

Integro-Differential Algebras as a Natural Setting for Boundary Problems

$$u'' = f$$

$$u(0) = u(1) = 0$$

Boundary problems

$$\partial(fg) = f \partial(g) + g \partial(f) \quad \int f \cdot \int g = \int f \int g + \int g \int f$$

Leibniz

Baxter

Georg Regensburger and Markus Rosenkranz

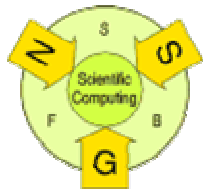
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Differential Algebras and Equations

(\mathcal{F}, ∂) commutative differential algebra over a field K

$$\partial: \mathcal{F} \rightarrow \mathcal{F} \quad K\text{-linear} \quad f' = \partial(f) \quad \mathbb{Q} \leq K$$

$$\text{Leibniz rule} \quad \partial(fg) = f \partial(g) + g \partial(f)$$

$\mathcal{F}[\partial]$ differential operators Multiplication: $\partial f = f' + f\partial$

Differential equation (DE) $Tu = f$ $T \in \mathcal{F}[\partial], f \in \mathcal{F}$

Solution operators $G: f \mapsto u$

Simplest equation $u' = f$

Solution operators \int are sections (right inverses) of ∂
$$\partial \int = 1$$

(sections do not always exist)

Sections

Characterization of sections from Linear Algebra:

Every section f of ∂ $\partial f = 1$

corresponds to a projector $P = 1 - \int \partial$

and to a direct sum $\mathcal{F} = \mathcal{C} \dot{+} \mathcal{I}$

where $\mathcal{C} = \text{Ker}(\partial) = \text{Im}(P)$ $\mathcal{I} = \text{Im}(f) = \text{Ker}(P)$
constants *initialized*

Example: $\mathcal{F} = C^\infty[a, b]$ with usual derivation

Fix $\alpha \in [a, b]$ section $\int f = \int_\alpha^x f(\xi) d\xi$

$$f \in \mathcal{I} \Leftrightarrow f(\alpha) = 0$$

Baxter Algebras

Iterate integral for higher order DE

$$\text{Baxter axiom} \quad \int f \cdot \int g = \int f \int g + \int g \int f$$

Integration by parts where $\int f \int g = \int (f \int g)$

(\mathcal{F}, \int) Rota-Baxter algebra (of weight 0) Baxter [Bax60], Rota [Rot69]

“flattens” products

“integrate out” coefficients

$$\text{Diff. Baxter axiom} \quad \int f g = f \int g - \int f' \int g$$

Diff. Baxter axiom $\xRightarrow{f = \int f}$ Baxter axiom

\Leftarrow

Integro-Differential Algebras

Baxter axiom + Homogeneity over the constants:

$$\int cf = c \int f \quad c \in \mathcal{C}, f \in \mathcal{F}$$

\Leftrightarrow Diff. Baxter axiom

A section \int of ∂ is an *integral* if it satisfies the diff. Baxter axiom

$(\mathcal{F}, \partial, \int)$ *integro-differential algebra* [RR08b]

differential Rota-Baxter algebras Guo and Keigher [GK08]

Examples: $\mathcal{F} = C^\infty[a, b]$ or exponential polynomials (usual ∂)

$$\int^* : f \mapsto \int_a^x f(\xi) d\xi \quad \int_* : f \mapsto \int_x^b f(\xi) d\xi \quad f \mapsto \int_a^b \int_\tau^x f(\xi) d\xi d\tau$$

$(\mathcal{F}, \partial, \int^*)$

$(\mathcal{F}, \partial, -\int_*)$

section

$$P^* : f \mapsto f(a)$$

$$P_* : f \mapsto f(b)$$

$$P : f \mapsto \int_a^b f(\xi) d\xi$$

multiplicative

nonmultiplicative

Power series, Hurwitz series Keigher and Pritchard [KP00]

Multiplicative Projectors

A section \int of ∂ is an integral

$\Leftrightarrow P = 1 - \int \partial$ is multiplicative

$$P(fg) = P(f)P(g)$$

$\Leftrightarrow \mathcal{I} = \text{Im}(\int) = \text{Ker}(P)$ is an ideal

Linear structure and algebra structure fit together

Variant of differential Baxter axiom (Integration by parts):

$$\int f g' = f g - \int f' g - P(f)P(g)$$

Ordinary Integro-Differential Algebras

Ordinary differential Equations

$$\dim_K \text{Ker}(\partial) = 1$$

$(\mathcal{F}, \partial, \int)$ Ordinary (integro)-differential algebra

$$\mathcal{C} = \text{Ker}(\partial) = K$$

Diff. Baxter axiom \Leftrightarrow Baxter axiom

Projection: $\mathbf{E} = 1 - \int \partial$

Multiplicative linear
functional (character)

Evaluation

$\mathcal{M}(\mathcal{F})$ Multiplicative functionals φ, ψ

Differential Operators

$\mathcal{F}[\partial]$ differential operators:

Free K -algebra generated by the symbol(s) ∂

the „functions“ $f \in \mathcal{F}$

modulo the rewrite system

$fg \rightarrow f \cdot g$	$\partial f \rightarrow \partial \cdot f + f\partial$
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 . action on \mathcal{F}

Normal forms $\sum_i f_i \partial^i$

Noncommutative Gröbner Basis Bergman [Ber78]

Proposition. The rewrite system is Noetherian and confluent

Integro-Differential Operators

$\mathcal{F}[\partial, \int]$ integro-differential operators:

Free K -algebra generated by the symbol(s) ∂ and \int

the „functions“ $f \in \mathcal{F}$ and the „functionals“ $\varphi \in \mathcal{M}(\mathcal{F})$

modulo the rewrite system

fg	\rightarrow	$f \cdot g$	∂f	\rightarrow	$\partial \cdot f + f\partial$
$\varphi\psi$	\rightarrow	ψ	$\partial\varphi$	\rightarrow	0
φf	\rightarrow	$(\varphi \cdot f)\varphi$	$\partial\int$	\rightarrow	1
$\int f\int$	\rightarrow	$(\int \cdot f)\int - \int(\int \cdot f)$			
$\int f\partial$	\rightarrow	$f - \int(\partial \cdot f) - (\mathbf{E} \cdot f)\mathbf{E}$			
$\int f\varphi$	\rightarrow	$(\int \cdot f)\varphi$			

• action on \mathcal{F}

Noncommutative Gröbner Basis

Bergman [Ber78]

Proposition. The rewrite system is Noetherian and confluent

Normal Forms for Integro-Differential Operators

Proposition. Every normal form can be uniquely written as

$$T + G + B$$

differential

integral

boundary

operator

$$T = \sum f_i \partial^i$$

$$G = \sum f_i \int g_i$$

$$B = \sum f_i \beta_i$$

$$\beta_i \in \mathcal{S}(\mathcal{F}) = \mathcal{M}(\mathcal{F}) \mathcal{F}[\partial, f]$$

Subalgebras

Stieltjes boundary conditions

$$\mathcal{F}[\partial]$$

$$\mathcal{F}[f]$$

$$\mathcal{F}[\mathbf{E}]$$

Stieltjes boundary condition

$$\beta = \sum_{\varphi} \left(\sum a_{\varphi,i} \varphi \partial^i + \varphi \int f_{\varphi} \right) \quad u \mapsto u(0) + u''(0) + \int_0^1 u(\xi) d\xi$$

Boundary Problems

$T \in \mathcal{F}[\partial]$ monic with degree n

$\beta_1, \dots, \beta_n \in \mathcal{S}(\mathcal{F})$ Stieltjes boundary conditions

Given $f \in \mathcal{F}$ find $u \in \mathcal{F}$ *regular* $\forall f \exists! u$

$$Tu = f$$

$$\beta_1 u = \dots = \beta_n u = 0$$

Criterion $[\beta_i(u_j)]$ *regular*

$$\text{Ker}(T) = [u_1, \dots, u_n]$$

fundamental system

Boundary Problem:

$$(T, \mathcal{B})$$

$$\text{Ker}(T) \dot{+} \mathcal{B}^\perp = \mathcal{F}$$

$\mathcal{B} \leq \mathcal{S}(\mathcal{F})$ Dimension n

Abstract setting:

Boundary problem = Surjective linear Map + Subspace of dual
(Includes LPDEs, Systems)

[RR08]

Green's Operators

$$\begin{array}{ll} \text{Green's operator: } G: \mathcal{F} \rightarrow \mathcal{F} & TG = 1 \\ f \mapsto u & \text{Im}(G) = \mathcal{B}^\perp \end{array}$$

$$G = (T, \mathcal{B})^{-1}$$

Theorem. Given a fundamental system with regular Wronskian, the Green's operator of a regular boundary problem can be computed as integro-differential operator. Its normal form gives the Green's function.

State and solve boundary problems
with integro-differential operators.

Composing and Factoring Boundary Problems

Composition of boundary problems:

$$(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 T_2 + \mathcal{B}_2)$$

(Semidirect product of monoids)

Composition of regular problems is regular

$$((T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2))^{-1} = (T_2, \mathcal{B}_2)^{-1} \cdot (T_1, \mathcal{B}_1)^{-1}$$

Theorem. Given a regular boundary problem (T, \mathcal{B}) ,

every factorization $T = T_1 T_2$ can be lifted to a factorization

$$(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T, \mathcal{B}) \text{ into regular problems.}$$

unique

$\mathcal{B}_2 \leq \mathcal{B}$ such that it is regular

abstract setting

Integro-Differential Polynomials

$(\mathcal{F}, \partial, \int)$ Integro-differential algebra

adjoin indeterminate u

Example:

$$x^3 u(0) \int (x^4 u u''^2 \int (u''(0)^2 x e^{3x} u'^3 \int u)) \cdot (x^2 \int x u'' \int x u'(0)^2 u)$$

Baxter rule $\int f \cdot \int g = \int f \int g + \int g \int f$ Section $\partial \int = 1$

Evaluation $1 - \int \partial$ Notation: $(1 - \int \partial)(u) = u(0)$

Sum of terms of the form

$$f u(0)^\alpha u^\beta \int f_1 u^{\gamma_1} \int \dots \int f_n u^{\gamma_n} \quad n \text{ depth}$$

$$u^\beta = u_0^{\beta_0} \dots u_k^{\beta_k} \quad \text{differential monomial} \quad \beta_k \neq 0$$

u_n nth derivative

Canonical Simplifier

Represent a polynomial as sum of terms of the form

$$f u(0)^\alpha u^\beta \int f_1 u^{\gamma_1} \int \dots \int f_n u^{\gamma_n}$$

Is this representation unique (a canonical form)?

Do we have a canonical simplifier?

Buchberger, Loos [BL83]

Integration by parts:

$$\int f u' \equiv f u - \int f' u - f(0)u(0)$$

Highest derivative appears linearly:

u^β with order > 0 *quasilinear*
and *functional* otherwise

Canonical Forms

\mathcal{R} set of all linear combinations of terms

$$bu(0)^\alpha u^\beta \int b_1 u^{\gamma_1} \int \dots \int b_n u^{\gamma_n}$$

b_i basis elements u^{γ_i} functional

Theorem. Every integro-differential polynomial can be uniquely represented by a term in \mathcal{R}

Main part of the proof:

Show that different terms represent different polynomials

Integro-Differential Algebra on Canonical Forms

Strategy:

Equip \mathcal{R} with the structure of an integro-differential algebra

Substitutionhomomorphism polynomials in universal algebra

Coproduct of free algebra and coefficient algebra

Lausch, Nöbauer [LN73] Buchberger, Loos [BL83]

Multiplication: shuffle product (Li Guo's talk) [Ree58]

Derivation: Leibniz rule, section axiom and basis expansion

Integral: Recursively for quasiconstant, quasilinear, functional

Difficult part is verifying the Baxter axiom

Conclusion

Algebraic setting with differential and integral operator

Allows us to treat local conditions (evaluations)

Integro-differential operators (state and solve BPs)

Integro-differential polynomials (extensions, quotients)

Implementation in Theorema based on functors

Buchberger, R., R., Tec [BRRT08]

Extension to several indeterminates

Systems of integro-differential polynomial equations

PDEs

Thank you!

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