (iv) For each j there exists some $\lambda \in \Lambda$ and a term u_j of f such that

$$lt(\lambda f) = \lambda u_j \in \Lambda_j E.$$

Furthermore, the term u_j of f is unique: if $lt(\lambda_1 f) = \lambda_1 u_{j_1} \in \Lambda_j E$ and $lt(\lambda_2 f) = \lambda_2 u_{j_2} \in \Lambda_j E$ then $u_{j_1} = u_{j_2}$. We will write $lt_j(f)$ for the term u_j .

Theorem 3.1: Let \prec and \prec' be two generalized term orders on ΛE . Let $g_1, \dots, g_p \in F \setminus \{0\}$ and $f \in F$. Then

$$f = h_1 g_1 + \dots + h_p g_p + r \quad (3.2)$$

for some elements $h_1, \dots, h_p \in D$ and $r \in F$ such that

- (i) $h_i = 0$ or $lt_{\prec}(h_i g_i) \preceq lt_{\prec}(f)$, $i = 1, \dots, p$;
- (ii) $r = 0$ or $lt_{\prec}(r) \preceq lt_{\prec}(f)$ and

$$lt_{\prec}(r) \notin \{lt_{\prec}(\lambda g_i) \mid lt_{\prec'}(\lambda g_i) \preceq' lt_{\prec'}(r), \lambda \in \Lambda, i = 1, \dots, p\}$$

Definition 3.2: Let \prec and \prec' be two generalized term orders on ΛE . Let $g_1, \dots, g_p \in F \setminus \{0\}$ and $f \in F$. Suppose that the equation (3.2) holds and that the conditions (i), (ii) in Theorem 3.1 are satisfied. If $r \neq f$ we say that f is \prec -reducible to r modulo $\{g_1, \dots, g_p\}$ relative to \prec' .

In case $r = f$ and $h_i = 0, i = 1, \dots, p$, we say that f is \prec -reduced modulo $\{g_1, \dots, g_p\}$ relative to \prec' . \square

Example 3.1: Let the sets Δ and Σ consist of a single δ and a single σ , and let D be the ring of Δ - Σ -operators over R . Choose the canonical orthant decomposition on \mathbb{Z} as in Example 2.1 and define the generalized term orders \prec and \prec' on terms of D as follows:

$$\delta^k \sigma^l \prec \delta^r \sigma^s \iff (|l|, k, l) <_{lex} (|s|, r, s),$$

$$\delta^k \sigma^l \prec' \delta^r \sigma^s \iff (k, |l|, l) <_{lex} (r, |s|, s).$$

Given $f = \delta^3 \sigma - \sigma^{-1}$, $g = \delta^2 + \sigma$, then

$$lt_{\prec}(f) = \delta^3 \sigma = lt_{\prec}(\delta^3 g) = lt_{\prec}(\delta^5 + \delta^3 \sigma).$$

But

$$lt_{\prec'}(\delta^3 g) = \delta^5 \succ' lt_{\prec'}(f) = \delta^3 \sigma.$$

So f is not \prec -reduced modulo g in the usual meaning, but f is \prec -reduced modulo g relative to \prec' . \square

Definition 3.3: Let

- W be a submodule of the finitely generated free D -module F ,
- \prec and \prec' two generalized term orders on ΛE ,
- $G = \{g_1, \dots, g_p\}$ a subset of $W \setminus \{0\}$.

Then G is called a \prec -Gröbner basis of W relative to \prec' iff every $f \in W \setminus \{0\}$ is \prec -reducible to 0 modulo G relative to \prec' . We will call it shortly a **relative Gröbner basis** of W if no confusion is possible. \square

Proposition 3.1: Let \prec and \prec' be two generalized term orders on ΛE , $G \subset W \setminus \{0\}$ a \prec -Gröbner basis of W relative to \prec' , and $f \in F$. Then

- (i) G is a Gröbner basis of W w.r.t. \prec and \prec' . So G generates the D -module W .
- (ii) $f \in W$ if and only if $f = 0$ or f can be \prec -reduced to 0 modulo G relative to \prec' .
- (iii) $f \in W$ is \prec -reduced modulo G relative to \prec' if and only if $f = 0$.

Definition 3.4: Let F be a finitely generated free D -module and $f, g \in F \setminus \{0\}$. Let \prec be a generalized term order on ΛE . For every Λ_j let $V(j, f, g)$ be a finite system of generators of the $R[\Lambda_j]$ -module

$${}_{R[\Lambda_j]} \langle \text{lt}(\lambda f) \in \Lambda_j E \mid \lambda \in \Lambda \rangle \cap {}_{R[\Lambda_j]} \langle \text{lt}(\eta g) \in \Lambda_j E \mid \eta \in \Lambda \rangle.$$

Then for every generator $v \in V(j, f, g)$

$$S(j, f, g, v) = \frac{v}{\text{lt}_j(f)} \frac{f}{\text{lc}_j(f)} - \frac{v}{\text{lt}_j(g)} \frac{g}{\text{lc}_j(g)}$$

is called an **S-polynomial** of f and g with respect to j and v . \square

Theorem 3.2 (Generalized Buchberger Theorem in Zhou and Winkler (2006)):

Let F be a free D -module and \prec be a generalized term order on ΛE , G be a finite subset of $F \setminus \{0\}$ and W be the submodule in F generated by G . Then G is a Gröbner basis of W if and only if for all Λ_j , for all $g_i, g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G . \square

Theorem 3.3: Let F be a free D -module, \prec and \prec' be two generalized term orders on ΛE , G be a finite subset of $F \setminus \{0\}$ and W be the submodule in F generated by G . Then G is a \prec -Gröbner basis of W relative to \prec' if and only if G is a Gröbner basis with respect to \prec' of W and for all Λ_j , for all $g_i, g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ with respect to \prec can be \prec -reduced to 0 modulo G relative to \prec' .

In other words, G is a \prec -Gröbner basis relative to \prec' if and only if all $S'(j, g_i, g_k, v)$ can be reduced (w.r.t. \prec') to 0 by G and all $S(j, g_i, g_k, v)$ can be \prec -reduced to 0 modulo G relative to \prec' .

Theorem 3.4 (Buchberger's Algorithm for computing relative Gröbner bases):

Let F be a free D -module, \prec and \prec' be two generalized term order on ΛE , G be a finite subset of $F \setminus \{0\}$ and W be the submodule in F generated by G . For each Λ_j and $f, g \in F \setminus \{0\}$ let $V(j, f, g)$, $S(j, f, g, v)$ and $S'(j, f, g, v)$ be as in Definition 3.4 w.r.t. \prec and \prec' , respectively. Then by the following algorithm a \prec -Gröbner basis of W relative to \prec' can be computed:

input: $G = \{g_1, \dots, g_\mu\}$, a set of generators of W
 \prec and \prec' , two generalized term orders on ΛE
output: $G'' = \{g''_1, \dots, g''_\nu\}$, a \prec -GB of W rel. to \prec'
begin
 $G' := G$;
while exist $f, g \in G', v \in V(j, f, g)$ s.t.
 $S'(j, f, g, v)$ is reduced (w.r.t. \prec') to $r \neq 0$ by G'
do $G' := G' \cup \{r\}$
endwhile ;
 $G'' := G'$;
while exist $f, g \in G'', v \in V(j, f, g)$ s.t.
 $S(j, f, g, v)$ is \prec -reduced to $r \neq 0$ by G'' rel. to \prec'
do $G'' := G'' \cup \{r\}$
endwhile
end \square

4. Computing difference-differential dimension polynomials

Let

- R be a Δ - Σ -field,
- D the ring of Δ - Σ -operators over R ,
- F a finitely generated free Δ - Σ -module,
- M a finitely generated Δ - Σ -module (i.e. a finitely generated difference-differential-module).

Now we consider difference-differential dimension polynomials $\psi_M(t_1, t_2)$ in two variables t_1 and t_2 by the approach of relative difference-differential Gröbner bases.

Consider the canonical orthant decomposition of \mathbb{Z}^n as in Example 2.1 and define the generalized term orders " \prec " and " \prec' " on ΛE of the terms of F as follows:
for $\lambda e = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} e$ we set

$$|\lambda|_1 := k_1 + \cdots + k_m \quad \text{and} \quad |\lambda|_2 := |l_1| + \cdots + |l_n|;$$

furthermore

$$\begin{aligned} \langle \lambda e \rangle &:= (|\lambda|_2, |\lambda|_1, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n) \\ \langle \lambda e \rangle' &:= (|\lambda|_1, |\lambda|_2, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n) \end{aligned}$$

Now we define

$$\begin{aligned} \lambda e_i \prec \mu e_j &:\iff \langle \lambda e_i \rangle <_{lex} \langle \mu e_j \rangle \\ \lambda e_i \prec' \mu e_j &:\iff \langle \lambda e_i \rangle' <_{lex} \langle \mu e_j \rangle' \end{aligned}$$

For $u = \sum_{\lambda \in \Lambda} a_\lambda \lambda \in D$ and $i \in \{1, 2\}$ we define

$$|u|_i := \max\{|\lambda|_i \mid a_\lambda \neq 0\}.$$

D is a bifiltered ring with the bifiltration $(D_{rs})_{r,s \in \mathbb{Z}}$ s.t.

$$D_{rs} = \{u \in D \mid |u|_1 \leq r, |u|_2 \leq s\} \text{ for } (r, s) \in \mathbb{N}^2$$

$$\text{and } D_{rs} = \{\} \text{ for } (r, s) \in \mathbb{Z}^2 \setminus \mathbb{N}^2.$$

Obviously

$$\bigcup \{D_{rs} \mid r, s \in \mathbb{Z}\} = D,$$

$$D_{rs} \subseteq D_{r+1,s}, \quad D_{rs} \subseteq D_{r,s+1} \text{ for } r, s \in \mathbb{Z}, \text{ and}$$

$$D_{kl}D_{rs} = D_{r+k,s+l} \text{ for } r, s, k, l \in \mathbb{Z}.$$

Let M be a finitely generated left D -module with generators h_1, \dots, h_q . Let

$$M_{rs} = D_{rs}h_1 + \dots + D_{rs}h_q$$

for any $r, s \in \mathbb{Z}$. Then $(M_{rs})_{r,s \in \mathbb{Z}}$ is an excellent bifiltration of M , i.e. every (M_{rs}) is a finitely generated R -module and $D_{kl}M_{rs} = M_{r+k,s+l}$.

Definition 4.1: A polynomial $\psi(t_1, t_2)$ in $\mathbb{Q}[t_1, t_2]$ is called a **(bivariate) numerical** if $\psi(t_1, t_2) \in \mathbb{Z}$ for all sufficiently large $(r_1, r_2) \in \mathbb{Z}^2$, i.e. there exists a tuple $(s_1, s_2) \in \mathbb{Z}^2$ such that $\psi(r_1, r_2) \in \mathbb{Z}$ for all integers $r_1, r_2 \in \mathbb{Z}$ with $r_i \geq s_i$ ($1 \leq i \leq 2$).

The numerical polynomial $\psi(t_1, t_2)$ is called **(bivariate) difference-differential dimension polynomial** associated with M , iff

- (i) $\deg \psi \leq m + n$, $\deg_{t_1} \psi \leq m$, and $\deg_{t_2} \psi \leq n$ and
- (ii) $\psi(t_1, t_2) = \dim_R M_{t_1, t_2}$ for all sufficiently large t_1, t_2 .

Theorem 4.1: Let R be a Δ - Σ -field, D and M be as above, in particular let M have the generators h_1, \dots, h_q . Let F be a free Δ - Σ -module with a basis e_1, \dots, e_q and $\pi : F \longrightarrow M$ the natural Δ - Σ -epimorphism of F onto M ($\pi(e_i) = h_i$ for $i = 1, \dots, q$).

Let \prec and \prec' be the generalized term orders on ΛE of the terms of F defined above. Consider the submodule $N = \ker(\pi)$ of F and let $G = \{g_1, \dots, g_p\}$ be a \prec -Gröbner basis of N relative to \prec' . Let

$$\begin{aligned}
U_{r,s} = & \{w \in \Lambda E \mid |w|_1 \leq r, |w|_2 \leq s, \text{ and} \\
& w \neq lt_{\prec}(\lambda g_i) \text{ for all } \lambda \in \Lambda, g_i \in G\} \\
& \cup \\
& \{w \in \Lambda E \mid |w|_1 \leq r, |w|_2 \leq s, \text{ and} \\
& |lt_{\prec'}(\lambda g_i)|_1 > r \text{ for all } \lambda \in \Lambda, \\
& g_i \in G \text{ s.t. } w = lt_{\prec}(\lambda g_i)\} .
\end{aligned}$$

Then the bivariate difference-differential dimension polynomial ψ associated with M is the cardinality of U , i.e.

$$\psi(r, s) = |U_{r,s}| .$$

The difference-differential dimension polynomial $\psi(t_1, t_2)$ carries more information than the “one variable” dimension polynomial $\phi(t)$. From the point of view of strength of systems of difference-differential equations, the polynomial $\psi(t_1, t_2)$ determines the strength of systems w.r.t. each of the sets of operators Δ and Σ while the polynomial $\phi(t)$ determines just the general strength of the systems w.r.t. the set $\Delta \cup \Sigma$.

Example 4.1: Let R be a difference-differential field whose basic sets Δ and Σ consist of a single δ and a single σ . Furthermore, let D be the ring of Δ - Σ -operators over R and $M = Dh$ be a Δ - Σ -module whose generator h satisfies the defining equation

$$(\delta\sigma + \sigma^{-2})h = 0 .$$

In other words, M is isomorphic to the factor module of a free Δ - Σ -module F with a free generator e by its Δ - Σ -submodule N which is a cyclic submodule with a generator $\{g = \delta\sigma + \sigma^{-2}\}$. We compute the difference-differential dimension polynomial $\psi(r, s)$.

By Theorem 4.1, we need to compute a relative Gröbner basis of N and then $\psi(r, s) = |U_{r,s}|$. Clearly the relative Gröbner basis is

$$\{g = \delta\sigma + \sigma^{-2}\}.$$

We have $lt(g) = \sigma^{-2} \in \Lambda_2$. Because the leading term of $\sigma g = \delta\sigma^2 + \sigma^{-1}$ is $\delta\sigma^2 \in \Lambda_1$, we have

$$lt(\lambda g) = \Lambda_1\delta\sigma^2 \cup \Lambda_2\sigma^{-2} .$$

Put

$$U'_{r,s} := \{w \in \Lambda \mid |w|_1 \leq r, |w|_2 \leq s, \text{ and } w \neq lt_{\prec}(\lambda g) \text{ for all } \lambda \in \Lambda\} ,$$

$$U''_{r,s} := \{w \in \Lambda \mid |w|_1 \leq r, |w|_2 \leq s, \text{ and } |lt_{\prec}(\lambda g)|_1 > r \text{ for all } \lambda \in \Lambda \text{ s.t. } w = lt_{\prec}(\lambda g)\} .$$

Then

$$|U_{r,s}| = |U'_{r,s}| + |U''_{r,s}| = (3r + s + 2) + (s - 1)$$

and

$$\psi(r, s) = \dim_R M_{r,s} = |U_{r,s}| = 3r + 2s + 1 .$$