

# Lecture 9

## Dynamical Systems and Bifurcations

James Lu

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- ▶ Lack of complete mathematical description of model and behavior
  - ▶ biological data is often only qualitative in nature

# Dynamical Systems and Bifurcations

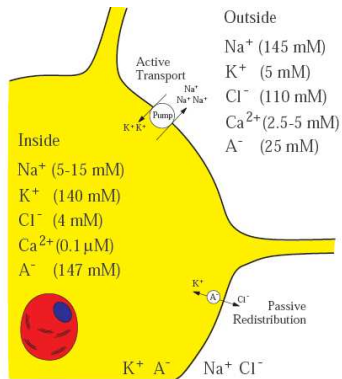
- ▶ Due to their high dimensionality and nonlinearity, biological models cannot be understood by intuition alone
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# Dynamical Systems and Bifurcations

- ▶ Due to their high dimensionality and nonlinearity, biological models cannot be understood by intuition alone
- ▶ Lack of complete mathematical description of model and behavior
  - ▶ biological data is often only qualitative in nature
  - ▶ precise estimates of parameter values in the model are often unavailable
  
- ▶ Can we understand the dynamics of biological models using the theory of bifurcations?

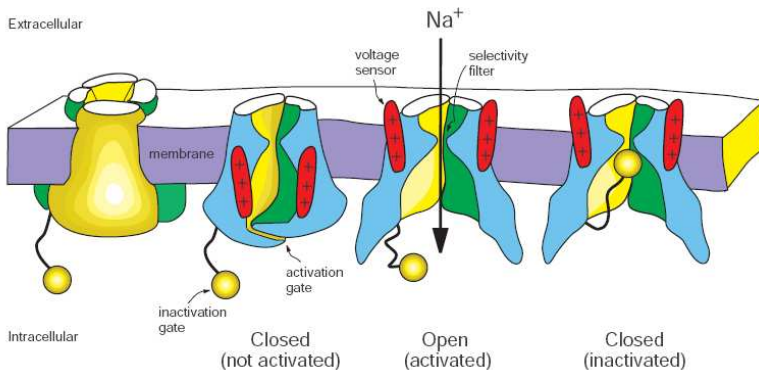
# Dynamical Systems and Bifurcations

- ▶ Neuron models: movement of ions across membrane



# Dynamical Systems and Bifurcations

- ▶ Neuron models: movement of ions via voltage-gated channels



# Dynamical Systems and Bifurcations

- ▶ Consider the following 2-dimensional model for the voltage  $V$  and gating variable  $n$ , involving  $Na^+$  and  $K^+$  currents:

$$\begin{aligned}C\dot{V} &= I - \overbrace{g_L(V - E_L)}^{\text{leak } I_L} - \overbrace{g_{Na} m_\infty(V) (V - E_{Na})}^{\text{instantaneous } I_{Na,p}} - \overbrace{g_K n (V - E_K)}^{I_K} \\ \dot{n} &= (n_\infty(V) - n) / \tau(V),\end{aligned}$$

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- ▶ There are many different physiological mechanisms for bringing the neuron from the resting to spiking states

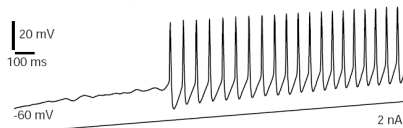
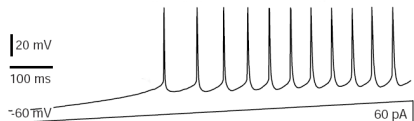
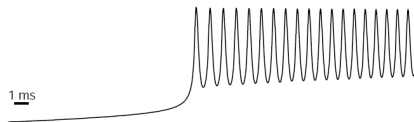
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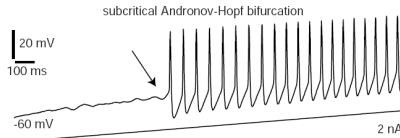
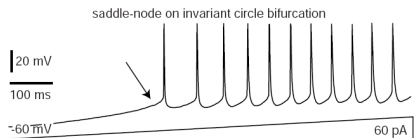
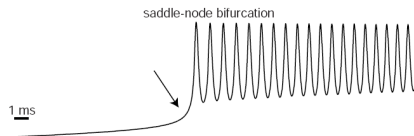
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- ▶ There are many different physiological mechanisms for bringing the neuron from the resting to spiking states
- ▶ However, there are only 4 types of bifurcations the system can undergo

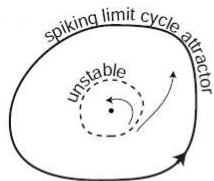
# Dynamical Systems and Bifurcations



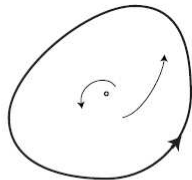
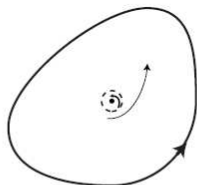
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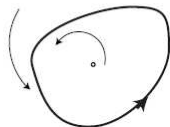
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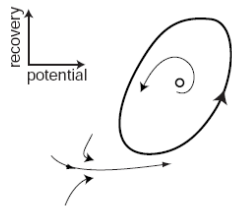
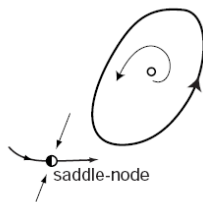
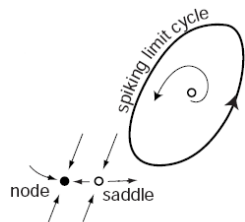
subcritical Andronov-Hopf bifurcation



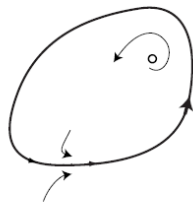
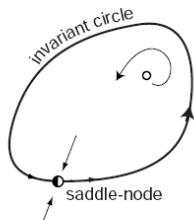
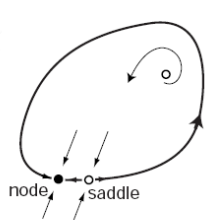
supercritical Andronov-Hopf bifurcation



# Dynamical Systems and Bifurcations



saddle-node bifurcation



saddle-node on invariant circle (SNIC) bifurcation

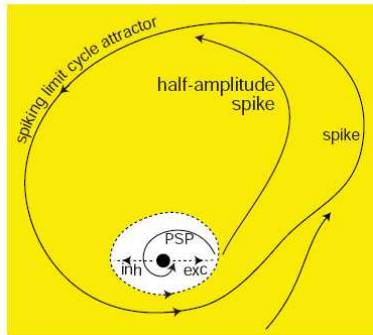


- ▶ Neurons can be classified as *resonators* or *integrators* according to the bifurcation type

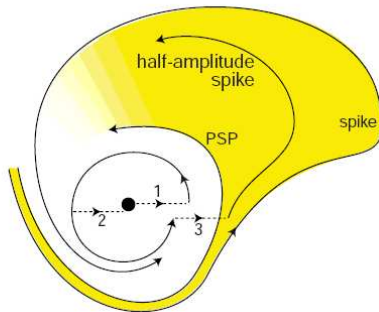


# Dynamical Systems and Bifurcations

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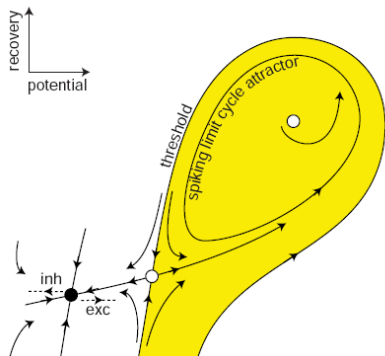


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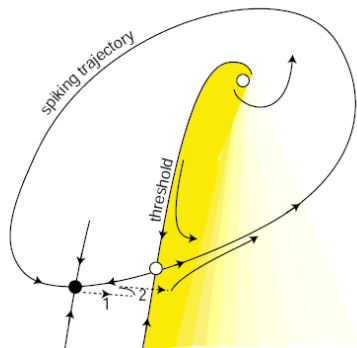


# Dynamical Systems and Bifurcations

saddle-node bifurcation



saddle-node on invariant circle bifurcation



# Dynamical Systems and Bifurcations

- ▶ Qualitative dynamics of system near are captured by **co-dimension 1** bifurcation points in the phase space
  - ▶ found by varying parameters along a direction: point on a line

# Dynamical Systems and Bifurcations

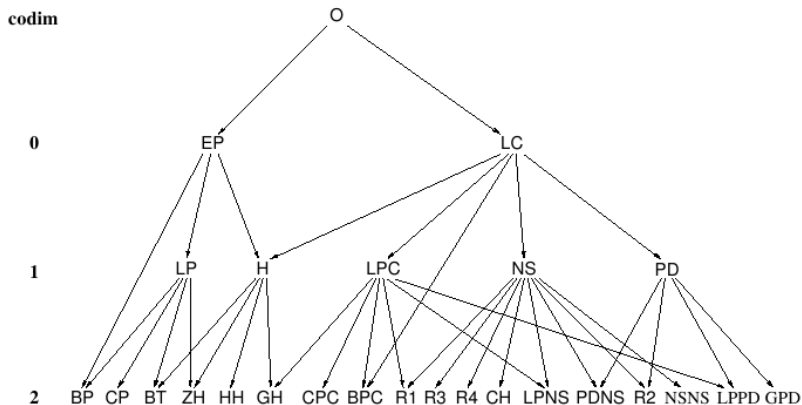
- ▶ Qualitative dynamics of system near are captured by **co-dimension 1** bifurcation points in the phase space
  - ▶ found by varying parameters along a direction: point on a line
- ▶ What characterizes the transition between different types of behaviors?
  - ▶ e.g., a neuron switching from being an resonator to an integrator

# Dynamical Systems and Bifurcations

- ▶ Qualitative dynamics of system near are captured by **co-dimension 1** bifurcation points in the phase space
  - ▶ found by varying parameters along a direction: point on a line
- ▶ What characterizes the transition between different types of behaviors?
  - ▶ e.g., a neuron switching from being an resonator to an integrator
- ▶ These transitions occur at the **co-dimension 2** bifurcations
  - ▶ e.g., the transition from being a resonator to an integrator occurs at the Bogdanov-Takens (BT) point
  - ▶ found by varying parameters in a 2-dimensional space: point on a plane

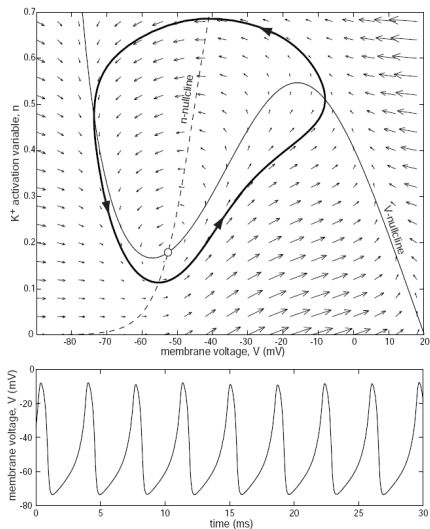
# Dynamical Systems and Bifurcations

## ► Tree of bifurcations



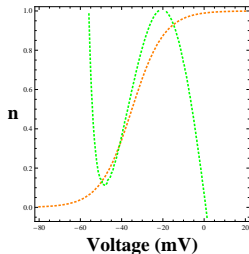
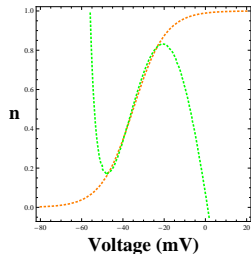
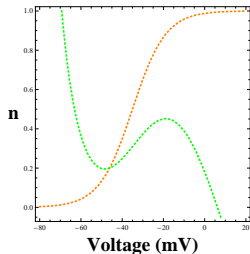
# Dynamical Systems and Bifurcations

## ► Null-clines for oscillations:



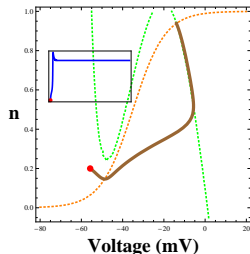
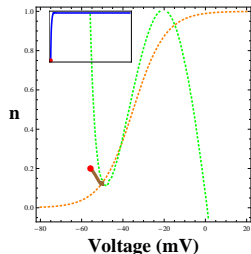
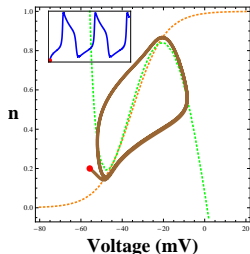
# Dynamical Systems and Bifurcations

- ▶ Nullclines of neuron system
  - ▶ oscillations → **Bogdanov-Takens** → bistability



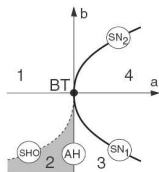
# Dynamical Systems and Bifurcations

- ▶ From the BT point, perturbations give the following solutions:



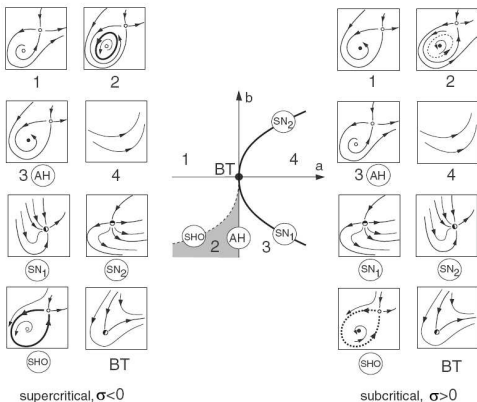
# Dynamical Systems and Bifurcations

- ▶ Bifurcation theory shows that Bogdanov-Takens bifurcation is an *organizing center*:
  - ▶ gives rise to: saddle-node, Hopf and saddle-homoclinic orbit bifurcations



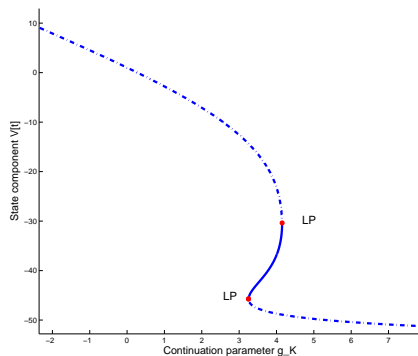
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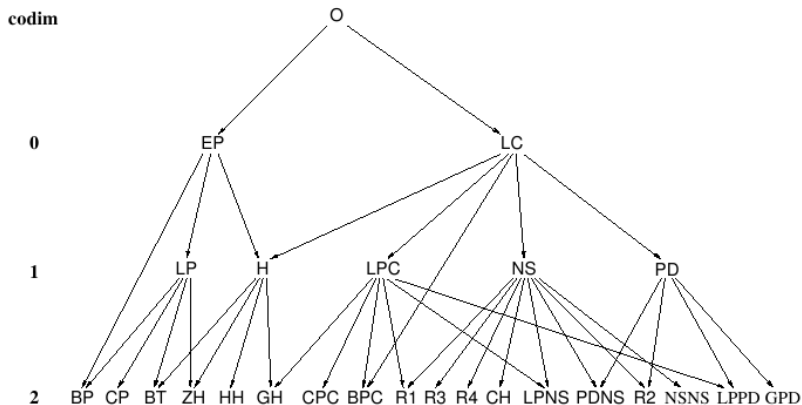
# Dynamical Systems and Bifurcations

- ▶ Changes in the steady-state solution as the maximal conductance  $g_K$  is varied



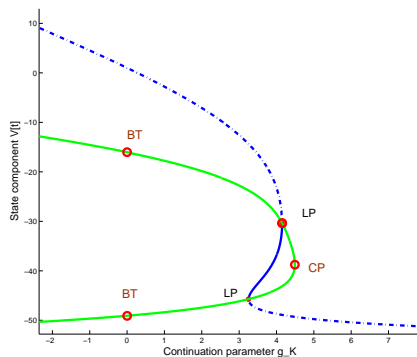
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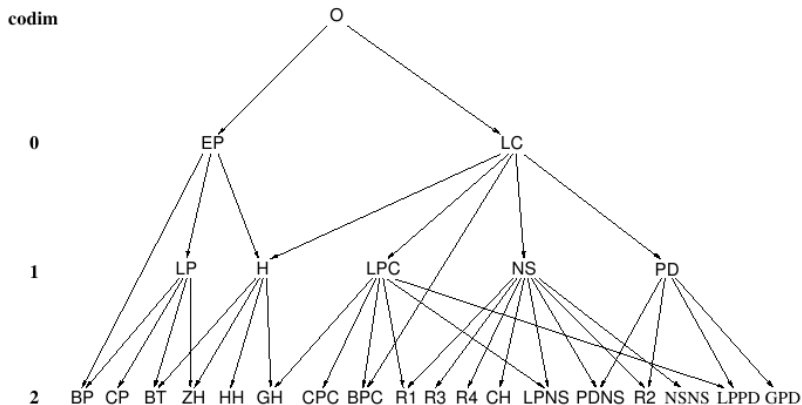
# Dynamical Systems and Bifurcations

- ▶ Steady state solutions when both  $g_K$  and  $E_L$  are varied



# Dynamical Systems and Bifurcations

## ► Tree of bifurcations



# Dynamical Systems and Bifurcations

- ▶ From BT point, a line of Hopf points can be extracted

