On Differential Rota-Baxter Algebras [1]

Li GUO

Rutgers University at Newark

[1] Joint work with W. Keigher and W. Sit

- **1. Rota-Baxter algebras:**
  Fix λ in the base ring $\mathbf{k}$. A Rota-Baxter operator or a Baxter operator of weight λ on a $\mathbf{k}$-algebra R is a linear map $P: R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall x, y \in R.$$

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  Then P is a weight 0 Rota-Baxter operator:
- $$F(x) := P[f](x) = \int_0^x f(t)\,dt, \quad G(x) := P[g](x) = \int_0^x g(t)\,dt.$$
  Then the integration by parts formula states
  $$\int_0^x F(t)G'(t)dt = F(x)G(x) - \int_0^x F'(t)G(t)dt$$
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- that is,

$$P[P[f]g](x) = P[f](x)P[g](x) - P[fP[g]](x).$$

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$$P[f](x)P[g](x) = \Big(\sum_{n\geq 1} f(x+n)\Big)\Big(\sum_{m\geq 1} g(x+m)\Big)$$

$$= \sum_{n\geq 1, m\geq 1} f(x+n)g(x+m)$$

$$= \Big(\sum_{n>m\geq 1} + \sum_{m>n\geq 1} + \sum_{m=n\geq 1}\Big) f(x+n)g(x+m)$$

$$= \sum_{m\geq 1} \big(\sum_{k\geq 1} f(x+\underbrace{k+m}_{=n})\big)g(x+m) + \sum_{n\geq 1}\big(\sum_{k\geq 1} g(x+\underbrace{k+n}_{=m})\big)f(x+n)$$

$$+ \sum_{n\geq 1} f(x+n)g(x+n)$$

$$= P(P(f)g)(x) + P(fP(g))(x) + P(fg)(x).$$

- **Partial sum:** Let R be the set of sequences $\{a_n\}$ with values in **k**. Then R is a **k**-algebra with termwise addition, multiplication and scalar product. Define

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  Then P is a Rota-Baxter operator of weight 1.
- **Matrices:** On the algebra of upper triangular $n \times n$ matrices $M_n^u(\mathbf{k})$, define

$$P((c_{k\ell}))_{ij} = \delta_{ij} \sum_{k \geq i} c_{ik}.$$

  Then P is a Rota-Baxter operator of weight -1.

- **Scalar product:** Let R be a **k**-algebra. For a given $\lambda \in \mathbf{k}$, define

$$P_\lambda : R \to R, x \mapsto -\lambda x, \forall x \in R.$$

  Then (R, P_λ) is a Rota-Baxter algebra of weight λ. In particular, id is a Rota-Baxter operator of weight -1 and any **k**-algebra is a Rota-Baxter algebra.

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- **QFT dimensional regularization:** Let $R = \mathbb{C}[t^{-1}, t]]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n t^n,\ k \geq 0$. Define

$$P(\sum_{n=-k}^{\infty} a_n t^n) = \sum_{n=-k}^{-1} a_n t^n.$$

Then P is a Rota-Baxter operator of weight -1.

- **Classical Yang-Baxter equation:** Let $\mathfrak{g}$ be a Lie algebra with a self-duality $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathbf{k}) \cong \mathfrak{g}$. Then $\mathfrak{g}^{\otimes 2} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End}(\mathfrak{g})$. Let $r_{12} \in \mathfrak{g}^{\otimes 2}$ be anti-symmetric. Then r_{12} is a solution (r-matrix) of the classical Yang-Baxter equation (CYB)

$$CYB(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

if and only if the corresponding $P \in \mathrm{End}(\mathfrak{g})$ is a (Lie algebra) Rota-Baxter operator of weight 0:

$$[P(x), P(y)] = P[P(x), y] + P[x, P(y)].$$

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- **Associative Yang-Baxter equations** (Aguiar) Let A be an associative algebra and let $r := \sum_i u_i \otimes v_i \in R \otimes R$ be a solution of the associative Yang-Baxter equation

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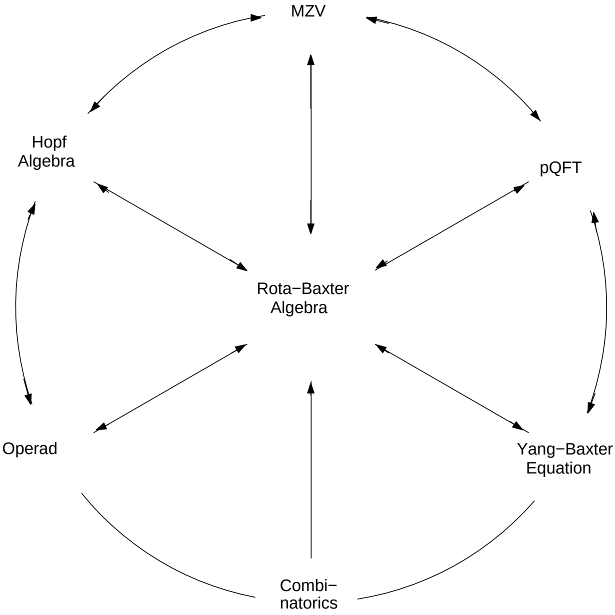
- **Others** Divided powers, Hochschild homology ring, dendriform algebras, rooted trees, quasi-shuffles, Chen integral symbols, ...


MZV
Hopf Algebra
pQFT
Rota−Baxter Algebra
Operad
Yang−Baxter Equation
Combi−natorics

- **2. Free commutative Rota-Baxter algebras**
  Let A be a commutative $\mathbf{k}$-algebra. Let $Ш^+(A) = \bigoplus_{n\geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $Ш^+(A)$. Define $\mathbf{1_k} \in \mathbf{k}$ to be the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$.

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- **Quasi-shuffle product:** Hoffman (2000) on multiple zeta values. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Recursively define

$$(a_1\otimes\mathfrak{a}')*(b_1\otimes\mathfrak{b}') = a_1\otimes(\mathfrak{a}'*(b_1\otimes\mathfrak{b}'))+b_1\otimes((a_1\otimes\mathfrak{a}')*\mathfrak{b}')+a_1b_1\otimes(\mathfrak{a}'*\mathfrak{b}'),$$

with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity. It defines the shuffle product without the third term.

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$$(a_1 \otimes \mathfrak{a}') * (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes \mathfrak{b}'))) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') * \mathfrak{b}') + a_1 b_1 \otimes (\mathfrak{a}' * \mathfrak{b}'),$$

  with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity. It defines the shuffle product without the third term.
- **Example.**
  $a_1 * (b_1 \otimes b_2) = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes b_2)) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes (\mathfrak{a}' * b_2)$
  $= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes b_2.$
  $= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes a_1 b_2 + a_1 b_1 \otimes b_2.$

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- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.
  Define $(a_1 \otimes \ldots \otimes a_m) \diamond (b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.

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- **Example:**

$$
\begin{aligned}
& a_1 \diamond (b_1 \otimes b_2) \\
& = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\
& + a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad \text{(merged shuffles)}.
\end{aligned}
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- Let $\overline{S}_c(m,n)$ be the set of triples (k,α,β) where $\max\{m,n\} \leq k \leq m+n$ and $\alpha : [m] \to [k]$, $\beta : [n] \to [k]$ are order preserving injections with $\mathrm{im}(\alpha) \cup \mathrm{im}(\beta) = [k]$.

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- For a given $(k,\alpha,\beta) \in \overline{S}_c(m,n)$ and $1 \leq i \leq k$, $\alpha^{-1}(i)$ is either a singleton $\{j\}$ or $\emptyset$. Then accordingly define $a_{\alpha^{-1}(i)} = a_j$ or 1. Similarly define $b_{\beta^{-1}(i)}$. Call $a_{\alpha^{-1}(1)} b_{\beta^{-1}(1)} \otimes \cdots \otimes a_{\alpha^{-1}(k)} b_{\beta^{-1}(k)}$ a stuffle.

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- Define a product

$$\mathfrak{a}\diamond\mathfrak{b}=\sum_{(k,\alpha,\beta)\in\overline{S}_c(m,n)} a_{\alpha^{-1}(1)}b_{\beta^{-1}(1)}\otimes\cdots\otimes a_{\alpha^{-1}(k)}b_{\beta^{-1}(k)}$$

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- **Example:** $\mathfrak{a}=a_1, m=1$, $\mathfrak{b}=b_1\otimes b_2$, $n=2$.

k	α	β	stuffles
3	$\alpha(1)=1$	$\beta(1)=2,\beta(2)=3$	$a_1\otimes b_1\otimes b_2$
3	$\alpha(1)=2$	$\beta(1)=1,\beta(2)=3$	$b_1\otimes a_1\otimes b_2$
3	$\alpha(1)=3$	$\beta(1)=1,\beta(2)=2$	$b_1\otimes b_2\otimes a_1$
2	$\alpha(1)=1$	$\beta(1)=1,\beta(2)=2$	$a_1b_1\otimes b_2$
2	$\alpha(1)=2$	$\beta(1)=1,\beta(2)=2$	$b_1\otimes a_1b_2$

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- Let $D(m,n)$ be the set of lattice paths from $(0,0)$ to (m,n) consisting steps either to the right, to the above, or to the above-right. For $d \in D(m,n)$ define $d(\mathfrak{a},\mathfrak{b})$ to be the path d with $\mathfrak{a} = (a_1, \cdots, a_m)$ (resp. $\mathfrak{b} = (b_1, \cdots, b_n)$) sequentially labeling the horizontal (resp. vertical) and diagonal segments of d. Define

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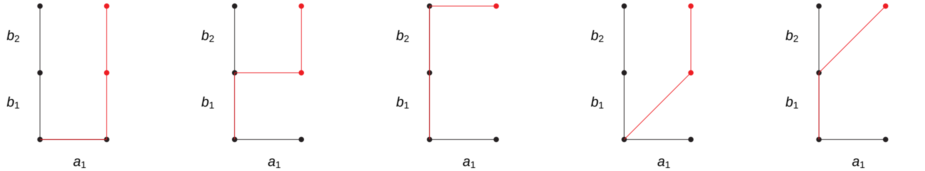

$a_1 \otimes b_1 \otimes b_2$ $\quad$ $b_1 \otimes a_1 \otimes b_2$ $\quad$ $b_1 \otimes b_2 \otimes a_1$ $\quad$ $a_1 b_1 \otimes b_2$ $\quad$ $b_1 \otimes a_1 b_2$

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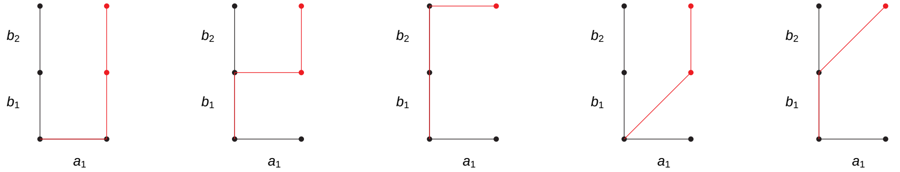

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- **Theorem** All the above products define the same algebra on $Ш^+(A)$ (of weight $\lambda = 1$).

- A **free commutative Rota-Baxter algebra over another commutative algebra** A is a commutative Rota-Baxter algebra $Ш(A)$ with an algebra homomorphism $j_A : A \to Ш(A)$ such that for any commutative Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\;j_A\;} & Ш(A) \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}} \\ & & R \end{array}$$

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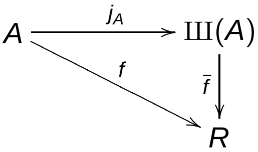

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- When $A = \mathbf{k}[X]$, we have the free commutative Rota-Baxter algebra over X.
- Recall $(\text{Ш}^+(A), \diamond)$ is a commutative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is a commutative A-algebra.

  Theorem (Guo-Keigher) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative RBA over A.

- Let $\lambda \in \mathbf{k}$ be given. A differential operator of weight λ on a **k**-algebra R is a linear map $\delta : R \to R$ such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \lambda\delta(x)\delta(y), \quad x, y \in R.$$

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- An integral-differential algebra $(R, \partial, \int)$ (from last talk) is a DRBA of weight 0.
- For fixed $0 < \lambda \in \mathbb{R}$, define $\delta_\lambda(f)(x) = (f(x+\lambda) - f(x))/\lambda$ and $P_\lambda(f)(x) = -\lambda \sum_{n\geq 0} f(x + n\lambda)$ on a suitable algebra R of functions (say $f \searrow 0$ and $\int_1^\infty f(t)dt < \infty$). Then $(R, \delta_\lambda, P_\lambda)$ is a DRBA of weight λ.

- Consider $R = A^{\mathbb{N}} = \{f : \mathbb{N} \to A\}$ with multiplication

$$(fg)(n) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k f(n-j) g(k+j).$$

Define the operators $\delta(f)(n) = f(n+1), n \geq 0$ and $P(f)(n) = f(n-1), n \geq 1, P(f)(0) = 0$. Then (R, δ, P) is a DRBA of weight λ.

- Free commutative DRBA is defined by the same kind of universal property as for free commutative RBA.
- Construction. Let X be a set of variables. Let $A = \mathbf{k}\{X\} = \mathbf{k}[X \times \mathbb{N}] = \mathbf{k}[x^{(n)} \mid x \in X, n \in \mathbb{N}]$ be the ring of differential polynomials (of weight λ) with derivation $d(x^{(n)}) = x^{(n+1)}$. Define the free commutative RBA $(Ш(A), P_A)$ on A as before, namely, $Ш(A) = A \otimes Ш^+(A)$ where $Ш^+(A) = \oplus_{n\geq 1} A^{\otimes n}$ with its product defined by the mixable shuffle (or quasi-shuffle, or stuffle, or Delannoy paths). Define $\delta(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = d(a_0) \otimes a_1 \otimes \cdots a_k + (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_k + \lambda (d(a_0) a_1) \otimes a_2 \otimes \cdots \otimes a_k$.
- Theorem. $(Ш(A), \delta_A, P_A)$ is the free commutative DRBA on X (or differential Rota-Baxter polynomial on X.

- Enumeration in free commutative RBA $\text{Ш}(\{x\})$ generated by x. A linear basis of $\text{Ш}(\{x\})$ is the set

$$R := \{\mathfrak{x} := x^{i_0} \otimes \cdots \otimes x^{i_n} \mid i_j \geq 0, 1 \leq j \leq n, n \geq 0\}.$$

Call n the depth (degree) of $\mathfrak{x}$ and $w := i_0 + \cdots + i_n$ the weight (arity) of $\mathfrak{x}$. Let $R(n,w)$ be the subset of basis elements of degree n and weight w. Let $r(n,w) = |R(n,w)|$. Then

$$\mathbf{R}(z,t) := \sum_{n,w\geq 0} r(n,w) z^n t^w = 1/(1-z-t).$$

- **Free (noncommutative) Rota-Baxter algebras of rooted forests.** Let $Ш^{\mathrm{NC}}(X)$ be the free Rota-Baxter algebra on a set X. Let $\mathcal{F}(X)$ be the set of rooted forests with leafs decorated by X,

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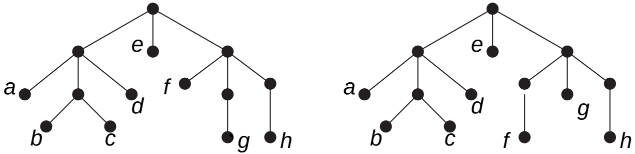
- 

  For example, the left tree is not leaf-spaced since the two right most branches are not separated by a leaf branch. But the right tree is leaf-spaced.

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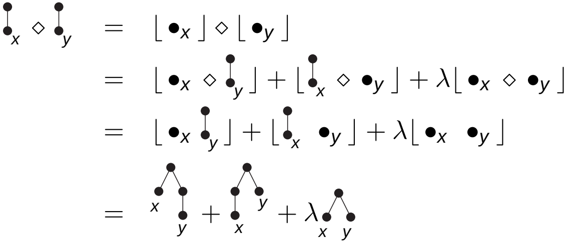
  2. If $F = F_1 \cdots F_b$ and $F' = F'_1 \cdots F'_{b'}$ are in $\mathcal{F}_\ell(X)$ with their corresponding decompositions into leaf decorated trees. Then

$$F \diamond F' = F_1 \cdots (F_b \diamond F'_1) \cdots F_{b'}.$$



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- Theorem. The quadruple $(\mathbf{k}\mathcal{F}_\ell(X), \diamond, \lfloor\ \rfloor, j_X)$ is the free nonunitary (noncommutative) Rota-Baxter algebra on X.

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$$f(z,t) = \frac{1-\sqrt{1-4zt(1+t)}}{2zt} = (1+t)\mathbf{C}(zt(1+t)).$$

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3. Free (non-unitary) DRBA generated by x is the free RBA $\mathbf{k}\mathfrak{F}_\ell(x^{(n)}, n\geq 0)$, namely the linear span of the set of leaf-spaced forests with the leaf vertices decorated by the set $\{x^{(n)}, n\geq 0\}$. Now define $\deg(x^{(n)}) = n$. Define $e(n,d)$ to be the number of such forests with n non-leaf vertices, and with total degree of the leaf vertices to be d.

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4. Theorem
$E(z,t) := \sum_{n,d\geq 0} e(n,d) z^n s^d$
$$= f(\tfrac{z}{1-z}, \tfrac{s}{1-2s}) + \left(\tfrac{1}{1-z}\right)\left(\tfrac{1-s-s^2}{(1-s)(1-2s)}\right) - \left(\tfrac{1-s}{1-2s}\right).$$