On the use of fixed point iterations for the regularization of nonlinear ill-posed problems

Ronny Ramlau
University of Bremen, Germany

September 9, 2004

Abstract

We report on a new iterative method for regularizing a nonlinear operator equation in Hilbert spaces. The proposed algorithm is a combination of Tikhonov regularization and a fixed point algorithm for the minimization of the Tikhonov–functional. Under the assumptions that the operator F is twice continuous Fréchet–differentiable with Lipschitz–continuous first derivative and that the solution of the equation $F(x) = y$ fulfills a smoothness condition we will give a convergence rate result. Numerical results with data from Single Photon Emission Computed Tomography (SPECT) show the rapid convergence of the proposed algorithms.

AMS Classification. 65J15, 65J20, 65J22, 44A12

1 Introduction

In this paper we consider the computation of an approximation to a solution of a nonlinear operator equation

$$F(x) = y \tag{1.1}$$

from noisy measurements y^δ with

$$\|y^\delta - y\| \le \delta \; . \tag{1.2}$$

If the problem is *ill-posed*, then (1.1) cannot be solved in a stable way and *regularization methods* have to be applied.

In recent years, many methods have been proposed for the regularization of nonlinear ill-posed problems. Prominent examples are Tikhonov regularization [17, 18, 5] and iterative algorithms like Landweber methods [8, 10], Levenberg–Marquardt methods [6], Gauss–Newton [1, 2], conjugate gradient [7] and other Newton–like methods [9]. In many practical applications, iterative methods show a good performance. On the other hand, convergence results can only be obtained under severe restrictions on the operator, and for applications it seems often impossible to meet these conditions.

Probably the best understood regularization method is *Tikhonov regularization*. As an approximation to a solution, a global minimizer x_α^δ of the Tikhonov functional

$$\Phi_\alpha(x) = \|y^\delta - F(x)\|^2 + \alpha\|x - \bar{x}\|^2 \tag{1.3}$$

with regularization parameter α is taken. If x_* denotes a solution of $F(x) = y$ and α is chosen properly, then it can be shown that $x_\alpha^\delta \to x_*$ as $\delta \to 0$ provided the operator F fulfills some slight restrictions (mainly, it has to be assumed that the operator has a Lipschitz continuous Fréchet derivative). Moreover, if we assume that the solution x_* fulfills a smoothness condition $x_* - \bar{x} = (F'(x_*)^* F'(x_*))^\nu \omega$, then an estimate

$$\|x_* - x_\alpha^\delta\| = \mathcal{O}(\delta^{2\nu/(2\nu+1)})$$

holds. For more details we refer to [4].
Besides choosing the proper regularization parameter, a main difficulty in Tikhonov regularization is the actual computation of a global minimizer of the functional (1.3). As the operator F is nonlinear, the functional is not convex any more and might thus have several (even local) minimizer. A chosen optimization routine has to make sure that a global minimizer is reconstructed. To this end we have introduced the TIGRA–algorithm which combines Tikhonov regularization with Morozov's discrepancy principle as parameter choice rule and a gradient method for the computation of a minimizer of the functional, see [12, 14] and Section 2. Under relatively mild restrictions on the operator we were able to show that the method is of optimal order for $\nu = 1/2$. Namely, we have to assume that

I) F is twice continuous Fréchet-differentiable

II) the Fréchet derivative F' is (globally) Lipschitz-continuous,

$$\|F'(x) - F'(\tilde{x})\| \le L\|x - \bar{x}\| \tag{1.4}$$

III) and the solution x_* of (1.1) fulfills a smoothness condition

$$x_* - \bar{x} = F'(x_*)^* \omega \tag{1.5}$$

with $\|\omega\| \le \varrho$ and ϱ small enough.

In the following, we will refer to these conditions by I–III.
The speed of a reconstruction of the TIGRA algorithm depends mainly on the gradient method used to reconstruct a global minimizer of the Tikhonov functional with fixed regularization parameter. It is a well known fact that the speed of convergence for the gradient method is sometimes slow, so it might be of great interest to replace the gradient method by a faster algorithm. Moreover, gradient methods for the minimization of a functional $\Phi(x)$ have the structure $x_{k+1} = x_k - \beta_k \nabla\Phi(x_k)$ with step size parameter β_k that has to be determined additionally. In [14] a rule for the choice of β_k that uses the knowledge of $\|\omega\|$, ω as in (1.5), was proposed. In many practical applications $\|\omega\|$ will not be known explicitly and β_k has to be determined by other methods, i.e. $\beta_k = \arg\min_\beta\{\Phi(x_k - \beta_k\nabla\Phi(x_k))\}$, which increases again the computational time. Thus the aim of this paper is to introduce faster methods for the reconstruction of a global minimizer of the Tikhonov functional. In particular, we will focus

on fixed point iterations. As for the TIGRA algorithm, we have to impose conditions I-III on the nonlinear operator F, and will thus discuss them shortly. It is a well known fact that the convergence of regularization methods can be arbitrarily slow. In order to obtain convergence rate results, a smoothness condition of type (1.5) is always necessary. The main difference between linear and nonlinear problems is that in the latter case an additional closeness condition is needed. For example, Tikhonov regularization requires (1.4) and (1.5) with $L\|\omega\| \leq 1$, cf. [4]. Thus we have $\|x_* - \bar{x}\| \leq \|F'(x_*)\|\|\omega\| \leq 1/L$, i.e. the a priori guess $\bar{x}$ has to be close enough to the solution x_*. Now, for the TIGRA algorithm as well as for the here considered fixed point iterations, we need only slightly stronger restrictions. Firstly, the operator has to be twice continuously differentiable, which holds in many applications, e.g. for the attenuated Radon transform [3, 14]. Secondly, the a priori guess $\bar{x}$ has to be closer to the solution (for TIGRA, we require $L\|\omega\| \leq 0.241$) [14]. Thus, our method works under almost the same conditions as Tikhonov regularization but reconstructs additionally a global minimizer of the Tikhonov functional. This task is usually not considered in the classical theory of Tikhonov regularization with nonlinear operator but arises immediately when Tikhonov regularization has to be used in practice. As our proposed algorithms will be iterative, we have to compare them with other iterative methods like Landweber, Levenberg-Marquardt or Gauss-Newton. To obtain convergence rates, all these methods need smoothness conditions of type (1.5) as well as a closeness condition. Additionally, they require severe restrictions to the operator and its Fréchet-derivative. In the simplest case, for Landweber iteration, a condition

$$\|F(x) - F(\tilde{x}) - F'(x)(\tilde{x} - x)\| = O(\|x - \tilde{x}\|\|F(x) - F(\tilde{x})\|)$$

is needed, that is already very difficult to show for real life applications. Thus we might claim that our methods work under less restrictive conditions than most of the other iterative algorithms.

The structure of the paper is as follows. In Section 2 we will recall some useful properties of the TIGRA-algorithm and the Tikhonov functional. In Sections 3 and 4 we will introduce and analyze two different fixed point iterations for the computation of a minimizer of the Tikhonov functional with given parameter α. Based on these results, a fixed point algorithm for the regularization of nonlinear problems will be proposed in Section 5. Finally we will illustrate our results in Section 6 with a numerical example from Single Photon Emission Computed Tomography (SPECT).

2 Some results on TIGRA

The TIGRA–algorithm combines Tikhonov regularization with a gradient method for the iterative construction of a minimizer of the Tikhonov functional. The algorithm has been investigated extensively in [14]. In principle it works as follows:

- given y^δ, δ, q, $\bar{x}$, x_0 and α_0
- set $k = 0$, $x_{\alpha_{-1}} = x_0$
- Repeat
  - $\alpha_k = q^k \alpha_0$
  - compute $x^\delta_{\alpha_k} = \arg\min \Phi_{\alpha_k}(x)$ by the gradient method with starting value $x^\delta_{\alpha_{k-1}}$
  - $k = k + 1$

  until $\|y^\delta - F(x^\delta_{\alpha_k})\| \le 5\delta$

The analysis in the following sections is based on results from the paper mentioned above, and thus we will summarize them now. The first result shows that the Tikhonov functional with nonlinear operator F is still locally convex in a neighborhood of a global minimizer x^δ_α.

Theorem 1 *Let the conditions I–III be fulfilled, and assume that ϱ with $\|\omega\| \le \varrho$ is small enough. For a global minimizer x^δ_α of the Tikhonov functional $\Phi_\alpha(x)$ we define the function*

$$\phi(t) = \Phi_\alpha(x^\delta_\alpha + th) \ , \qquad \|h\| = 1 \ . \tag{2.1}$$

The function $\phi''(t)$ is strictly positive for all $0 \le t \le r(\alpha)$, $\phi''(t) > \gamma\alpha$, with

$$r(\alpha) = \frac{1}{1+\sqrt{2}} \min\left\{ \sqrt{\frac{2\kappa\alpha}{3}}, \frac{2\kappa\alpha}{K} \right\} \ , \tag{2.2}$$

with constant $K > 0$, $\kappa = 1 - 3L\varrho - \gamma$ and a free parameter $\gamma > 0$ that has to be chosen such that $\kappa > 0$ holds. Thus ϕ is strictly convex in

$$K_{r(\alpha)}(x^\delta_\alpha) := \left\{ x \in X : x = x^\delta_\alpha + h, \ \|h\| \le r(\alpha) \right\} \ . \tag{2.3}$$

(see [14], p.441). Based on this result, it was shown that the gradient method converges to a global minimizer of the Tikhonov functional if it is started with $x_0 \in K_{r(\alpha)}(x^\delta_\alpha)$. The next two Theorems are concerned with the choice of α_0 and q.

Theorem 2 *Let x_0 be given, and assume that the conditions of Theorem 1 are fulfilled. Then there exists α_0 such that $x_0 \in K_{r(\alpha_0)}(x^\delta_{\alpha_0})$.*

(see [14], p.452)

Theorem 3 *Let the conditions of Theorem 1 hold and assume that $\alpha_k = q^k\alpha_0$. Then $q < 1$ can be chosen such that*

$$x^\delta_{\alpha_{k-1}} \in K_{r(\alpha_k)}(x^\delta_{\alpha_k}) \ . \tag{2.4}$$

Moreover, if we assume $\|y^\delta - F(\bar{x})\| > 5\delta$*, then there exists* $k_* \in \mathbb{N}$ *with*

$$\delta \le \|y^\delta - F(x^\delta_{\alpha_{k_*}})\| \le 5\delta < \|y^\delta - F(x^\delta_{\alpha_{k_*-1}})\| \ , \tag{2.5}$$

and

$$\|x_* - x^\delta_{\alpha_{k_*}}\| = \mathcal{O}(\sqrt{\delta}) \ . \tag{2.6}$$

For a proof, see again [14]. With these results, the convergence proof for the TIGRA method works as follows. For a given x_0 (in most cases, it is convenient to set $x_0 = \bar{x}$) we chose α_0 according to Theorem 2. As $x_0 \in K_{r(\alpha_0)}(x^\delta_{\alpha_0})$, the gradient method converges to $x^\delta_{\alpha_0}$. If q is chosen according to Theorem 3, then $x^\delta_{\alpha_0} \in K_{r(\alpha_1)}(x^\delta_{\alpha_1})$, and the gradient method for minimizing $\Phi_{\alpha_1}(x)$ and starting value $x^\delta_{\alpha_0}$ converges towards $x^\delta_{\alpha_1}$ and so forth. If the outer iteration finally stops, then (2.6) guarantees the order optimality of the method. As we have pointed out earlier, the main goal of this paper will be the replacement of the gradient method by a fixed point iteration. As the above given proof for the order optimality shows, it will be sufficient to show that the new method converges if it is started with a starting function in the convexity area of the Tikhonov functional.

3 Tikhonov regularization and Fixed Point iterations: A first attempt

The starting point for the development of our first fixed point algorithm is the necessary condition for a minimum of (1.3):

$$\alpha(x - \bar{x}) = F'(x)^*(y^\delta - F(x)) \ . \tag{3.1}$$

Setting

$$S_\alpha(x) = \frac{1}{\alpha} F'(x)^*(y^\delta - F(x)) + \bar{x} \ , \tag{3.2}$$

a minimizer of (1.3) clearly is a fixed point of S_α . However, S_α will be a contraction for large α only: In case of a linear operator A, we get

$$\|S_\alpha(x) - S_\alpha(\tilde{x})\| = \frac{1}{\alpha}\|A^*A(x - \tilde{x})\| \le \frac{\|A\|^2}{\alpha}\|x - \tilde{x}\| \ ,$$

i.e. S_α is only a contraction if $\|A\|^2 < \alpha$ holds. As $\alpha(\delta) \to 0$ for $\delta \to 0$, it depends on the data error whether the fixed point iteration $x_{k+1} = S_\alpha(x_k)$ can be used for the reconstruction of a solution of (3.1). A slightly more complicated but otherwise similar result holds in the case of an arbitrary nonlinear operator.

Proposition 4 *Assume that the minimizer* x^δ_α *of (1.3) and* $x, \tilde{x}$ *belong to a ball with radius* r *and center* $\bar{x}$*. With an operator* F *fulfilling I-II we obtain an estimate*

$$\|S_\alpha(x) - S_\alpha(\tilde{x})\| \le \frac{c}{\alpha}\|x - \tilde{x}\| \ , \tag{3.3}$$

with

$$c = L\|y^\delta\| + L(\|F(\bar{x})\| + r\|F'(\bar{x})\| + Lr^2) + (Lr + \|F'(\bar{x})\|)^2 + Lr(Lr + \|F'(\bar{x})\|) \ . \tag{3.4}$$

Proof:
For an operator with I-II we get the Taylor expansion

$$F(\tilde{x}) = F(x) + F'(x)(\tilde{x} - x) + R(x, \tilde{x}) \ , \tag{3.5}$$

and R fulfills an estimate

$$\|R(x, \tilde{x})\| \le L\|x - \tilde{x}\|^2 \ . \tag{3.6}$$

We obtain with (3.2)

$$\begin{aligned} S_\alpha(x) - S_\alpha(\tilde{x}) &= \frac{1}{\alpha}\left(F'(x) - F'(\tilde{x})\right)^* y^\delta + \frac{1}{\alpha}\left(F'(\tilde{x})^* F(\tilde{x}) - F'(x)^* F(x)\right) \\ &\overset{(3.5)}{=} \frac{1}{\alpha}\left(F'(x) - F'(\tilde{x})\right)^* y^\delta + \frac{1}{\alpha}\Big((F'(\tilde{x}) - F'(x))^* F(x) \\ &\qquad + F'(\tilde{x})^* F'(x)(\tilde{x} - x) + F'(\tilde{x})^* R(x, \tilde{x})\Big) \ . \end{aligned} \tag{3.7}$$

By using (1.4), the first term can be estimated by

$$\|\frac{1}{\alpha}\left(F'(x) - F'(\tilde{x})\right)^* y^\delta\| \le \frac{L\|y^\delta\|}{\alpha}\|x - \tilde{x}\| \ . \tag{3.8}$$

As $x, \bar{x} \in B_r(\bar{x})$, we get

$$\begin{aligned} \|F'(x)\| &\le \|F'(x) - F'(\bar{x})\| + \|F'(\bar{x})\| \le L\|x - \bar{x}\| + \|F'(\bar{x})\| \\ &\le Lr + \|F'(\bar{x})\| \ , \\ \|F(x)\| &\le \|F(\bar{x})\| + \|F'(\bar{x})\|\|x - \bar{x}\| + L\|x - \bar{x}\|^2 \\ &\le \|F(\bar{x})\| + \|F'(\bar{x})\| r + Lr^2 \ . \end{aligned}$$

Using these estimates, the second term in (3.7) can be estimated by

$$\begin{aligned} &\frac{1}{\alpha}\|(F'(\tilde{x}) - F'(x))^* F(x) + F'(\tilde{x})^* F'(x)(\tilde{x} - x) + F'(\tilde{x})^* R(x, \tilde{x})\| \\ &\quad \le \frac{1}{\alpha}\left(L(\|F(\bar{x})\| + \|F'(\bar{x})\| r + Lr^2) + (Lr + \|F(\bar{x})\|)^2 + 2Lr(Lr + \|F(\bar{x})\|)\right)\|x - \tilde{x}\| \ . \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) finishes the proof.

□

If we assume $c/\alpha < 1$, then S_α is at least locally a contraction and a fixed point iteration $x_{k+1} = S_\alpha(x_k)$ can be used to reconstruct a solution of (3.1). However, we should keep in mind that (3.1) is only a necessary condition for a global minimizer of (1.3), and thus the iteration might converge to a critical point only. To ensure convergence to a global minimizer, we have to use some convexity properties of the Tikhonov functional. According to Theorem 1 the Tikhonov functional is convex in a neighborhood $K_{r(\alpha)}(x_\alpha^\delta)$ of a global minimizer (for the definition of $K_{r(\alpha)}(x_\alpha^\delta)$, see (2.3). Now, if we start the fixed point iteration with $x_0 \in K_{r(\alpha)}(x_\alpha^\delta)$, then it is easy to see that the iteration converges to the global minimizer:

Theorem 5 *Let the conditions I-III be fulfilled, and assume that* $x_0 \in K_{r(\alpha)}(x_\alpha^\delta)$*. If* $\frac{c}{\alpha} < 1$*, then the sequence of fixed point iterates* $x_{k+1} = S_\alpha(x_k)$ *converges to the global minimizer of (1.3) in* $K_{r(\alpha)}(x_\alpha^\delta)$ *and the error estimate*

$$\|x_\alpha^\delta - x_k\| \leq \frac{\left(\frac{c}{\alpha}\right)^k}{1-\frac{c}{\alpha}}\|x_1 - x_0\| \tag{3.10}$$

holds.

<u>Proof:</u>
The global minimizer x_α^δ of (1.3) fulfills (3.1). Now let us assume there exists $\tilde{x} \in K_{r(\alpha)}(x_\alpha^\delta)$ with (3.1) and $\tilde{x} \neq x_\alpha^\delta$. We set $h = \tilde{x} - x_\alpha^\delta$ and

$$\phi(t) = \Phi_\alpha(x_\alpha^\delta + th), \qquad 0 \leq t \leq 1 \ .$$

In particular we have $\phi(0) = \Phi_\alpha(x_\alpha^\delta)$ and $\phi(1) = \Phi_\alpha(\tilde{x})$. The function $\phi(t)$ is twice continuous differentiable, and we have

$$\phi'(t) = \phi'(0) + \phi''(\xi)t, \qquad 0 \leq \xi \leq 1 \ .$$

According to Theorem 1, the function $\phi''(\xi)$ is strictly positive for $0 \leq \xi \leq 1$ if the conditions I–III hold, and

$$0 = \phi'(0) < \phi'(1) = -{\Phi_\alpha}'(\tilde{x})h \ .$$

Thus x_α^δ is the only point with (3.1) in $K_{r(\alpha)}(x_\alpha^\delta)$. By the contraction property of S_α we find

$$\|x_{k+1} - x_\alpha^\delta\| = \|S_\alpha(x_k) - S_\alpha(x_\alpha^\delta)\| \leq \frac{c}{\alpha}\|x_k - x_\alpha^\delta\| \ , \tag{3.11}$$

and all x_k stay within $K_{r(\alpha)}(x_\alpha^\delta)$ if only $x_0 \in K_{r(\alpha)}(x_\alpha^\delta)$. It follows by induction from (3.11) that

$$\|x_{k+1} - x_\alpha^\delta\| \leq \left(\frac{c}{\alpha}\right)^{k+1}\|x_0 - x_\alpha^\delta\| \ ,$$

and the sequence of iterates converges to the global minimizer x_α^δ. Now, as in Banach's fixed point theorem, we get the error estimate

$$\|x_{k+n} - x_k\| \leq \sum_{i=1}^{n}\|x_{k+i} - x_{k+i-1}\| \leq \frac{\left(\frac{c}{\alpha}\right)^k}{1-\frac{c}{\alpha}}\|x_1 - x_0\| \ ,$$

and by taking $n \to \infty$ follows (3.10).

□

The condition $c/\alpha < 1$ is of course a restriction. It is a well known fact that the regularization parameter tends to zero if the data error level tends to zero, and thus $c/\alpha \geq 1$ for δ small enough. However, in practical applications one usually has a fixed data error level, and it depends on the size of the error level and the operator F if the fixed point iteration with S_α converges.
Another problem is the choice of the starting value x_0, as it has to belong to the convexity area $K_{r(\alpha)}(x_\alpha^\delta)$. As $\alpha \to 0$ for $\delta \to 0$ and $r(\alpha) = \mathcal{O}(\alpha)$ holds, we need a real good guess x_0 for the minimizer x_α^δ and small error level. We will address this problem in Section 5.

4 A Fixed Point iteration for small regularization parameters

An advantage of the fixed point iteration $x_{k+1} = S_\alpha(x_k)$ with the operator S_α defined in (3.2) is that the evaluation of S_α is relatively cheap. Indeed, we have to evaluate the operator F and the adjoint of its Fréchet derivative only once. However, if we want to reconstruct a solution of (3.1) for small α, then we have to think of a different method. It turns out that we can find another fixed point formulation describing a minimizer of the Tikhonov functional that turns out to be a contraction. As nothing is for free, we have to pay with a higher numerical effort for the reconstruction. Moreover, we require the knowledge of a minimizer of the Tikhonov functional with a bigger regularization parameter.

Theorem 6 *Let x_α^δ be a minimizer of the Tikhonov functional (1.3), $x_{\tilde\alpha}^\delta$ a minimizer of $\Phi_{\tilde\alpha}$ with*

$$\alpha = q\tilde\alpha \ , \qquad q < 1 \tag{4.1}$$

and

$$F_\alpha(x) \;=\; F'(x)^* F'(x) + \alpha I \ , \tag{4.2}$$

$$B(x_{\tilde\alpha}^\delta - x)^2 \;=\; \frac{1}{2}\int_0^1 F''(x + \tau(x_{\tilde\alpha}^\delta - x))(x_{\tilde\alpha}^\delta - x, x_{\tilde\alpha}^\delta - x)\, d\tau \ . \tag{4.3}$$

Then x_α^δ is a fixed point of the equation $x = T_\alpha(x)$ with

$$T_\alpha(x) = F_\alpha(x)^{-1}F'(x)^*(y^\delta - F(x_{\tilde\alpha}^\delta) + B(x_{\tilde\alpha}^\delta - x)^2) - qF_\alpha(x)^{-1}F'(x_{\tilde\alpha}^\delta)^*(y^\delta - F(x_{\tilde\alpha}^\delta)) + x_{\tilde\alpha}^\delta \ . \tag{4.4}$$

Proof:
As F is twice continuous Fréchet-differentiable, we have the Taylor expansion

$$F(x_{\tilde\alpha}^\delta) = F(x) + F'(x)(x_{\tilde\alpha}^\delta - x) + B(x_{\tilde\alpha}^\delta - x)^2 \ , \tag{4.5}$$

where $B(x_{\tilde\alpha}^\delta - x)^2$ fulfills an estimate

$$\|B(x_{\tilde\alpha}^\delta - x)^2\| \le \frac{L}{2}\|x_{\tilde\alpha}^\delta - x\|^2 \tag{4.6}$$

see e.g. [19], Vol. I. Inserting this into the necessary condition (3.1) gives

$$\alpha(x - \bar x) = F'(x)^*(y^\delta - F(x_{\tilde\alpha}^\delta) + B(x_{\tilde\alpha}^\delta - x)^2) + F'(x)^*F'(x)(x_{\tilde\alpha}^\delta - x)$$

or

$$F'(x)^*F'(x)(x - x_{\tilde\alpha}^\delta) + \alpha(x - x_{\tilde\alpha}^\delta + x_{\tilde\alpha}^\delta - \bar x) = F'(x)^*(y^\delta - F(x_{\tilde\alpha}^\delta) + B(x_{\tilde\alpha}^\delta - x)^2) \ ,$$

which is equivalent to

$$\begin{aligned} x - x_{\tilde\alpha}^\delta &= F_\alpha(x)^{-1}F'(x)^*(y^\delta - F(x_{\tilde\alpha}^\delta) + B(x_{\tilde\alpha}^\delta - x)^2) - F_\alpha(x)^{-1}(\alpha(x_{\tilde\alpha}^\delta - \bar x)) \\ &\overset{(4.1)}{=} F_\alpha(x)^{-1}F'(x)^*(y^\delta - F(x_{\tilde\alpha}^\delta) + B(x_{\tilde\alpha}^\delta - x)^2) - qF_\alpha(x)^{-1}(\tilde\alpha(x_{\tilde\alpha}^\delta - \bar x)) \ . \end{aligned} \tag{4.7}$$

As $x_{\tilde{\alpha}}^{\delta}$ is a minimizer of $\Phi_{\tilde{\alpha}}$,

$$\tilde{\alpha}(x_{\tilde{\alpha}}^{\delta} - \bar{x}) = F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta}))$$

holds. Inserting this equality into (4.7) yields

$$x = F_\alpha(x)^{-1}F'(x)^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta}) + B(x_{\tilde{\alpha}}^{\delta} - x)^2) - qF_\alpha(x)^{-1}F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta})) + x_{\tilde{\alpha}}^{\delta} \ .$$

□

In the following we would like to show that T_α is a contraction. To this end, we have to estimate

$$\begin{aligned} T_\alpha(x) - T_\alpha(\tilde{x}) \ &= \ F_\alpha(x)^{-1}F'(x)^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta}) + B(x_{\alpha}^{\delta} - x)^2) \\ &\quad -F_\alpha(\tilde{x})^{-1}F'(\tilde{x})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta}) + B(x_{\alpha}^{\delta} - \tilde{x})^2) \\ &\quad -q(F_\alpha(x)^{-1} - F_\alpha(\tilde{x})^{-1})F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta})) \ . \end{aligned} \tag{4.8}$$

We are going to investigate the last term in (4.8) first.

Proposition 7 *Let $F_\alpha(x)$ be defined as in (4.2), $q < 1$ and let $x_{\tilde{\alpha}}^{\delta}$ be a minimizer of $\Phi_{\tilde{\alpha}}$ with*

$$\alpha \ = \ q\tilde{\alpha} \tag{4.9}$$

$$\|y^\delta - F(x_{\tilde{\alpha}}^{\delta})\| \ \leq \ 3\tilde{\alpha}\varrho \ . \tag{4.10}$$

Moreover, assume that $x_{\tilde{\alpha}}^{\delta}$, $x, \tilde{x} \in B_r(x_{\alpha}^{\delta})$, where $B_r(x_{\alpha}^{\delta})$ denotes a ball with center x_{α}^{δ} and radius

$$r = \tilde{c}\sqrt{\alpha} \ ,$$

$\tilde{c} > 0$. *Then*

$$\|(F_\alpha(x)^{-1} - F_\alpha(\tilde{x})^{-1})F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta}))\| \leq c_1\|x - \tilde{x}\| \tag{4.11}$$

holds, where the constant c_1 is given by

$$c_1 := \frac{3L\varrho(5 + 24L\tilde{c})}{4q} \ . \tag{4.12}$$

Here, L denotes the Lipschitz constant in II) and $\|\omega\| \leq \varrho$, ω as in III).

Proof:
Let $z, \tilde{z}$ be solutions of the equations

$$\begin{aligned} F_\alpha(x)z \ &= \ F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta})) \\ F_\alpha(\tilde{x})\tilde{z} \ &= \ F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta})) \ . \end{aligned}$$

As $F_\alpha(x)$, $F_\alpha(\tilde{x})$ are invertible for $\alpha > 0$ we have

$$F_\alpha(x)(\tilde{z} - z) = F_\alpha(x)\tilde{z} - F_\alpha(\tilde{x})\tilde{z} + \underbrace{F_\alpha(\tilde{x})\tilde{z} - F_\alpha(x)z}_{=0} = (F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z}$$

or

$$\tilde{z} - z = F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z} \ .$$

With $\tilde{z} = F_\alpha(\tilde{x})^{-1}F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))$ follows

$$\tilde{z} - z = F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))F_\alpha(\tilde{x})^{-1}F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}})) \ . \tag{4.13}$$

To simplify the notation we set

$$\begin{aligned} A &= F'(x) && (4.14)\\ \tilde{A} &= F'(\tilde{x}) \ . && (4.15) \end{aligned}$$

and

$$\begin{aligned} B(\tilde{x} - x)(\,\cdot\,) &= \int_0^1 F''(x + \tau(\tilde{x} - x))(\tilde{x} - x, \,\cdot\,)\, d\tau && (4.16)\\ \tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})(\,\cdot\,) &= \int_0^1 F''(\tilde{x} + \tau(x^\delta_{\tilde{\alpha}} - \tilde{x}))(x^\delta_{\tilde{\alpha}} - \tilde{x}, \,\cdot\,)\, d\tau \ . && (4.17) \end{aligned}$$

The operators $\tilde{A}$ and $F'(x^\delta_{\tilde{\alpha}})$ admit the Taylor expansions

$$\begin{aligned} \tilde{A} &= A + B(\tilde{x} - x) && (4.18)\\ F'(x^\delta_{\tilde{\alpha}}) &= \tilde{A} + \tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x}) \ , && (4.19) \end{aligned}$$

with A, $\tilde{A}$, $B(\tilde{x} - x)$ and $\tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})$ fulfilling the estimates

$$\begin{aligned} \|A - \tilde{A}\| &\leq L\|x - \tilde{x}\| && (4.20)\\ \|B(\tilde{x} - x)\| &\leq L\|\tilde{x} - x\| && (4.21)\\ \|\tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})\| &\leq L\|x^\delta_{\tilde{\alpha}} - \tilde{x}\| \overset{(4.11)}{\leq} 2L\tilde{c}\sqrt{\alpha} \ . && (4.22) \end{aligned}$$

It follows

$$\begin{aligned} F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))F_\alpha(\tilde{x})^{-1} &= F_\alpha(x)^{-1}(A^*A - \tilde{A}^*\tilde{A})F_\alpha(\tilde{x})^{-1}\\ &\overset{(4.18)}{=} F_\alpha(x)^{-1}(A^*A - A^*\tilde{A})F_\alpha(\tilde{x})^{-1} + F_\alpha(x)^{-1}B(\tilde{x} - x)^*\tilde{A}F_\alpha(\tilde{x})^{-1}\\ &= F_\alpha(x)^{-1}A^*(A - \tilde{A})F_\alpha(\tilde{x})^{-1} + F_\alpha(x)^{-1}B(\tilde{x} - x)^*\tilde{A}F_\alpha(\tilde{x})^{-1} && (4.23) \end{aligned}$$

and thus with (4.13)

$$\begin{aligned} \tilde{z} - z = {} & \underbrace{F_\alpha(x)^{-1}A^*(A - \tilde{A})F_\alpha(\tilde{x})^{-1}F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))}_{=:T_1}\\ & + \underbrace{F_\alpha(x)^{-1}B(\tilde{x} - x)^*\tilde{A}F_\alpha(\tilde{x})^{-1}F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}})) \ .}_{=:T_2} && (4.24) \end{aligned}$$

By (4.19), the first term can be further decomposed into

$$T_1 = \underbrace{F_\alpha(x)^{-1}A^*(A-\tilde{A})F_\alpha(\tilde{x})^{-1}\tilde{A}^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))}_{=:T_{11}} + \underbrace{F_\alpha(x)^{-1}A^*(A-\tilde{A})F_\alpha(\tilde{x})^{-1}\tilde{B}(x^\delta_{\tilde{\alpha}}-\tilde{x})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))}_{=:T_{12}} .$$

Using (4.10), (4.9), (4.20) and

$$\left.\begin{array}{l} \|F_\alpha(x)^{-1}A^*\| = \|(A^*A+\alpha I)A^*\| \\ \|F_\alpha(\tilde{x})^{-1}\tilde{A}^*\| = \|(\tilde{A}^*\tilde{A}+\alpha I)\tilde{A}^*\| \end{array}\right\} \le \frac{1}{2\sqrt{\alpha}}$$

$$\left.\begin{array}{l} \|F_\alpha(x)^{-1}\| = \|(A^*A+\alpha I)\| \\ \|F_\alpha(\tilde{x})^{-1}\| = \|(\tilde{A}^*\tilde{A}+\alpha I\| \end{array}\right\} \le \frac{1}{\alpha} \tag{4.25}$$

we can estimate T_{11} by

$$\begin{aligned} \|T_{11}\| &\le \|F_\alpha(x)^{-1}A^*\|\|A-\tilde{A}\|\|F_\alpha(\tilde{x})^{-1}\tilde{A}^*\|\|y^\delta - F(x^\delta_{\tilde{\alpha}})\| \\ &\le \frac{L}{2\sqrt{\alpha}}\|x-\tilde{x}\|\frac{3\tilde{\alpha}}{2\sqrt{\alpha}}\varrho \\ &= \frac{3L\varrho}{4q}\|x-\tilde{x}\| . \end{aligned} \tag{4.26}$$

T_{12} can be estimated in the same way:

$$\begin{aligned} T_{12} &\le \|F_\alpha(x)^{-1}A^*\|\|A-\tilde{A}\|\|F_\alpha(\tilde{x})^{-1}\|\|\tilde{B}(x^\delta_{\tilde{\alpha}}-\tilde{x})^*\|\|y^\delta - F(x^\delta_{\tilde{\alpha}})\| \\ &\overset{(4.22)}{\le} \frac{L}{2\sqrt{\alpha}}\|x-\tilde{x}\|\frac{2L\tilde{c}\sqrt{\alpha}}{\alpha}\cdot 3\tilde{\alpha}\varrho \\ &= \frac{3L^2\varrho\tilde{c}}{q}\|x-\tilde{x}\| . \end{aligned} \tag{4.27}$$

By (4.19), T_2 can be rewritten as

$$\begin{aligned} T_2 &= F_\alpha(x)^{-1}B(\tilde{x}-x)^*\tilde{A}F_\alpha(\tilde{x})^{-1}F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}})) \\ &= \underbrace{F_\alpha(x)^{-1}B(\tilde{x}-x)^*\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{A}^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))}_{T_{21}} \\ &\quad + \underbrace{F_\alpha(x)^{-1}B(\tilde{x}-x)^*\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{B}(x^\delta_{\tilde{\alpha}}-\tilde{x})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))}_{T_{22}} . \end{aligned}$$

Because of

$$\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{A}^* = \tilde{A}(\tilde{A}^*\tilde{A}+\alpha I)^{-1}\tilde{A}^* = (\tilde{A}\tilde{A}^*+\alpha I)^{-1}\tilde{A}\tilde{A}^* ,$$

it follows

$$\|\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{A}^*\| \le 1 ,$$

and thus

$$\begin{aligned}\|T_{21}\| &\le \|F_\alpha(x)^{-1}\|\|B(\tilde{x}-x)\|\|\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{A}^*\|\|y^\delta - F(x^\delta_{\tilde{\alpha}})\| \\ &\le \frac{1}{\alpha}L\|x-\tilde{x}\|3\tilde{\alpha}\varrho = \frac{3L\varrho}{q}\|x-\tilde{x}\| \ . \end{aligned} \tag{4.28}$$

Moreover, we have

$$\|\tilde{A}F_\alpha(\tilde{x})\| = \|\tilde{A}(\tilde{A}^*\tilde{A}+\alpha I)^{-1}\| = \|(\tilde{A}\tilde{A}^*+\alpha I)^{-1}\tilde{A}\| \le \frac{1}{2\sqrt{\alpha}} \ ,$$

and we obtain for T_{22}

$$\begin{aligned}\|T_{22}\| &\le \|F_\alpha(x)^{-1}\|\|B(\tilde{x}-x)\|\|\tilde{A}F_\alpha(\tilde{x})^{-1}\|\|\tilde{B}(x^\delta_{\tilde{\alpha}}-\tilde{x})^*\|\|y^\delta - F(x^\delta_{\tilde{\alpha}})\| \\ &\le \frac{1}{\alpha}L\|x-\tilde{x}\|\frac{1}{2\sqrt{\alpha}}2L\tilde{c}\sqrt{\alpha}3\varrho\tilde{\alpha} = \frac{3L^2\varrho\tilde{c}}{q}\|x-\tilde{x}\| \ . \end{aligned} \tag{4.29}$$

Putting (4.13), (4.24), (4.26), (4.27), (4.28) and (4.29) together we arrive finally at

$$\begin{aligned}\|(F_\alpha(x)^{-1} - F_\alpha(\tilde{x})^{-1})F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))\| &= \|z-\tilde{z}\| \\ &\le \|T_{11}\| + \|T_{12}\| + \|T_{21}\| + \|T_{22}\| \\ &\le \underbrace{\left[\frac{3L\varrho}{4q} + \frac{3L^2\varrho\tilde{c}}{q} + \frac{3L\varrho}{q} + \frac{3L^2\varrho\tilde{c}}{q}\right]}_{=:c_1}\|x-\tilde{x}\| \ , \end{aligned}$$

which concludes the proof.

□

Next we will have a closer look at the first two terms of (4.8).

Proposition 8 *Let the assumptions (4.9)-(4.11) hold. If $z, \tilde{z}$ denote the solutions of*

$$\begin{aligned} F_\alpha(x)z &= F'(x)^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + B(x^\delta_{\tilde{\alpha}} - x)^2) \\ F_\alpha(\tilde{x})\tilde{z} &= F'(\tilde{x})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + \tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})^2) \ , \end{aligned} \tag{4.30}$$

then

$$\|z-\tilde{z}\| \le c_2\|x-\tilde{x}\| \tag{4.31}$$

holds, where c_2 is defined by

$$c_2 = \frac{5L}{4q}\left(3\varrho + L\tilde{c}^2 q\right) + \frac{3\varrho L}{q} + \frac{3}{2}L\tilde{c} + L^2\tilde{c}^2 \ . \tag{4.32}$$

Proof:
By (4.30) follows

$$\begin{aligned} F_\alpha(x)(\tilde{z}-z) &= (F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z} + F_\alpha(\tilde{x})\tilde{z} - F_\alpha(x)z \\ &= (F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z} + F'(\tilde{x})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + \tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})^2) \\ &\qquad - F'(x)^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + B(x^\delta_{\tilde{\alpha}} - x)^2) \end{aligned}$$

and thus

$$\tilde{z} - z = \underbrace{F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z}}_{T_1} + \underbrace{F_\alpha(x)^{-1}(F'(\tilde{x}) - F'(x))^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))}_{T_2}$$
$$+ \underbrace{F_\alpha(x)^{-1}(F'(\tilde{x})^*\tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})^2 - F'(x)^*B(x^\delta_{\tilde{\alpha}} - x)^2)}_{T_3} \ . \qquad (4.33)$$

Once again, these three terms will be treated separately. Using definition (4.30) of $\tilde{z}$, we get

$$F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z} = F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))F_\alpha(\tilde{x})^{-1}F'(\tilde{x})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + \tilde{B}(x^\delta_{\tilde{\alpha}} - \tilde{x})^2) \ . \qquad (4.34)$$

Defining $A, \tilde{A}$ as in (4.14), (4.15), we get as in (4.23)

$$F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))F_\alpha(\tilde{x})^{-1}\tilde{A}^* = F_\alpha(x)^{-1}A^*(A - \tilde{A})F_\alpha(\tilde{x})^{-1}\tilde{A}^* + F_\alpha(x)^{-1}B(\tilde{x} - x)^*\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{A}^* \ . \qquad (4.35)$$

With

$$\begin{aligned} \|y^\delta - F(x^\delta_{\tilde{\alpha}}) + B(x^\delta_{\tilde{\alpha}} - x)^2\| &\overset{(4.10),(4.6)}{\leq} 3\varrho\tilde{\alpha} + \frac{L}{2}\|x^\delta_{\tilde{\alpha}} - \tilde{x}\|^2 \\ &\overset{(4.11)}{\leq} (3\varrho + L\tilde{c}^2 q)\tilde{\alpha} \end{aligned}$$

it follows from (4.34), (4.35)

$$\begin{aligned} \|T_1\| &= \|F_\alpha(x)^{-1}(F_\alpha(x) - F_\alpha(\tilde{x}))\tilde{z}\| \\ &\leq \Big(\|F_\alpha(x)^{-1}A^*\|\|A - \tilde{A}\|\|F_\alpha(\tilde{x})^{-1}\tilde{A}^*\| \\ &\qquad + \|F_\alpha(x)^{-1}\|\|B(\tilde{x} - x)^*\|\|\tilde{A}F_\alpha(\tilde{x})^{-1}\tilde{A}^*\|\Big) \|y^\delta - F(x^\delta_{\tilde{\alpha}}) + B(x^\delta_{\tilde{\alpha}} - x)^2\| \\ &\leq \left(\frac{1}{2\sqrt{\alpha}}L\|x - \tilde{x}\|\frac{1}{2\sqrt{\alpha}} + \frac{1}{\alpha}L\|x - \tilde{x}\|\right)(3\varrho + L\tilde{c}^2 q)\tilde{\alpha} \\ &\overset{(4.9)}{\leq} \frac{5L}{4q}\left(3\varrho + L\tilde{c}^2 q\right)\|x - \tilde{x}\| \ . \end{aligned} \qquad (4.36)$$

The term T_2 is estimated with(4.25) and (4.10) by

$$\begin{aligned} \|T_2\| = \|F_\alpha(x)^{-1}(F'(\tilde{x}) - F'(x))^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))\| &\leq \frac{L}{\alpha}\|x - \tilde{x}\|(3\varrho\tilde{\alpha}) \\ &= \frac{3\varrho L}{q}\|x - \tilde{x}\| \ , \end{aligned} \qquad (4.37)$$

and we are left with the last term in (4.33). By the Taylor expansion of F we obtain

$$F(x^\delta_{\tilde{\alpha}}) = F(x) + F'(x)(x^\delta_{\tilde{\alpha}} - x) + B(x^\delta_{\tilde{\alpha}} - x)^2$$

or

$$B(x^\delta_{\tilde{\alpha}} - x)^2 = F(x^\delta_{\tilde{\alpha}}) - F(x) - A(x^\delta_{\tilde{\alpha}} - x) \ .$$

Simultaneously

$$\tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2 = F(x^\delta_{\tilde\alpha}) - F(\tilde{x}) - \tilde{A}(x^\delta_{\tilde\alpha} - \tilde{x})$$

holds, and thus

$$\begin{aligned}
\tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2 - B(x^\delta_{\tilde\alpha} - x)^2 &= F(x) - F(\tilde{x}) + A(x^\delta_{\tilde\alpha} - x) - \tilde{A}(x^\delta_{\tilde\alpha} - \tilde{x}) \\
&= F(x) - F(\tilde{x}) + (A - \tilde{A})(x^\delta_{\tilde\alpha} - \tilde{x}) + A(\tilde{x} - x) \ . \qquad (4.38)
\end{aligned}$$

The Taylor expansion for F gives

$$F(x) - F(\tilde{x}) + A(\tilde{x} - x) = -B(\tilde{x} - x)^2 \ ,$$

and inserting these terms into (4.38) yields

$$\tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2 - B(x^\delta_{\tilde\alpha} - x)^2 = -B(\tilde{x} - x)^2 + (A - \tilde{A})(x^\delta_{\tilde\alpha} - \tilde{x}) \ . \qquad (4.39)$$

Now, using (4.14), (4.15) and (4.18) the term T_3 can be decomposed as follows:

$$\begin{aligned}
T_3 &= F_\alpha(x)^{-1}(F'(\tilde{x})^* \tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2 - F'(x)^* B(x^\delta_{\tilde\alpha} - x)^2) \qquad (4.40) \\
&= \underbrace{F_\alpha(x)^{-1} A^* \left(\tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2 - B(x^\delta_{\tilde\alpha} - x)^2\right)}_{=:T_{31}} + \underbrace{F_\alpha(x)^{-1} B(\tilde{x} - x)^* \tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2}_{=:T_{32}} \ . \qquad (4.41)
\end{aligned}$$

By (4.39), (4.25), (4.11) and (4.9) we obtain

$$\begin{aligned}
\|T_{31}\| &= \| - F_\alpha(x)^{-1} A^*(B(\tilde{x} - x)^2 + (A - \tilde{A})(x^\delta_{\tilde\alpha} - \tilde{x})\| \qquad (4.42) \\
&\le \|F_\alpha(x)^{-1} A^*(B(\tilde{x} - x)^2\| + \|F_\alpha(x)^{-1} A^*\| \|A - \tilde{A}\| \|x^\delta_{\tilde\alpha} - \tilde{x}\| \qquad (4.43) \\
&\le \frac{1}{2\sqrt{\alpha}} \left(\frac{L}{2}\|x - \tilde{x}\|^2 + L\|x - \tilde{x}\| 2\tilde{c}\sqrt{\alpha}\right) \qquad (4.44) \\
&\le \frac{3L\tilde{c}}{2}\|x - \tilde{x}\| \ . \qquad (4.45)
\end{aligned}$$

For T_{32} holds

$$\begin{aligned}
\|T_{32}\| &= \|F_\alpha(x)^{-1} B(\tilde{x} - x)^* \tilde{B}(x^\delta_{\tilde\alpha} - \tilde{x})^2\| \qquad (4.46) \\
&\le \frac{1}{\alpha} L\|x - \tilde{x}\| \frac{L}{2} \|x^\delta_{\tilde\alpha} - \tilde{x}\|^2 \qquad (4.47) \\
&\overset{(4.11)}{\le} L^2 \tilde{c}^2 \|x - \tilde{x}\| \ . \qquad (4.48)
\end{aligned}$$

Thus we have

$$\|T_3\| \le \|T_{31}\| + \|T_{32}\| \le \left(\frac{3}{2}L\tilde{c} + L^2\tilde{c}^2\right) \|x - \tilde{x}\| \ , \qquad (4.49)$$

and, putting together (4.33), (4.36), (4.37) and (4.49) we obtain finally

$$\begin{aligned}
\|z - \tilde{z}\| &\le \|T_1\| + \|T_2\| + \|T_3\| \\
&\le \left(\frac{5L}{4q}\left(3\varrho + L\tilde{c}^2 q\right) + \frac{3\varrho L}{q} + \frac{3}{2}L\tilde{c} + L^2\tilde{c}^2\right) \|x - \tilde{x}\| \ .
\end{aligned}$$

□

Theorem 9 *Let the conditions I-III and (4.9)–(4.11) hold. Then the operator T_α, defined in (4.4), fulfills an estimate*

$$\|T_\alpha(x) - T_\alpha(\tilde{x})\| \leq \hat{c}\|x - \tilde{x}\| \ , \tag{4.50}$$

with

$$\hat{c} := qc_1 + c_2 \tag{4.51}$$

and $c_1, \ c_2$ defined in (4.12), (4.32).

Proof:
According to (4.8),

$$\begin{aligned} T_\alpha(x) - T_\alpha(\tilde{x}) &= F_\alpha(x)^{-1}F'(x)^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + B(x^\delta_\alpha - x)^2) \\ &\quad -F_\alpha(\tilde{x})^{-1}F'(\tilde{x})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}) + B(x^\delta_\alpha - \tilde{x})^2) \\ &\quad -q(F_\alpha(x)^{-1} - F_\alpha(\tilde{x})^{-1})F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}})) \\ &= z - \tilde{z} - q(F_\alpha(x)^{-1} - F_\alpha(\tilde{x})^{-1})F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}})) \ , \end{aligned}$$

where $z, \ \tilde{z}$ are solutions of the equations given in (4.30). Lemma 8 states

$$\|z - \tilde{z}\| \leq c_2\|x - \tilde{x}\| \ .$$

From Lemma 7 follows

$$\|(F_\alpha(x)^{-1} - F_\alpha(\tilde{x})^{-1})F'(x^\delta_{\tilde{\alpha}})^*(y^\delta - F(x^\delta_{\tilde{\alpha}}))\| \leq qc_1\|x - \tilde{x}\|$$

and thus (4.50).

□

Proposition 10 *Let $r(\alpha)$ be defined by (2.2), and assume $x^\delta_{\tilde{\alpha}}, x, \tilde{x} \in K_{r(\alpha)}(x^\delta_\alpha)$. Then $x^\delta_{\tilde{\alpha}}, x, \tilde{x}$ fulfill (4.11) with*

$$\tilde{c} = \tilde{c}(\alpha, \kappa, L) = \frac{1}{L(1+\sqrt{2})} \min\left\{\sqrt{\frac{2\kappa}{3}}, \frac{2\kappa}{K}\sqrt{\alpha}\right\} \ . \tag{4.52}$$

The proof follows immediately from the definition of $r(\alpha)$.

So far, it is not clear at all whether T_α is a contraction or not, as we have no information on the size of the constant $\hat{c}$. However, this constant depends on several parameters, $\hat{c} = \hat{c}(L, \varrho, q, \kappa)$, and we might show that, if the parameters are properly chosen, T_α is a contraction:

Theorem 11 *Let $\hat{c}$ be the constant defined in (4.51), $0 < q_{min} < q < 1$, and assume that $\tilde{c}$ is given by (4.52). If the solution x_* of $F(x) = y$ fulfills a smoothness condition*

$$x_* - \bar{x} = F'(x_*)^* \omega$$

with $\|\omega\|$ small enough, then the operator T_α is a contraction in $K_{r(\alpha)}(x_\alpha^\delta)$.

Proof:
It remains to show that $\hat{c} < 1$ holds for small $\|\omega\| \leq \varrho$. First, we would like to rearrange $\hat{c} = qc_1 + c_2$ in terms of $L\tilde{c}$ and $L\varrho$. We get $\hat{c} = \hat{c}_1 + \hat{c}_2$,

$$\begin{aligned} \hat{c}_1 &= L\varrho \left(\frac{21 + 36L\tilde{c}}{2q} \right) \\ \hat{c}_2 &= L\tilde{c} \left(\frac{9 + 6L\tilde{c}}{4} \right) . \end{aligned}$$

Let us start with $\hat{c}_1$. First we observe that $L\tilde{c}$ is bounded, and because of $0 < q_{min} < q < 1$ we do observe that

$$\frac{21 + 36L\tilde{c}}{2q}$$

is bounded from above. Thus, if $\|\omega\| \leq \varrho$ is small enough, we find

$$\hat{c}_1 < \frac{1}{2} .$$

Now let us consider $\hat{c}_2$. From the definition (4.52) of $\tilde{c}$ follows

$$L\tilde{c} = \frac{1}{1 + \sqrt{2}} \min \left\{ \sqrt{\frac{2\kappa}{3}}, \frac{2\kappa}{3K} \sqrt{\alpha} \right\} \leq \sqrt{\frac{2\kappa}{3}} .$$

From Theorem 1 we recall $\kappa = 1 - 3L\varrho - \gamma$, where ϱ has to be small enough to fulfill $3L\varrho < 1$. Additionally, γ is a free parameter that has to be chosen such that $\kappa > 0$ holds. Thus we can choose γ such that κ and consequently $\hat{c}_2$ get arbitrarily small. In particular γ can be chosen such that

$$\hat{c}_2 < \frac{1}{2} .$$

Finally we get

$$\hat{c} < 1$$

and T_α is a contraction.

□

It might be of interest to discuss the influence of the parameter γ. A large γ means that κ is close to zero, and thus $r(\alpha)$ is small as well. On the other hand, we have $\phi''(t) > \gamma\tilde{\alpha}$ (cf. (2.1)), i.e. a large γ gives a larger lower bound on the second derivative of ϕ in $K_{r(\tilde{\alpha})}(x_{\tilde{\alpha}}^\delta)$.
Next, we can give a convergence result for our fixed point iteration.

Theorem 12 *Let the conditions I-III be fulfilled, with* $\|\omega\| \le \varrho$ *and* γ *are chosen such that* $\hat{c} < 1$. *Moreover, let* x_α^δ *and* $x_{\tilde{\alpha}}^\delta$ *be minimizers of the Tikhonov functional with* $\alpha = q\tilde{\alpha}$, $q < 1$. *If* $x_{\tilde{\alpha}}^\delta$, $x_0 \in K_{r(\alpha)}(x_\alpha^\delta)$, *then the sequence of fixed point iterates* $x_{k+1} = T_\alpha(x_k)$ *converges to the global minimizer of (1.3) in* $K_{r(\alpha)}(x_\alpha^\delta)$ *and the error estimate*

$$\|x_\alpha^\delta - x_k\| \le \frac{\hat{c}^k}{1-\hat{c}}\|x_1 - x_0\| \tag{4.53}$$

holds.

Proof:
The proof is similar to the proof of Theorem 5.

Because of $\|x_* - \bar{x}\| \le \|F'(x_*)\|\|\omega\|$ a smallness assumption on $\|\omega\|$ automatically induces the assumption that $\bar{x}$ has already been close to the solution. However, as the estimate for $\|T_\alpha(x) - T_\alpha(\tilde{x})\|$ in Theorem 9 is far from being sharp, we expect convergence of the fixed point iteration even in cases where $\|\omega\|$ is bigger then it is allowed according to Theorem 11. In addition, in many practical applications it will not be possible to estimate the constant $\hat{c}$, as $\|\omega\|$ and thus ϱ can be only estimated roughly or might be even unknown.

As the evaluation of the operator T_α requires the knowledge of a minimizing function $x_{\tilde{\alpha}}^\delta$ of $\Phi_{\tilde{\alpha}}$ we will present an algorithm that successively computes the minimizing functions $x_{\alpha_k}^\delta$ for a given sequence $\{\alpha_k\}_{k\in\mathbb{R}}$ of regularization parameters.

5 A fixed point based algorithm for the regularization of nonlinear operator equations

In this section we will use both fixed point iterations to form a regularization method for nonlinear ill-posed operators. From the numerical point of view the first fixed point iteration (with operator S_α) seems to be more effective, as the evaluation of $S_\alpha(x)$ requires only the computation of $F(x)$ and $F'(x)^*$. In contrast, the evaluation of T_α needs the solution of a linear operator equation. However, the fixed point iteration with S_α converges only if α is large enough (see Theorem 5). Thus it seems a good idea to use the iteration with S_α as long as $\frac{c}{\alpha} < 1$, and to switch to the iteration with T_α if the condition is violated.

As we have pointed out in the last section, the evaluation of $T_\alpha(x)$ needs a minimizer $x_{\tilde{\alpha}}^\delta$ of $\Phi_{\tilde{\alpha}}(x)$ with $x_{\tilde{\alpha}}^\delta \in K_{r(\alpha)}(x_\alpha^\delta)$. To recognize this dependence, we will further write $T_\alpha(\cdot, x_{\tilde{\alpha}}^\delta)$.

Another important question is the selection of the regularization parameter. As for the TIGRA algorithm, we will use Morozov's discrepancy principle, i.e. we will choose a regularization parameter such that

$$\delta \le \|y^\delta - F(x_\alpha^\delta)\| \le 5\delta \tag{5.1}$$

holds. We wish to remark that a regularization parameter with (5.1) does not exist for arbitrary nonlinear operators; its existence for the case of twice Fréchet differentiable operators was shown in [11]. To find a parameter with (5.1), we are going to compute the minimizers of the

Tikhonov functional for a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ of regularization parameters with $\alpha_{k+1} = q\alpha_k$ and $q < 1$.

Let $\bar{\alpha}$ be the regularization parameter with

$$\frac{c}{\bar{\alpha}} = 1 \ ,$$

c as in (3.4). Thus we can use the fixed point iteration with S_α for all $\alpha > \bar{\alpha}$, see Theorem 5 For simplicity, we define the operator $R_\alpha(x, x^\delta_{\tilde{\alpha}})$ by

$$R_\alpha(x, x^\delta_{\tilde{\alpha}}) := \begin{cases} S_\alpha(x) & \text{for } \alpha > \bar{\alpha} \ , \\ T_\alpha(x, x^\delta_{\tilde{\alpha}}) & \text{for } \alpha \le \bar{\alpha} \ . \end{cases} \tag{5.2}$$

The proposed fixed point regularization algorithm reads as follows.

- given y^δ, δ, $\bar{x}$, x_0
- choose $\alpha_0 > \bar{\alpha}$, $q < 1$
- set $k = 0$ and $x^\delta_{\alpha_{-1}} = x_0$
- Repeat
  - ◇ compute $x^\delta_{\alpha_k}$ as fixed point of $R_{\alpha_k}(x, x^\delta_{\alpha_{k-1}})$, use $x^\delta_{\alpha_{k-1}}$ as starting value for the iteration
  - ◇ $\alpha_{k+1} = q\alpha_k$
  - ◇ $k = k + 1$

  item[]until $\|y^\delta - F(x^\delta_{\alpha_k})\| \le 5\delta$

In order to obtain a convergence rate result, we have to ensure

i) $x_0 \in K_{r(\alpha_0)}(x^\delta_{\alpha_0})$

ii) $x^\delta_{\alpha_{k-1}} \in K_{r(\alpha_k)}(x^\delta_{\alpha_k})$

iii) The algorithm stops with $x^\delta_{\alpha_{k_*}}$ and $\delta \le \|y^\delta - F(x^\delta_{\alpha_{k_*}})\| \le 5\delta$.

This can be done by using the following results.

Proposition 13 *If the regularization parameter α is chosen large enough, then $\bar{x} \in K_{r(\alpha)}(x^\delta_\alpha)$. Thus we can set $x_0 := \bar{x}$.*

Proof:
We have

$$\alpha\|x^{\delta}_{\alpha} - \bar{x}\|^2 \leq \|y^\delta - F(x^{\delta}_{\alpha})\|^2 + \alpha\|x^{\delta}_{\alpha} - \bar{x}\|^2 = \Phi_\alpha(x^{\delta}_{\alpha}) \leq \Phi_\alpha(\bar{x}) = \|y^\delta - F(\bar{x})\|^2 \ ,$$

i.e.

$$\|x^{\delta}_{\alpha} - \bar{x}\|^2 \leq \frac{1}{\alpha}\|y^\delta - F(\bar{x})\|^2 \ .$$

As $\bar{x} \in K_{r(\alpha)}(x^{\delta}_{\alpha})$ if $\|x^{\delta}_{\alpha} - \bar{x}\| \leq r(\alpha)$, it is sufficient to show

$$\frac{1}{\alpha}\|y^\delta - F(\bar{x})\|^2 \leq r(\alpha) \tag{5.3}$$

for large α. But as the left hand side of (5.3) tends to zero for $\alpha \to \infty$, and the right hand side to infinity, this inequation always holds for large α.

□

Proposition 14 *Let $\alpha_{k} = q\alpha_{k-1}$. If $\|L\omega\| \leq 0.241$, then there exists $\bar{q} < 1$ s.t. $x^{\delta}_{\alpha_{k-1}} \in K_{r(\alpha_k)}(x^{\delta}_{\alpha_k})$ for all $\bar{q} \leq q < 1$.*

The proof of the Proposition has been given in [14], (see Proposition 6.2.) Now we can give a final convergence rate result.

Theorem 15 *Let the conditions I-III hold. Then the parameters $\alpha_0 > \bar{\alpha}$ and $2/3 < \bar{q} \leq q < 1$ in the fixed point regularization algorithm can be chosen s.t. the algorithm terminates within a finite number of outer iterations. The last iterate, $x^{\delta}_{\alpha_{k_*}}$, fulfills the estimates*

$$\delta \leq \|y^\delta - F(x^{\delta}_{\alpha_{k_*}})\| \ \leq \ 5\delta \ , \tag{5.4}$$

$$\|x_* - x^{\delta}_{\alpha_{k_*}}\| \ = \ \mathcal{O}(\sqrt{\delta}) \ . \tag{5.5}$$

Proof:
According to Propositions 13 and 14, α_0 and q can be chosen with $x_0 \in K_{r(\alpha_0)}(x^\delta_{\alpha_0})$ and $x^{\delta}_{\alpha_{k-1}} \in K_{r(\alpha_k)}(x^{\delta}_{\alpha_k})$. As $\alpha_0 > \bar{\alpha}$ holds, at least the first fixed point iteration is carried out by using the operator S_α (which does not need another minimizer). Because of $x_0 \in K_{r(\alpha_0)}(x^\delta_{\alpha_0})$, the iteration converges towards $x^\delta_{\alpha_0}$. Due to the choice of q we have $x^\delta_{\alpha_0} \in K_{r(\alpha_1)}(x^\delta_{\alpha_1})$. By induction, we find that the fixed point iteration with operator $R_{\alpha_k}(\cdot, x^{\delta}_{\alpha_{k-1}})$ converges towards $x^{\delta}_{\alpha_k}$, with $x^{\delta}_{\alpha_k} \in K_{r(\alpha_{k+1})}(x^\delta_{\alpha_{k+1}}$. In [14], Proposition 6.4. and Theorem 6.5. it was shown that the outer iteration terminates after a finite number of iteration steps as long as the inner iteration finds a minimizer of the Tikhonov functional with the actual regularization parameter, and that (5.4) holds if q is properly chosen. Now, it is a well known fact that a minimizer $x^{\delta}_{\alpha_{k_*}}$ of the Tikhonov functional that fulfills (5.5) and a smoothness condition III also admits an error estimate (5.5), see e.g. [11].

□

We wish to remark that the choice of the parameters α_0 and q depends on ϱ, the estimate for $\|\omega\|$. Although it is possible to determine both parameters exactly (in dependence of ϱ), we have omitted these calculations because ϱ will be unknown in many practical applications. In these cases, the algorithm should be carried out with q close to 1 and a large α_0.

Let us finish this section with a remark on the numerical realization of the fixed point iteration with T_α. According to (4.4), we have to evaluate $B(x_{\tilde{\alpha}}^{\delta} - x)^2$ for the computation of the iterates. Looking at (4.3), this requires the evaluation of an integral over an operator and is thus difficult to implement. However, using (4.5), we get

$$B(x_{\tilde{\alpha}}^{\delta} - x)^2 = F(x_{\tilde{\alpha}}^{\delta}) - F(x) - F'(x)(x_{\tilde{\alpha}}^{\delta} - x) \ ,$$

and inserting this equation in (4.4) we get

$$T_\alpha(x) = F_\alpha(x)^{-1} F'(x)^*(y^\delta - F(x) - F'(x)(x_{\tilde{\alpha}}^{\delta} - x)) - qF_\alpha(x)^{-1} F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta})) + x_{\tilde{\alpha}}^{\delta} \ .$$

Moreover, as $x_{\tilde{\alpha}}^{\delta}$ is a minimizer of $\Phi_{\tilde{\alpha}}(x)$, we have $\tilde{\alpha}(x_{\tilde{\alpha}}^{\delta} - \bar{x}) = F'(x_{\tilde{\alpha}}^{\delta})^*(y^\delta - F(x_{\tilde{\alpha}}^{\delta}))$, and with $\alpha = q\tilde{\alpha}$ we finally get

$$T_\alpha(x) = F_\alpha(x)^{-1} F'(x)^*(y^\delta - F(x) - F'(x)(x_{\tilde{\alpha}}^{\delta} - x)) - \alpha F_\alpha(x)^{-1}(x_{\tilde{\alpha}}^{\delta} - \bar{x}) + x_{\tilde{\alpha}}^{\delta} \ . \tag{5.6}$$

which is much easier to implement.

6 Numerical results

We will present first numerical results from Single Photon Emission Computed Tomography (SPECT). In this medical application, the data $g(s,\omega) \in L_2(\mathbb{R} \times S^1)$ is described by the attenuated Radon transform $R(f,\mu)$,

$$g(s,\omega) = R(f,\mu)(s,\omega) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) e^{-\int_t^\infty \mu(s\omega^\perp + \tau\omega)\, d\tau}\, dt \ , \tag{6.1}$$

with the two unknown functions $(f,\mu) \in L_2(\Omega) \times L_2(\Omega)$ and a bounded domain $\Omega \subset \mathbb{R}^2$. Fréchet differentiability and mapping properties of the attenuated Radon transform have been investigated extensively in [3], and numerical results for the TIGRA algorithm both with synthetic and real data have been presented in [13, 14]. It turns out that, due to the non-uniqueness of the attenuated Radon Transform as operator acting on (f,μ), only the activity function f can be reconstructed accurately. However, as the task in SPECT imaging is the reconstruction of the activity function f from measurements $g(s,\omega)$ and unknown μ, a wrong reconstruction for the density function causes no problems.

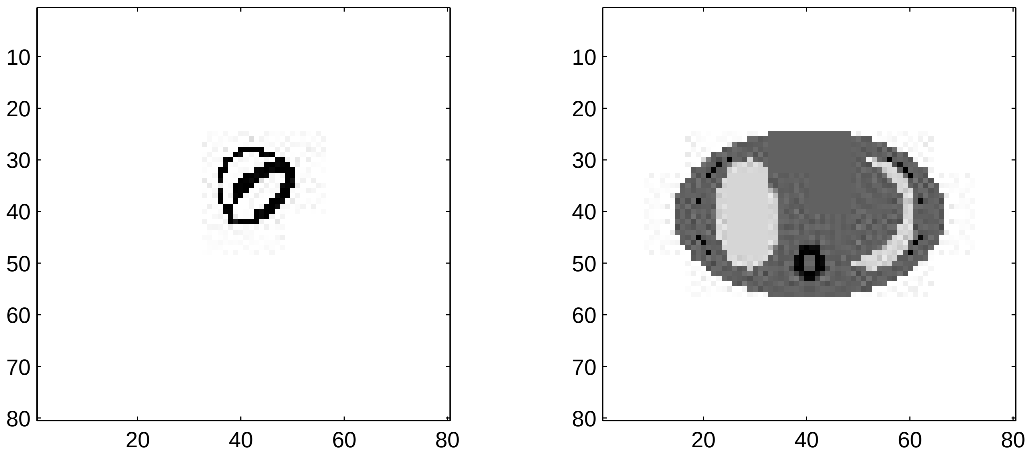

Figure 1: Activity function f_* (left) and attenuation function μ_* (right)

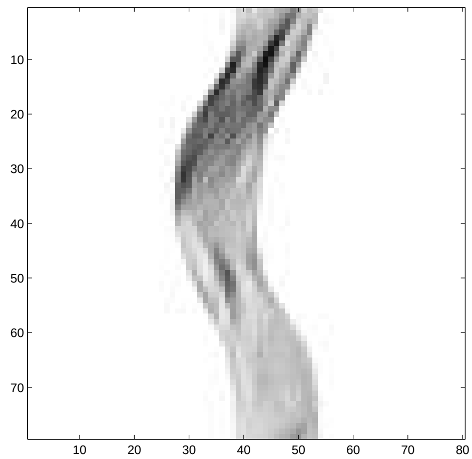

Figure 2: Generated data $g(s, \omega) = R(f_*, \mu_*)(s, \omega)$.

6.1 A feasibility test with synthetic data

For the first test we will use the so called MCAT phantom [16], that models a cut through the human torso for the density function μ_*. The activity f_* is concentrated in the heart, see Figure 1. The data $g(s, \omega) = R(f_*, \mu_*)(s, \omega)$ is shown in Figure 2; for the inversion it was contaminated with 5% noise multiplicative gaussian noise, i.e. $\|g^\delta - g\|/\|g\| \approx 0.05$. We wish to remark that the raw SPECT data is more likely to be affected with Poisson noise, and thus the chosen noise model is not very realistic. Therefore we will present a reconstruction from measured data within the next section. However, deterministic regularization methods depend not heavily on the noise model, as they only use information on the error bound $\|g^\delta - g\|$.

We will here focus on the reconstruction of the minimizers of the Tikhonov functional with given regularization by the fixed point iterations with operators S_α and T_α. As we have seen in Section 3, a fixed point iteration with operator S_α will only converge if the regularization parameter α is chosen large enough. In our example, it turns out that the fixed point iteration $x_{k+1} = S_\alpha(x_k)$, $x = (f, \mu)$ converges for $\alpha \geq \alpha_0 = 7000$. If $\alpha_1 = q\alpha_0$ with $q = 0.7$, i.e.

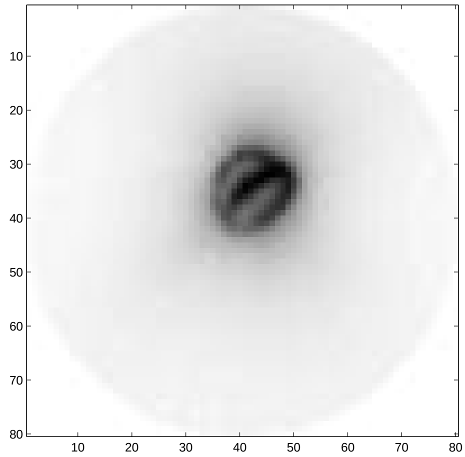

Figure 3: f_{α}^{δ} for $\alpha = 7000$ and $\delta = 5\%$.

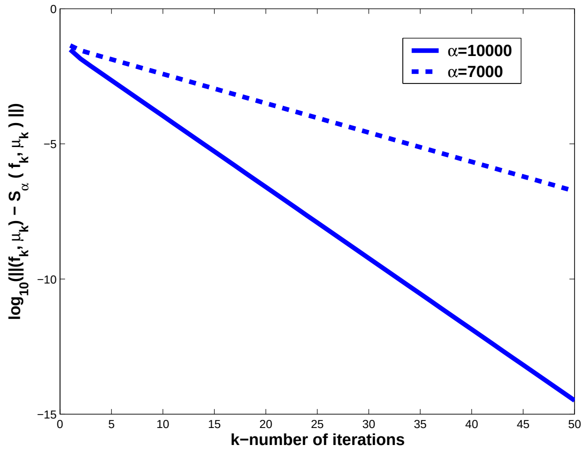

Figure 4: Logarithmic plot of the iteration error.

$\alpha_1 = 4900$, then the iteration with S_α does not converge anymore, and we have to switch to the fixed point iteration with operator T_α.

Let us start with some reconstructions for $\alpha \in \{20408,\ 10000,\ 7000\}$. As mentioned above, the fixed point iteration with operator S_α can be used for the reconstruction of the minimizers $(f_{\alpha}^{\delta}, \mu_{\alpha}^{\delta})$ of the belonging Tikhonov functional. In Figure 3, the minimizer f_{α}^{δ} of the Tikhonov functional for $\alpha = 7000$ can be seen.

In Figure 4 we have plotted the speed of the convergence of the iteration towards the fixed point of S_α, i.e. the values of $\|(f_k, \mu_k) - S_\alpha(f_k, \mu_k)\|$ for $\alpha \in \{10000, 7000\}$ in a logarithmic plot. We do observe that the convergence is faster for bigger α (and it is again faster for $\alpha = 20408$). The reason for this observation lies in the fact that the speed of convergence of a fixed point iteration depends on the size of the contraction factor for the fixed point operator. For S_α, we have found that the contraction factor can be estimated by c/α with a constant with the constant c given in (3.4). Thus, if α decreases, the contraction factor is getting bigger and the convergence speed decreases, too.

α	p_α	αp_α
20 408	0.2654	5416.3
10 000	0.5436	5436
7 000	0.7766	5436.3

Table 1: Contraction factors p_α for different α

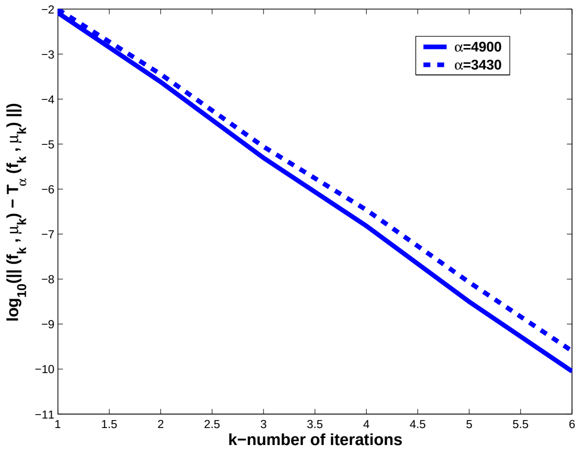

Figure 5: Logarithmic plot of the iteration error for T_α. The iteration error for $\alpha = 2401$ is almost the same as for $\alpha = 3430$.

Additionally we estimated the contraction factor p from the numerical results. From the classical error estimates for the fixed point iteration follows

$$\|(f_k, \mu_k) - (f_{k+1}, \mu_{k+1})\| \leq p\|(f_{k-1}, \mu_{k-1}) - (f_k, \mu_k)\| \ ,$$

i.e. we might estimate p by

$$p \approx \frac{1}{N} \sum_{k=1}^{N} \frac{\|(f_k, \mu_k) - (f_{k+1}, \mu_{k+1})\|}{\|(f_{k-1}, \mu_{k-1}) - (f_k, \mu_k)\|} \ . \tag{6.2}$$

On the other hand, as we have estimated $p \leq c/\alpha$, we do expect $p\alpha$ to be constant, as can be seen in Table 1 .

For $\alpha = 4900$, S_α is not a contraction anymore, and T_α is used to compute the minimizer. According to (5.6), we need a minimizer of the Tikhonov functional with bigger regularization parameter for the evaluation of the operator. To this end, we employed $(f_\alpha^\delta, \mu_\alpha^\delta)$ for $\alpha = 7000$, which has been already computed by the first fixed point iteration. The evaluation of T_α is much more complicated as it is the case for S_α, because a linear system has to be solved in every iteration step.

In our implementation, the conjugate gradient method for the solution of the linear system was used; in all our tests 2-8 cg-iterations were sufficient to compute a good approximation to the solution of the linear system. As the linear system was solved with an accuracy of 10^{-9}, we expect the fixed point equation $(f, \mu) = T_\alpha(f, \mu)$ to be approximated within the same level. Indeed, the numerical tests for $\alpha \in \{4900, 3430, 2401\}$ show that this accuracy level

α	p_α
4900	0.045
3430	0.0656
2401	0.0656
1680	0.0394

Table 2: Contraction factors p_α for T_α and different α

is approached within a few fixed point iteration steps, see Figure 5. The rapid convergence suggests a small contraction factor. Indeed, if we estimate the contraction factor as in (6.2), we get the factors as shown in Table 2. The numerical tests confirm that our method is highly recommendable for the reconstruction of a minimizer of the Tikhonov functional and therefore for the approximation of a solution of $F(x) = y$. The tests suggest in particular that fixed point iteration with operator T_α converges rapidly. However, we have to take into account that the solution of the linear equation, which is necessary for the evaluation of T_α, consumes additional computing time. A detailed comparison of the numerical effort of the suggested fixed point methods as well as a comparison with other algorithms like TIGRA will be presented in a forthcoming article.

6.2 A reconstruction with real data

In this section a reconstruction from measured SPECT data by using the fixed point regularization algorithm will be presented. The measurements were taken from the Jaszczak torso phantom at the Medical Imaging Research Laboratory at the University of Utah, Salt Lake City, Utah, USA. The Jaszczak phantom models a human torso, and has inserts for organs like liver and lung. A cardiac insert, filled with approximately 4 μCi/ml of Tc-99m, was used to model the activity function f.

The phantom was imaged on a Picker P2000 SPECT camera which gathered emission measurements for 120 angles over 360 degrees, with a projection matrix of size 128×128. One slice of the scan data was extracted to provide an emission sinogram $g^\delta(s, \omega)$, see Figure 6. Clearly, the emission sinogram is heavily contaminated with noise. The absolute value of the noise level, δ, is in principle unknown, but the relative error in the data is usually estimated as $10\% - 15\%$. Based on this estimate, we have computed the absolute noise level as $\delta \approx 94$. We might remark that the choice of the error level is very delicate, as both a too small or too big δ causes a bad reconstruction.

The task was to reconstruct a U-shaped object from the emission scan. Figure 7 shows the reconstructed image, it has a size of 128×128 pixels. A visual inspection of the image indicates a similar reconstruction quality as for the standard approach, where additionally a transmission scan is used to invert SPECT data, cf. the reconstructions with the same data in [15, 13]. The fixed point regularization algorithm terminated after 12 outer iterations, meaning that the regularization parameter α was twelve times updated. Again, we have used $q = 0.7$. The starting regularization parameter was chosen as $\alpha_0 = 1E+7$. Figure 7 shows a plot of the residual $\|g^\delta - R(f^\delta_{\alpha_{k,l}}, \mu^\delta_{\alpha_{k,l}})\|$ for all inner and outer iterations. In the beginning, the residual stays well above the termination level and, for the last two regularization parameters, drops

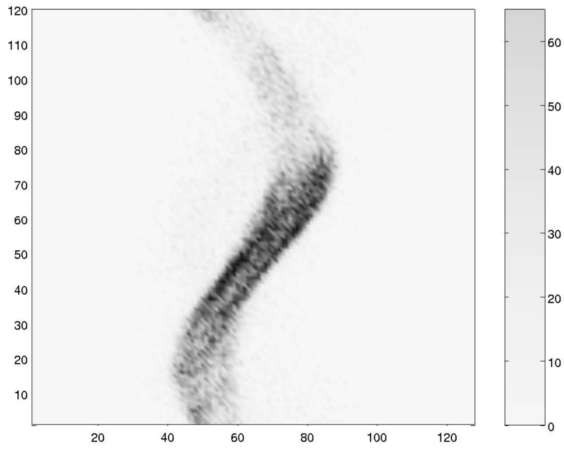

Figure 6: Emission data $g(s, \omega)$

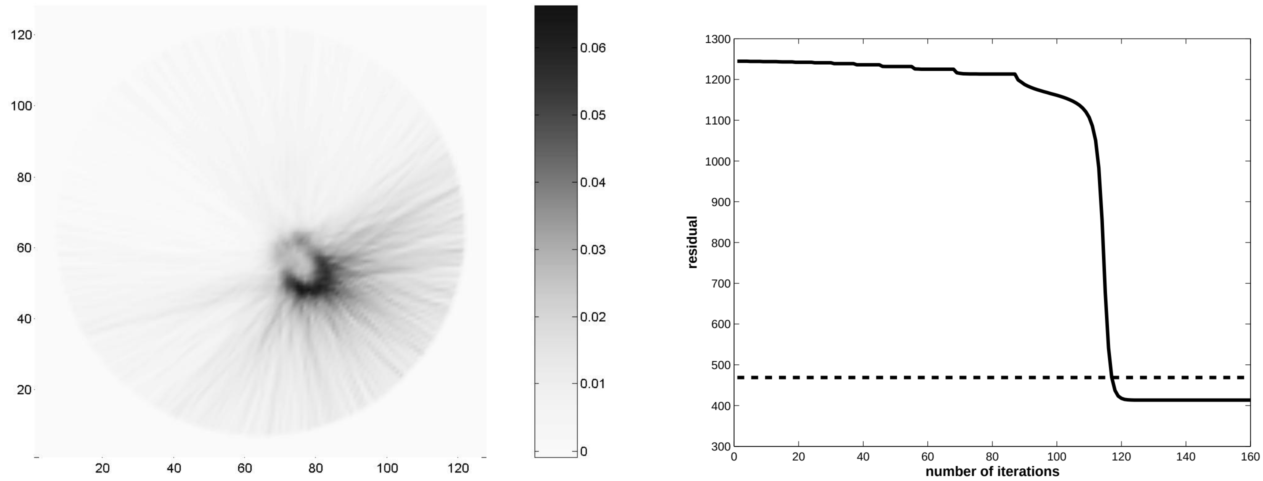

Figure 7: Reconstruction of a slice of the heart insert (left) and plot of the residual $\|g^\delta - R(f^\delta_{\alpha_k,l}, \mu^\delta_{\alpha_k,l})\|$ (right)

fast towards it.

References

[1] A. W. Bakushinskii. The problem of the convergence of the iteratively regularized Gauss–Newton method. *Comput. Maths. Math. Phys.*, (32):1353–1359, 1992.

[2] B. Blaschke, A. Neubauer, and O. Scherzer. On convergence rates for the iteratively regularized Gauss–Newton method. *IMA Journal of Numerical Analysis*, (17):421–436, 1997.

[3] V. Dicken. A new approach towards simultaneous activity and attenuation reconstruction in emission tomography. *Inverse Problems*, 15(4):931–960, 1999.

[4] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.

[5] H.W. Engl, K. Kunisch, and A. Neubauer. Convergence rates for Tikhonov regularization of nonlinear ill-posed problems. *Inverse Problems*, (5):523–540, 1989.

[6] M. Hanke. A regularizing Levenberg–Marquardt scheme, with applications to inverse groundwater filtration problems. *Inverse Problems*, (13):79–95, 1997.

[7] M. Hanke. Regularizing properties of a truncated Newton–cg algorithm for nonlinear ill-posed problems. *Num. Funct. Anal. Optim.*, (18):971–993, 1997.

[8] M. Hanke, A. Neubauer, and O. Scherzer. A convergence analysis of the Landweber iteration for nonlinear ill-posed problems. *Numerische Mathematik*, (72):21–37, 1995.

[9] B. Kaltenbacher. Some Newton–type methods for the regularization of nonlinear ill–posed problems. *Inverse Problems*, (13):729–753, 1997.

[10] R. Ramlau. A modified Landweber–method for inverse problems. *Numerical Functional Analysis and Optimization*, 20(1& 2), 1999.

[11] R. Ramlau. Morozov's discrepancy principle for Tikhonov regularization of nonlinear operators. *Numer. Funct. Anal. and Optimiz*, 23(1&2):147–172, 2002.

[12] R. Ramlau. A steepest descent algorithm for the global minimization of the Tikhonov–functional. *Inverse Problems*, 18(2):381–405, 2002.

[13] R. Ramlau. Regularization of nonlinear ill–posed operator equations: Methods and applications. Habilitationsschrift, Universität Bremen, 2003.

[14] R. Ramlau. TIGRA–an iterative algorithm for regularizing nonlinear ill–posed problems. *Inverse Problems*, 19(2):433–467, 2003.

[15] R. Ramlau, R. Clackdoyle, F. Noo, and G. Bal. Accurate attenuation correction in SPECT imaging using optimization of bilinear functions and assuming an unknown spatially–varying attenuation distribution. *Z. Angew. Math. Mech.*, 80(9):613–621, 2000.

[16] J. A. Terry, B. M. W. Tsui, J. R. Perry, J. L. Hendricks, and G. T. Gullberg. The design of a mathematical phantom of the upper human torso for use in 3-d spect imaging research. In *Proc. 1990 Fall Meeting Biomed. Eng. Soc. (Blacksburg, VA)*, pages 1467–74. New York University Press, 1990.

[17] A.N. Tikhonov and V.Y. Arsenin. *Solutions of ill posed problems*. Winston Wiley, New York, 1977.

[18] A.N. Tikhonov, A.S. Leonov, and A.G. Yagola. *Nonlinear Ill-posed Problems*. Chapman & Hall, London, 1998.

[19] E. Zeidler. *Nonlinear Functional Analysis and its Applications*. Springer, New York, 1985.