Mathematical Modelling and Scientific Computing in the Biosciences

12 June 2007

Lecture 9: Overview

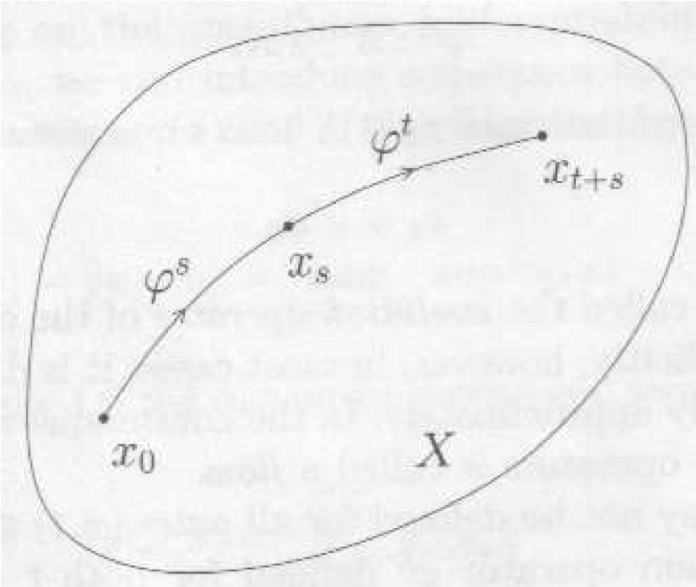
- **Dynamical Systems**
  - **topological equivalence, normal forms**
  - **bifurcations**
    saddle−node
    supercritical/subcritical Hopf

Dynamical Systems: Definition

Definition: a *dynamical system* consists of a space of possible states (i.e., state space X) and a law of evolution that describes how the state changes in time

• The evolution operator φ^t: X → X shows how an initial state $x(0) \in X$ is transformed to the state at time t:

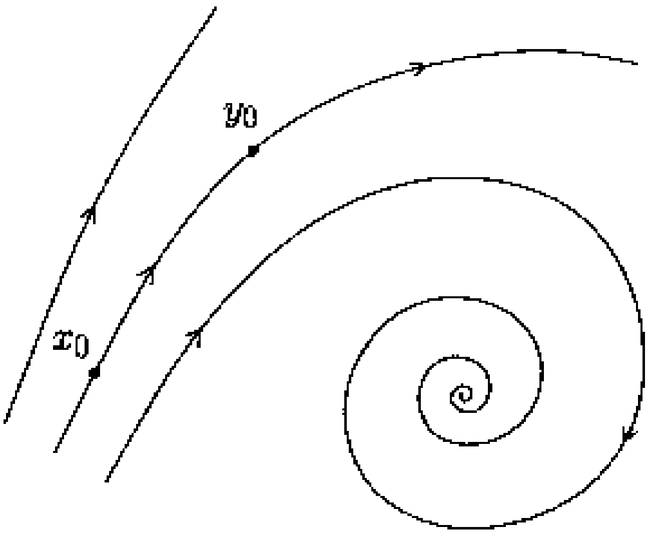
$$x_t = \varphi^t x_0$$



Dynamical Systems: Orbits

Definition: an *orbit* starting at x_0 is an ordered subset of the state space X,

$$Or(x_0)=\{x \in X:\; x = \varphi^t\, x_0,\ \text{for all } t \in T\}$$



Dynamical Systems: Types of Orbits

Definition: an *equilibrium* (or *fixed point*) is a point in the state space such that $x_0 = \varphi^t\, x_0$ for all $t \in T$

Definition: an *cycle* is a periodic orbit, namely a nonequilibrium orbit (L_0) such that each point $x_0 \in L_0$ satisfies $\varphi^{t+T_0}\, x_0 = \varphi^t\, x_0$ for some (period) $T_0 > 0$ and all $t \in T$

• We study dynamical systems defined by ordinary differential equations:

$$\dot{x} = f(x)$$

• Although the dynamical behaviors can be analyzed by running numerical simulations, one can use bifurcation theory to predict qualitative features of phase portraits without resorting to first solve the ODE system

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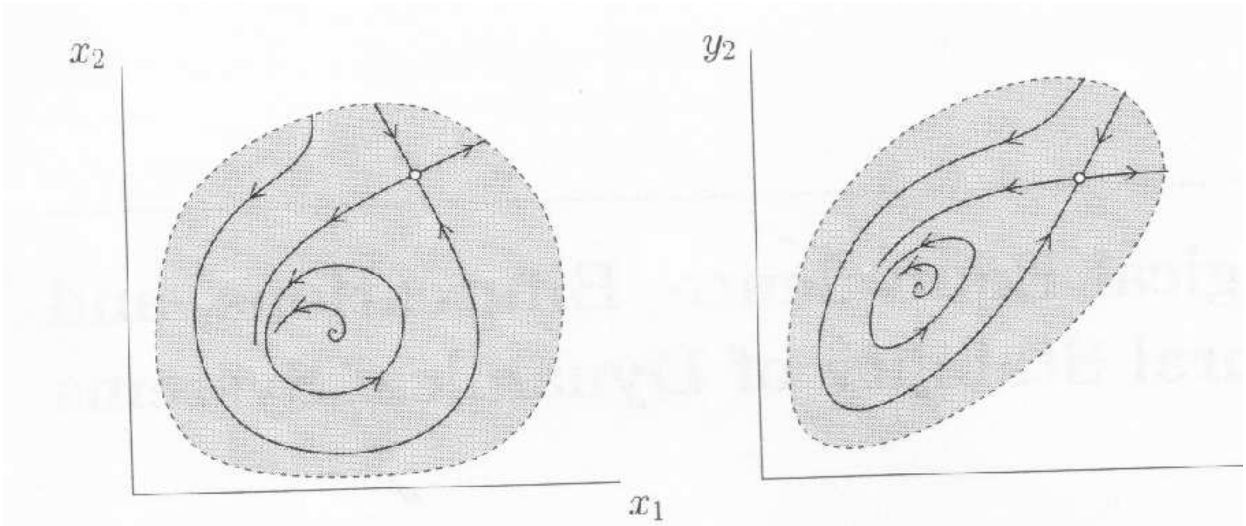
Dynamical Systems: Topological Equivalence

• Goal: study the qualitative features of dynamical systems

• Define equivalence classes of dynamical behaviors, and study transitions between these different classes

• Example of equivalent systems: same number of equilibria, each of which has the same stability type

• We consider 2 dynamical systems as being equivalent if phase portrait of one may be continuously transformed to another:



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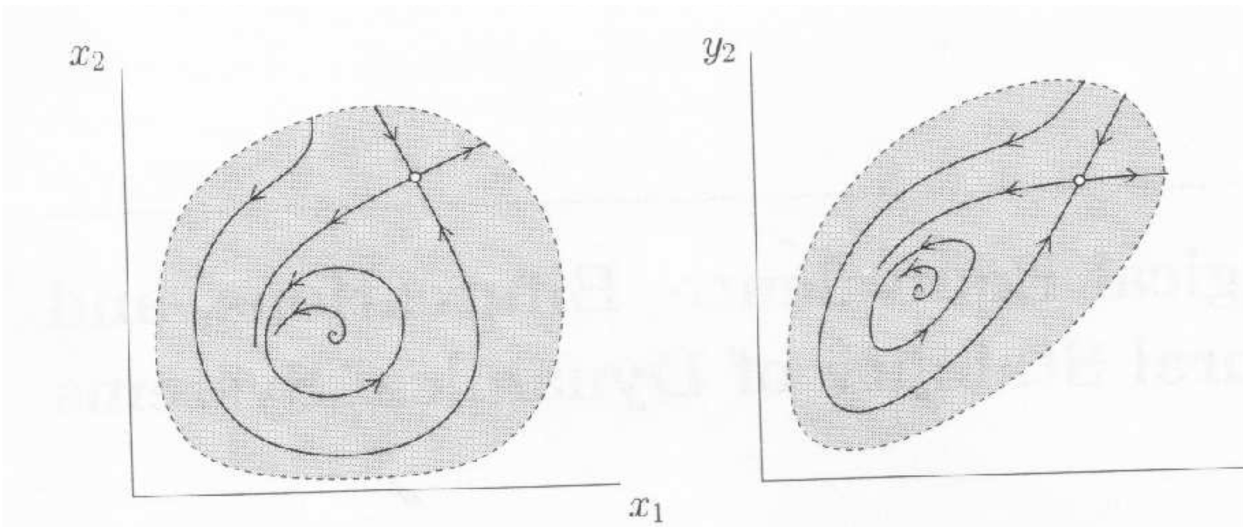
Dynamical Systems: Topological Equivalence

Definition: a dynamical system is *topologically equivalent* to another dynamical system if there is a homeomorphism *h*, mapping orbits of the first system to orbits of the second



• Homeomorphism: invertible map, both the map and its inverse are continuous

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Dynamical Systems: Topological Equivalence

• Example: consider 2 topologically−equivalent dynamical systems,

$$\dot{x} = f(x),$$
$$\dot{y} = g(y)$$

• Suppose one can find an invertible, smooth map $y = h(x)$ such that $f(x) = M^{-1}(x)\, g(\,h(x)\,)$, where $M(x) = \frac{d\,h(x)}{\mathrm{dx}}$ is the Jacobian matrix, then $h(x)$ is a homeomorphism mapping x into y

In fact, if the flows corresponding to $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are denoted as φ^t and ψ^t, respectively, then $h(\varphi^t\, x) = \psi^t\, h(x)$

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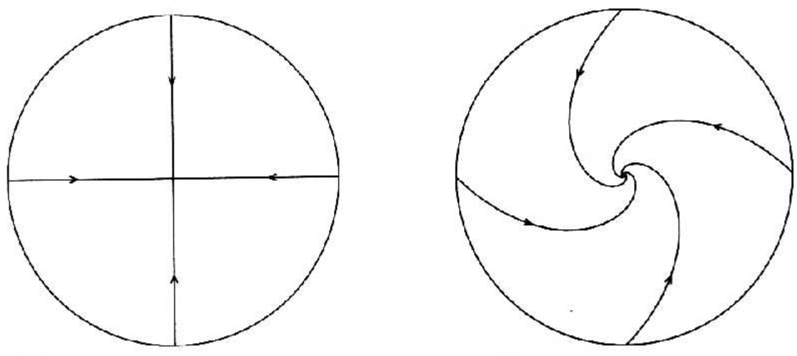
Dynamical Systems: Topological Equivalence

• Consider the following 2 dynamical systems,

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \end{cases} \qquad \begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$$

• In polar coordinates (ϱ, θ),

$$\begin{cases} \dot{\varrho} = -\varrho \\ \dot{\theta} = 0 \end{cases} \qquad \begin{cases} \dot{\varrho} = -\varrho \\ \dot{\theta} = 1 \end{cases}$$



• Left: node; perturbations decay monotonously, $\lambda_1 = \lambda_2 = -1$
Right: focus; perturbations decay oscillatorily, $\lambda_1 = \lambda_2 = -1 \pm i$

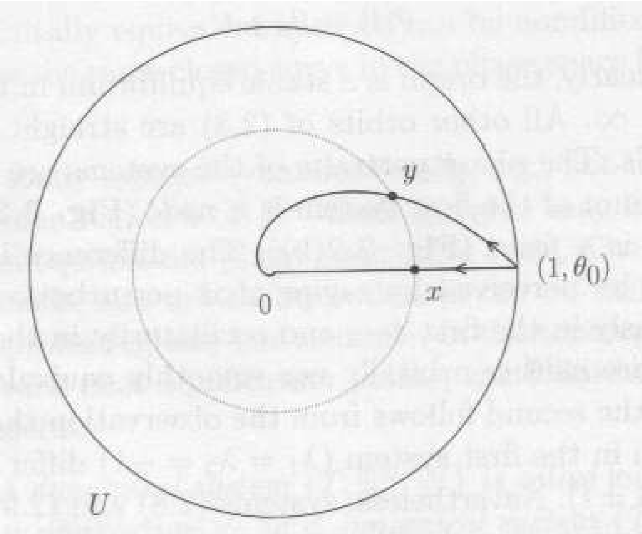
Dynamical Systems: Topological Equivalence

• Despite the difference in eigenvalues, the systems are in fact topologically equivalent

• Consider unit disc U={(ϱ, θ): $\varrho \leq 1$}, we construct a homeomorphism explicitly:

$$h: \begin{cases} \varrho = \varrho \\ \theta = \theta - \log(\varrho) \end{cases}$$

That is, h rotates each circle ϱ=constant by a ϱ−dependent angle:



• The example illustrates how topological equivalence perserves information on the number, stability and topology of invariant sets, while losing information on time−dependent behavior

Dynamical Systems: Topological Equivalence

• We now study geometry of phase portrait near generic equilibria according to their topological classification

Definition: consider a dynamical system $\dot{x} = f(x)$ and let x_0 be an equilibrium. Let n_+ , n_-, n_0 represent the number eigenvalues with positive, negative and zero real parts respectively. x_0 is called a *hyperbolic equilibrium* if there are no eigenvalues on the imaginary axis, i.e., $n_0 = 0$

Theorem: the phase portraits near 2 hyperbolic equilibria x_0 and y_0 are locally topologically equivalent if and only if they both have the same number n_+ , n_- of eigenvalues with positive and negative real parts

• Idea behind the proof:

near a hyperbolic equilibrium, the system is locally topologically equivalent to its linearization

$\dot{\xi} = A\,\xi$ (by the Grobman−Hartman theorem)

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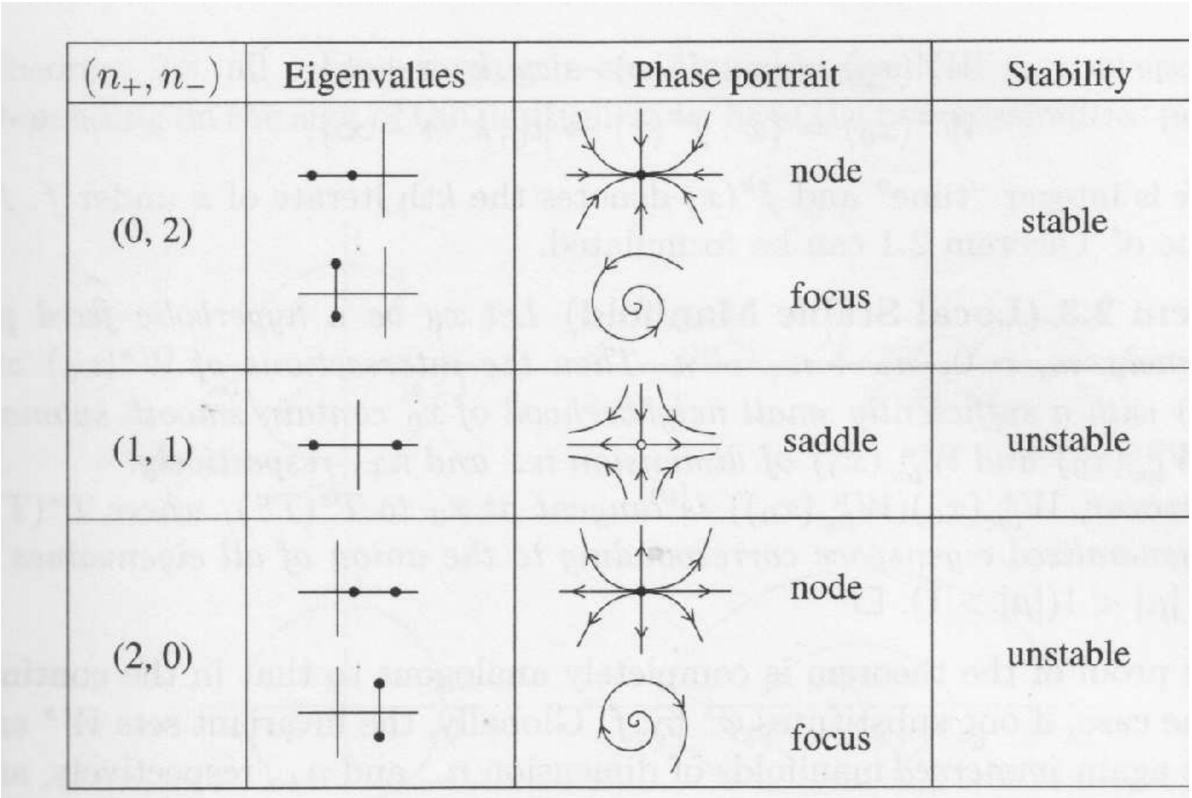
Dynamical Systems: Topological Equivalence

• Topological classification of hyperbolic equilibria for 2−dimensional, planar systems:



(n_+, n_-)	Eigenvalues	Phase portrait	Stability
(0, 2)		node	stable
		focus	
(1, 1)		saddle	unstable
(2, 0)		node	unstable
		focus	

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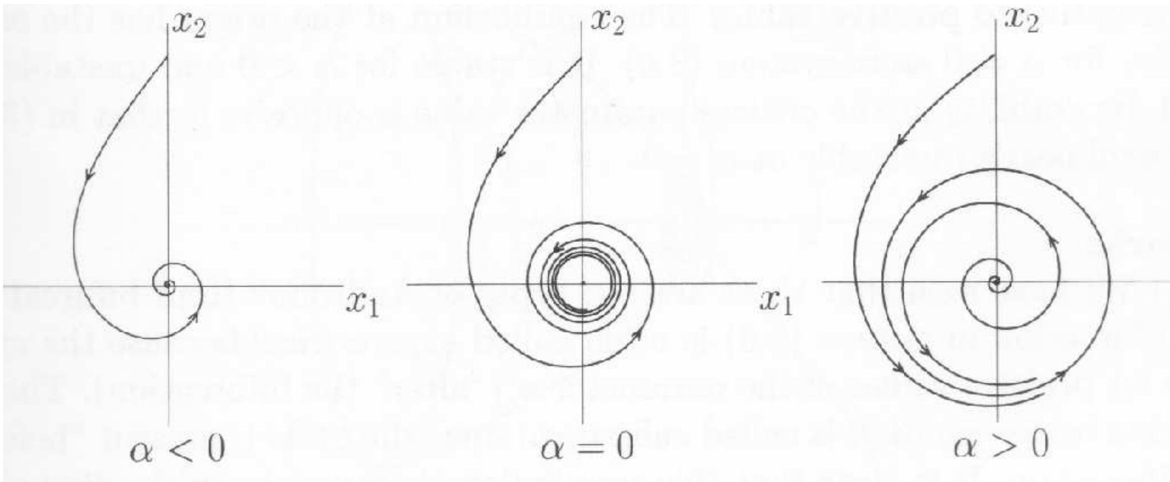
Dynamical Systems: Bifurcations

> Definition: *bifurcation* is the appearance of a topologically nonequivalent phase portrait under variation of parameters

• Consider the following dynamical system,

$$\begin{cases} \dot{\varrho} = \varrho(\alpha - \varrho^2) \\ \dot{\theta} = 1 \end{cases}$$

- For $\alpha \leq 0$, the equilbrium is a stable focus since $\dot{\varrho} < 0$
- For $\alpha > 0$: $\dot{\varrho} > 0$ for small ϱ
  $\dot{\varrho} < 0$ for large ϱ
- In fact, the system has periodic orbit of radius $\sqrt{\alpha}$



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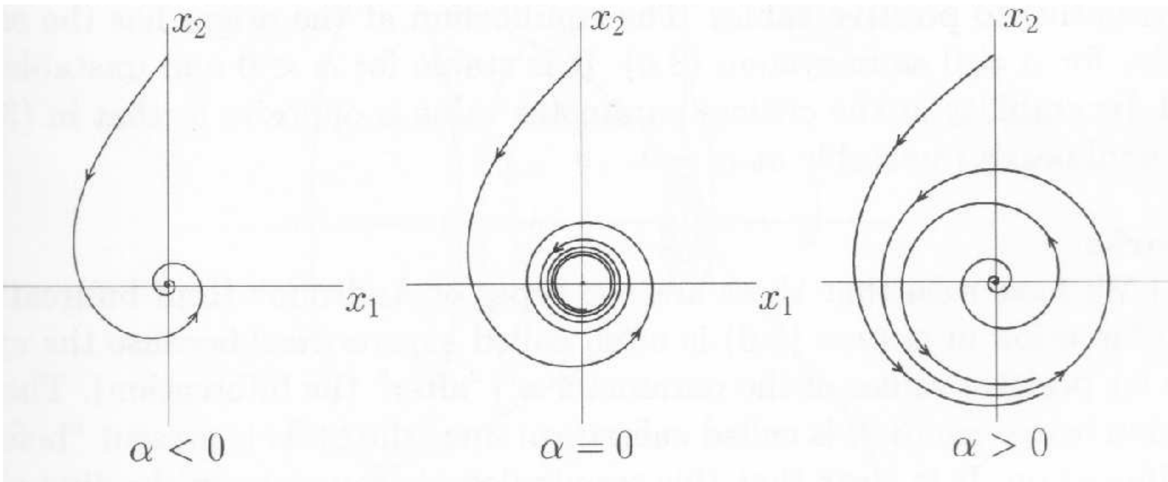
Dynamical Systems: Bifurcations

- Solution bifurcation occurs at $\alpha = 0$: a phase portrait with a limit cycle can−not be deformed by a one−to−one transformation into a phase portrait with a single equilbrium

- As α increases past 0, small−amplitude oscillations appear from the equilib−rium state via a Hopf bifurcation



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Dynamical Systems: Topological Normal Form

- By mapping dynamical systems to their topologically equivalent normal forms, one obtain the set of universal bifurcation diagrams
- Typically, normal forms are simple systems that are polynomials in ξ_i

$$\dot{\xi} = g(\xi, \alpha; \sigma)$$

and the coefficients σ are typically integer−valued coefficients giving rise to a finite number of topologically non−equivalent bifurcation diagrams

> Definition: the system $\dot{\xi} = g(\xi, \alpha; \sigma)$ is called a *topological normal form* if any generic system is locally topologically equivalent to it for some values of the coefficients σ

E.g.,

$$\begin{cases} \dot{\xi}_1 = \alpha\,\xi_1 - \xi_2 + \sigma\,\xi_1\,(\xi_1^2 + \xi_2^2) \\ \dot{\xi}_2 = \xi_1 - \alpha\,\xi_2 + \sigma\,\xi_2\,(\xi_1^2 + \xi_2^2) \end{cases}$$

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Dynamical Systems: One−Parameter Bifurcations of Equilibria

- By definition, at hyperbolic equilibrium, n_0=0

- Hyperbolicity condition can be lost via:

  − a simple real eigenvalue approaches the origin: $\lambda_1 = 0$

  − a pair of complex eigenvalues approach the imaginary axis: $\lambda_{1,2} = \pm i\,\omega$

- The corresponding bifurcations are:

  − saddle−node (also known as fold, limit point, turning point)

  − Hopf (also known as Andronov−Hopf)

- Now let's look at the associated normal forms

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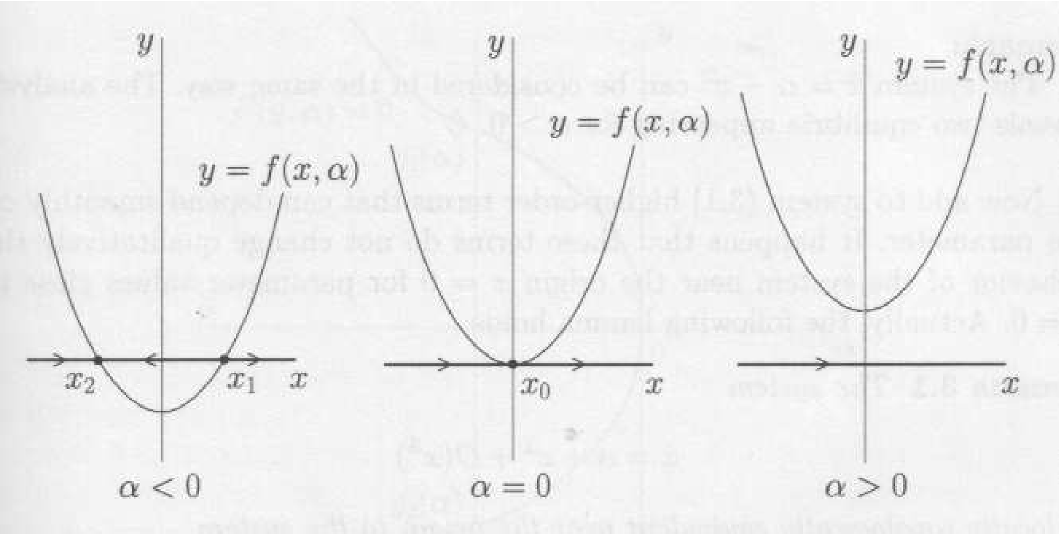
Dynamical Systems: Normal Form for Saddle−Node

- Consider the one−dimensional system:

  $\dot{x} = \alpha + x^2$

- For $\alpha < 0$, there are 2 equilibria: $x_{1,2}(\alpha) = \pm\sqrt{-\alpha}$



• The 2 equilibria have different stabilities:

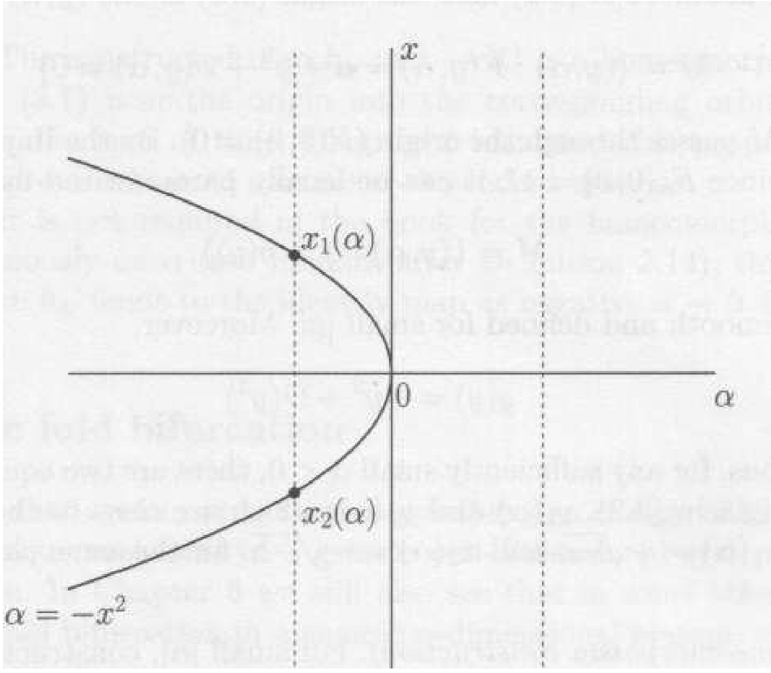
$x_1(\alpha) = -\sqrt{-\alpha}$ is stable

$x_2(\alpha) = \sqrt{-\alpha}$ is unstable

Dynamical Systems: Normal Form for Saddle−Node

• A different representation of solutions and dependence on α: plot the equilibrium manifold

$$\alpha = -x^2$$



• Hence, the projection of the equilibrium manifold onto the α−axis has a singularity of fold type

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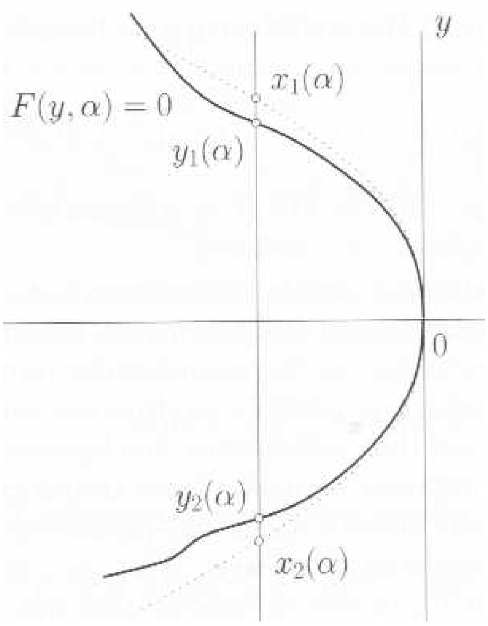
Dynamical Systems: Normal Form for Saddle−Node

Lemma: the system $\dot{x} = \alpha + x^2 + O(x^3)$
is topologically equivalent to the system $\dot{x} = \alpha + x^2$



• The proof is based on: first, showing that locally $O(x^3)$ does not change the number of equilibria; then construct a homeomorphism mapping equilibria to equilibria

Dynamical Systems: Normal Form for Saddle−Node

Theorem: any generic one−parameter system

$$\dot{x} = f(x, \alpha)$$

having at $\alpha = 0$ the equilibrium $x = 0$ and a 0 eigenvalue, is topologically equivalent near the origin to the saddle−node normal form

$$\dot{\xi} = \beta + \sigma \xi^2$$

where the coefficient $\sigma = \pm 1$

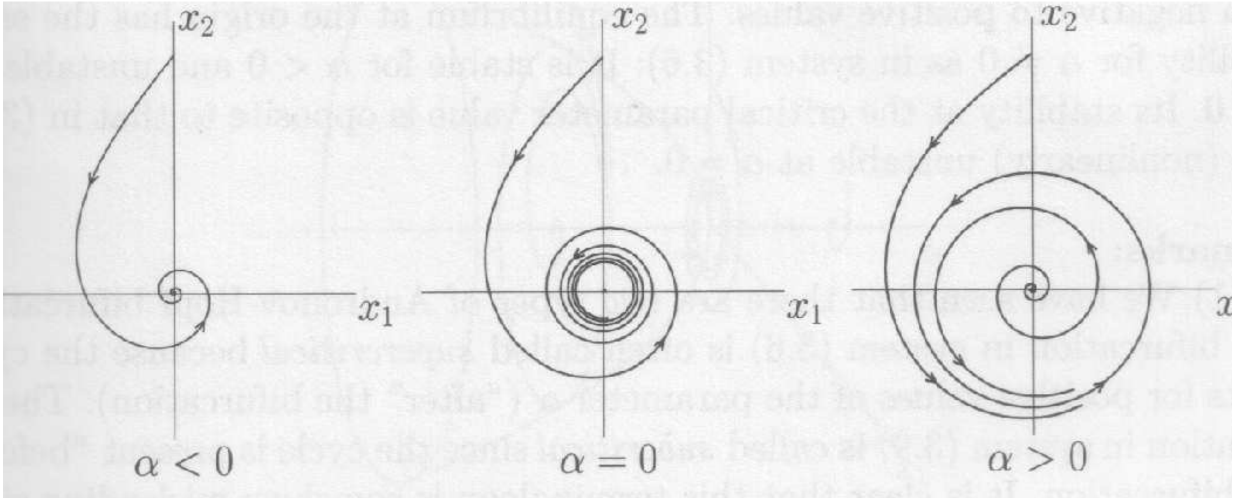
Dynamical Systems: Normal Form for Hopf

- Consider following 2−dimensional system

$$\begin{cases} \dot{\xi}_1 = \alpha\,\xi_1 - \xi_2 + \sigma\,\xi_1\,(\xi_1^2 + \xi_2^2) \\ \dot{\xi}_2 = \xi_1 - \alpha\,\xi_2 + \sigma\,\xi_2\,(\xi_1^2 + \xi_2^2) \end{cases}$$

for $\sigma = \pm 1$

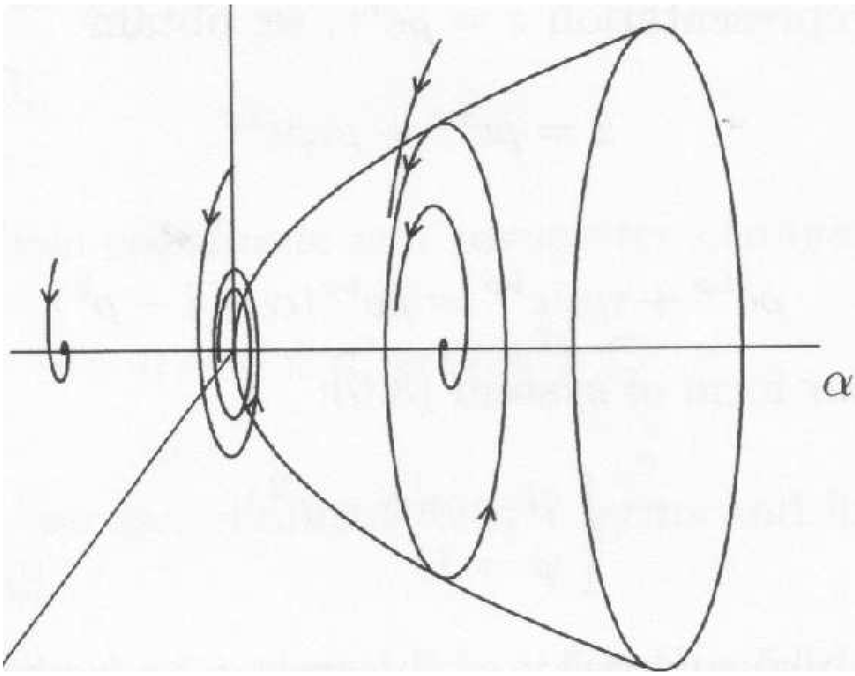
- The system has equilibrium $\xi_1 = \xi_2 = 0$ with eigenvalues $\lambda_{1,2} = \alpha \pm i$
- Hence, the system is linear stable for $\alpha < 0$, and linear unstable for $\alpha > 0$
- Case normal form coefficient $\sigma = 1$: supercritical Hopf bifurcation



Dynamical Systems: Normal Form for Hopf

$$\begin{cases} \dot{\xi}_1 = \alpha\,\xi_1 - \xi_2 + \sigma\,\xi_1\,(\xi_1^2 + \xi_2^2) \\ \dot{\xi}_2 = \xi_1 - \alpha\,\xi_2 + \sigma\,\xi_2\,(\xi_1^2 + \xi_2^2) \end{cases}$$

Case normal form coefficient $\sigma = 1$: supercritical Hopf bifurcation

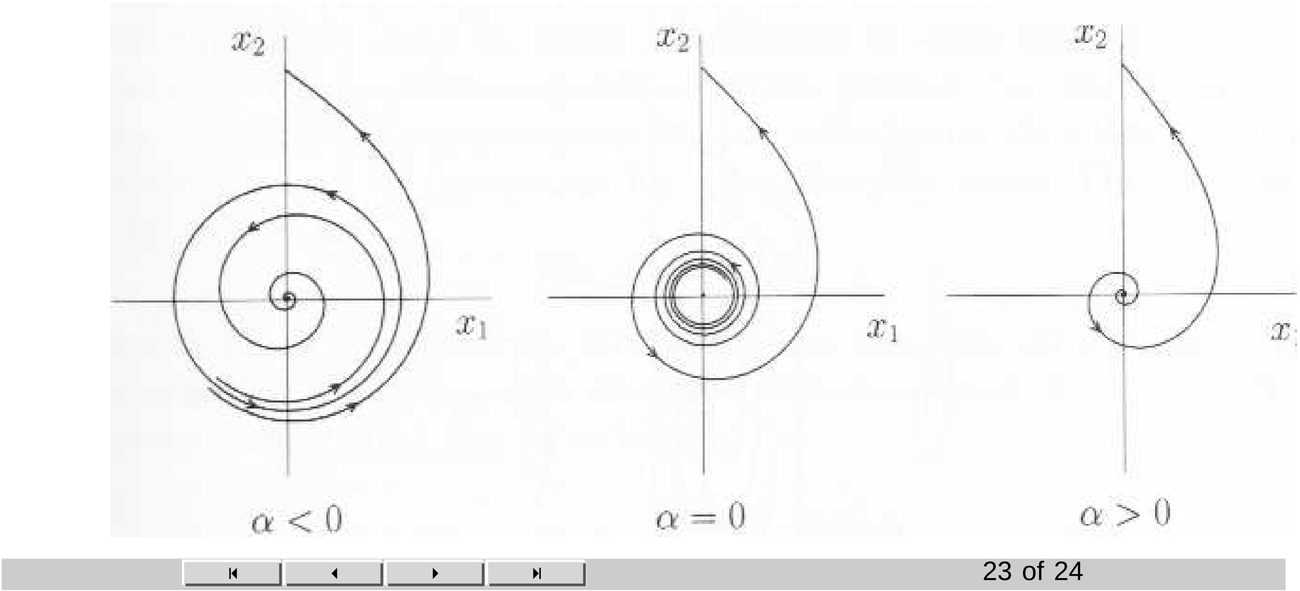


• For $\alpha > 0$, the equilibrium is surrounded by an isolated closed orbit (limit cycle) that is unique and stable

Dynamical Systems: Normal Form for Hopf

$$\begin{cases} \dot{\xi}_1 = \alpha\, \xi_1 - \xi_2 + \sigma\, \xi_1 (\xi_1^2 + \xi_2^2) \\ \dot{\xi}_2 = \xi_1 - \alpha\, \xi_2 + \sigma\, \xi_2 (\xi_1^2 + \xi_2^2) \end{cases}$$

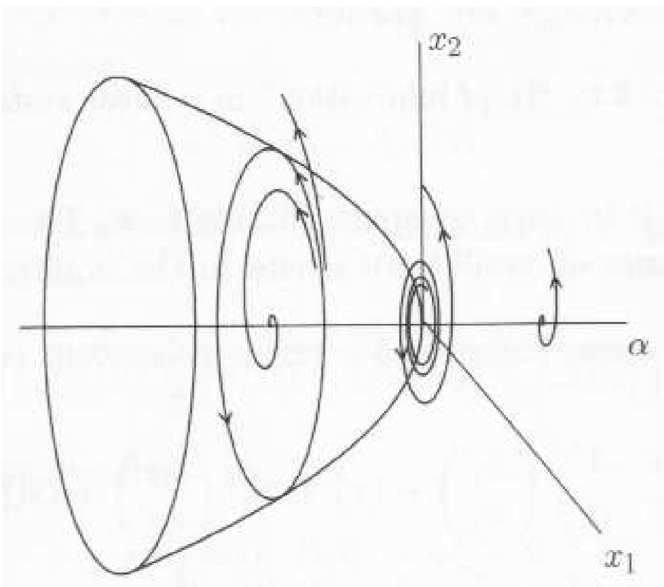
Case normal form coefficient $\sigma = -1$: subcritical Hopf bifurcation



Dynamical Systems: Normal Form for Hopf

$$\begin{cases} \dot{\xi}_1 = \alpha\,\xi_1 - \xi_2 + \sigma\,\xi_1\,(\xi_1^2 + \xi_2^2) \\ \dot{\xi}_2 = \xi_1 - \alpha\,\xi_2 + \sigma\,\xi_2\,(\xi_1^2 + \xi_2^2) \end{cases}$$

Case normal form coefficient $\sigma = -1$: subcritical Hopf bifurcation



• There is an unstable limit cycle (closed orbit) which disappears when α crosses zero from negative to positive values

• For $\alpha < 0$, the equilibrium is stable as it is surrounded by the unstable limit cycle

• Stability characteristics is reversed compared to supercritical Hopf bifurcation

Conclusions

• Topological equivalence captures information on the number, stability of invariant sets of dynamical systems

• Near equilibria, generic dynamical systems can be transformed to their (topologically equivalent) normal forms

• Normal form coefficients capture the certain dynamical characteristics (e.g., whether a Hopf bifucation is super− or sub−critical)

• Next lecture: numerical methods for detecting bifurcations