

Advances in Numerical Harmonic Analysis

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Outline

- 1 Back to Compressed Sensing and beyond l_1 -minimization
 - Variational limits
 - Beyond l_1 -minimization
- 2 Free-discontinuity problems
 - The Mumford-Shah functional
 - Approximation by discrete functionals
 - A general discrete functional
 - Iterative thresholding algorithms for Mumford-Shah minimization
- 3 Adaptive frame methods for PDEs

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Back to our functional: how do the parameters influence minimizers?

In our previous lecture we considered the following functional:

$$\begin{aligned}
 J(u, v) = & \underbrace{\sum_{j=1}^N \left\| \sum_{\ell=1}^M T_{\ell,j} u^\ell - g^j \right\|_{\mathcal{K}}^2}_{:=\mathcal{D}(u)} + \sum_{\lambda \in \Lambda} v_\lambda \left(\sum_{\ell=1}^M |u_\lambda^\ell|^q \right)^{1/q} \\
 & + \sum_{\lambda \in \Lambda} \omega_\lambda \sum_{\ell=1}^M |u_\lambda^\ell|^2 \\
 & + \sum_{\lambda \in \Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2.
 \end{aligned}$$

where $u = (u_\lambda^\ell)_{\lambda \in \Lambda, \ell=1, \dots, M}$, and $v = (v_\lambda)_\lambda \geq 0$.

Variational limits and De Giorgi's Γ -convergence

Given a sequence of functionals $(F_h)_{h \in \mathbb{N}}$ on a topological space X , we wonder whether there exists a target “limit” functional F such that

$$\underbrace{\lim_h F_h}_{\text{the limit is understood in some sense to be clarified}} = F$$

the limit is understood in some sense to be clarified

with the property that if $(x_h)_{h \in \mathbb{N}}$ is a sequence of minimizers of $(F_h)_{h \in \mathbb{N}}$ then

$$\lim_h x_h = x \in \operatorname{argmin}_y F(y).$$

For “nice” lower-semicontinuous functionals this convergence is given by the so-called Γ -convergence introduced by De Giorgi (1975). We emphasize that in general Γ -convergence does NOT coincide with pointwise convergence!

Just recalling notations

The second variable of the minimizers do depend on the first by the following formula:

$$\left(V_{\theta, \rho}^{(q)}(u) \right)_\lambda := \begin{cases} \rho\lambda - \frac{1}{2\theta\lambda} \|u_\lambda\|_q, & \|u_\lambda\|_q < 2\theta\lambda\rho\lambda \\ 0, & \text{otherwise} . \end{cases}$$

Approaching again ℓ_1 -minimization

$$\begin{aligned} J(u, v) &= J_{\theta, \rho, \omega}^{(q)}(u, v) \\ &= \|Tu - g\|_{\mathcal{X}}^2 + \sum_{\lambda \in \Lambda} \omega_\lambda \|u_\lambda\|_2^2 + \sum_{\lambda \in \Lambda} v_\lambda \|u_\lambda\|_q + \sum_{\lambda \in \Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2 \end{aligned}$$

Theorem (Fornasier and Rauhut '07)

Let $u_{\theta, \rho, \omega}^*$ be the first component of the minimizer of $J(u, v)$, we

have $u_{\theta, \rho, \omega}^* \xrightarrow[\omega \rightarrow 0]{\theta \rightarrow \infty} u^*$ which is a minimizer of $J_{1, \rho}$. In particular,

$$\Gamma - \lim_{\tau \rightarrow 0} \mathcal{J}(u, V_{\theta, \rho}^{(q)}(u)) = J_{1, \rho}(u),$$

$$J_{1, \rho}(u) = \mathcal{D}(u) + \sum_{\lambda} \rho_\lambda \|u_\lambda\|_q.$$

Approaching sparse recovery

We define

$$\underbrace{\Phi_{\theta, \rho, \omega}^{(q)}(u, v) = \sum_{\lambda \in \Lambda} v_\lambda \|u_\lambda\|_q + \sum_{\lambda \in \Lambda} \omega_\lambda \|u_\lambda\|_2^2 + \sum_{\lambda \in \Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2}_{\text{our joint-sparsity measure}}$$

and

$$\mathcal{J}_\tau^1(u, v) := \|Tu - g\|_{\mathcal{K}}^2 + \tau \Phi_{\theta, \rho, \omega}^{(q)}(u, v).$$

Theorem (Fornasier and Rauhut '07)

Let u_τ^* be the first component of the minimizer of $\mathcal{J}_\tau^1(u, v)$, then

$$u_\tau^* \xrightarrow{\tau \rightarrow 0} u^* \in \operatorname{argmin}_{Tu=g} \Phi_{\theta, \rho, \omega}^{(q)}(u, V_{\theta, \rho}^{(q)}(u)).$$

In particular, $\Gamma - \lim_{\tau \rightarrow 0} \mathcal{J}_\tau^1(u, V_{\theta, \rho}^{(q)}(u)) = \|Tu - g\|^2$.

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CS and ℓ_1 -minimization

We recall that for a matrix Φ which satisfies the RIP of order $2k$ for $\delta_{2k} \leq \delta < 1/3$. Then, for any $x \in \mathbb{R}^N$ with $\#(\text{supp}(x_0)) \leq k$ and $y = \Phi x$, the decoder

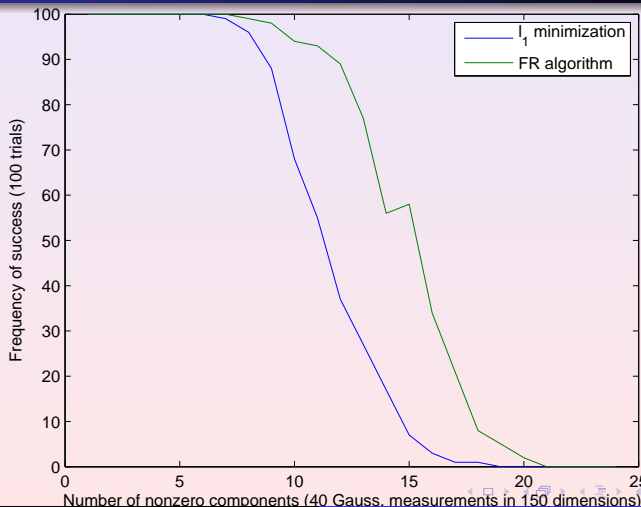
$$\Delta(y) := \operatorname{argmin}_{\Phi z = y} \|z\|_{\ell_1^N},$$

reconstructs $x = \Delta(y)$ exactly. What happens if we define as a decoder

$$\Delta(y) := \operatorname{argmin}_{\Phi z = y} \Phi_{\theta, \rho, \omega}^{(1)}(z, V_{\theta, \rho}^{(1)}(z)),$$

for some $M = 1$, and some parameters ρ, θ and

We surprisingly outperform the results of ℓ_1 -minimization!



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Free-discontinuity problems

The terminology “free-discontinuity problems” was introduced by E. De Giorgi ¹ to indicate a class of variational problems that consist in the minimization of a functional, involving both volume and a surface energy, depending on a closed set K , and a function u usually smooth outside of K .

- K is not fixed a priori and it is an unknown of the problem;
- K is not a boundary in general, but a free-surface inside the domain of the problem.

¹E. De Giorgi, *Free-discontinuity problems in calculus of variations*, Frontiers in pure and applied mathematics, a collection of papers dedicated to J.-L. Lions on the occasion of his 60th birthday, R. Dautray, ed. North Holland, 1991, 55-62.

A prominent example in visual analysis

The Mumford-Shah functional is defined by

$$J(u, K) := \int_{\Omega \setminus K} \left[|\nabla u|^2 + \alpha(u - g)^2 \right] dx + \beta \mathcal{H}^{d-1}(K \cap \Omega).$$

The set Ω is a bounded open subset of \mathbb{R}^d , $\alpha, \beta > 0$ are fixed parameters and $g \in L^\infty(\Omega)$.

We seek for a function $u \in W^{1,2}(\Omega \setminus K)$ which approximates the datum g , smooth out of the discontinuity set K .

In visual analysis the set K is used in order to *segment* the image into connected components.

The relevant unknown

If the set K were fixed, then the minimization of $J(u, K)$ with respect to u is a relatively easy problem whose solution is given by

$$\begin{aligned}\Delta u &= \alpha(u - g), & \text{in } \Omega \setminus K, \\ \frac{\partial u}{\partial \nu} &= 0, & \text{on } \partial\Omega \cup K.\end{aligned}$$

So, the relevant unknown in free-discontinuity problems is the set K !

SBV functions

The minimization of J in both u and K is not easy because there is NO topology on the closed sets which ensures

- compactness of minimizing sequences and
- lower semicontinuity of the Hausdorff measure.

However, we can think $u \in BV(\Omega) \cap W^{1,2}(\Omega \setminus K)$, with $K = \overline{S_u}$ (the discontinuity set of u), and minimize J in this space.

Unfortunately this space is TOO large. It contains Cantor-like functions such that $\nabla u = 0$ a.e. and $\mathcal{H}^{d-1}(S_u) = 0$, dense in $L^2(\Omega)$, so that the problem is trivialized.

SBV functions

Instead it is possible to give a meaningful formulation of the functional J in $SBV(\Omega)$ (the special $BV(\Omega)$ functions, with vanishing Cantor part),

$$\mathcal{F}(u) = \int_{\Omega \setminus S_u} \left[|\nabla u|^2 + \alpha(u - g)^2 \right] dx + \beta \mathcal{H}^{d-1}(S_u).$$

Γ -convergence

Definition

Let (X, d) be a metric space and let $f, f_n : X \rightarrow [0, \infty]$ be functions. We say that $(f_n)_n$ Γ -converges to f if the following two conditions are satisfied:

- i) for any sequence $(x_n)_n \subset X$ converging to x , the following holds:

$$\liminf_n f_n(x_n) \geq f(x);$$

- ii) for any $x \in X$ there exists a sequence $(x_n)_n \subset X$ converging to x such that

$$\limsup_n f_n(x_n) \leq f(x);$$

Eventually the minimizers of f_n do converge to a minimizer of f .

Γ -approximation by Ambrosio and Tortorelli

Let us define

$$J_\varepsilon(u, v) := \int_\Omega \left[v |\nabla u|^2 + \alpha(u - g)^2 \right] + \frac{\beta}{2} \left(\varepsilon |\nabla v|^2 + \frac{(1 - v)^2}{\varepsilon} \right) dx.$$

The function $0 \leq v \leq 1$ tends to indicate S_u for $\varepsilon \rightarrow 0$. We have

Theorem (Ambrosio-Tortorelli '92)

The functional $J_{\varepsilon_n}(u, v)$ (v depends on u in a suitable way) Γ -converges in $(L^2(\Omega))^2$ to $\mathcal{F}(u)$ for any infinitesimal sequence $(\varepsilon_n)_n$.

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The Blake-Zisserman's discrete functional

Let $d = 2$, $\Omega = [0, 1]^2$, and $u_{i,j} = u(hi, hj)$ a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}^2$, for $h > 0$. Define $W_h(t) = \min\{t^2, \beta/h\}$ the truncated quadratic potential and

$$\begin{aligned} J_h^\delta(u) &= h^2 \sum_{(hi, hj) \in \Omega_h} W_h \left(\frac{u_{i+1,j} - u_{i,j}}{h} \right) \\ &+ h^2 \sum_{(hi, hj) \in \Omega_h} W_h \left(\frac{u_{i,j+1} - u_{i,j}}{h} \right) \\ &+ \alpha h^2 \sum_{(hi, hj) \in \Omega_h} (u_{i,j} - g_{i,j})^2. \end{aligned}$$

The Blake-Zisserman's discrete functional

Theorem (Chambolle '95)

The functional $J_h^\delta(u)$ Γ -converges in $\mathcal{B}(\Omega)$ to

$$\mathcal{E}(u) = \int_{\Omega \setminus S_u} \left[|\nabla u|^2 + \alpha(u - g)^2 \right] dx + \beta \mathcal{C}(S_u),$$

for $h \rightarrow 0$, where \mathcal{C} is the so-called “cab-driver” measure.

Basically \mathcal{C} measures the length of a curve only through its projections along horizontal and vertical axes; for a regular C^1 curve $c = \gamma([0, 1])$, with $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \Omega$, we have

$$\mathcal{C}(c) = \int_0^1 (|\gamma_1'(t)| + |\gamma_2'(t)|) dt.$$

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The derivation

For simplicity assume $d = 1$. We consider $\Omega = [0, 1]$, and $u_i = u(hi)$ a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}$, for $h > 0$. Define $W_h(t) = \min\{t^2, \frac{1}{2h}\}$ and

$$\begin{aligned} J_h^\delta(u) &= h \sum_{(hi) \in \Omega_h} W_h \left(\frac{u_{i+1} - u_i}{h} \right) \\ &+ h \sum_{(hi) \in \Omega_h} (u_i - g_i)^2. \end{aligned}$$

Again, it is possible to show that this discrete functional Γ -converges to a suitable functional *a la* Mumford-Shah.

The derivation

Let us denote $S_u^\delta = \{hi : \left| \frac{u_{i+1} - u_i}{h} \right| \leq \frac{1}{\sqrt{h}}\}$ and

$$v(i) = \begin{cases} 1 & i \in \Omega_h \setminus S_u^\delta \\ 0 & i \in S_u^\delta \end{cases}$$

Hence, we have

$$\begin{aligned} J_h^\delta(u) &= h \sum_{(hi) \in \Omega_h} v(i) \left| \frac{u_{i+1} - u_i}{h} \right|^2 + \frac{1}{2} \sum_{(hi) \in \Omega_h} (1 - v(i))^2 \\ &+ h \sum_{(hi) \in \Omega_h} (u_i - g_i)^2. \end{aligned}$$

The derivation

Inspired by Ambrosio-Tortorelli, we can even dare to consider v as an additional variable

$$\begin{aligned} J_h^\delta(u, v) &= h \sum_{(hi) \in \Omega_h} v(i) \left| \frac{u_{i+1} - u_i}{h} \right|^2 + \frac{1}{2} \sum_{(hi) \in \Omega_h} (1 - v(i))^2 \\ &+ h \sum_{(hi) \in \Omega_h} (u_i - g_i)^2. \end{aligned}$$

Indeed the minimization w.r.t. $v \geq 0$ for u fixed gives though

$$v(i) = \begin{cases} 1 - h \left| \frac{u_{i+1} - u_i}{h} \right|^2, & i \in \Omega_h \setminus S_u^\delta \\ 0, & i \in S_u^\delta \end{cases}$$

The derivation

Plugging this minimal $v \geq 0$ into the functional we obtain

$$J_h^\delta(u, v) = J_h^\delta(u) - \underbrace{\frac{h^2}{2} \sum_{(hi) \in \Omega_h \setminus S_u^\delta} \left| \frac{u_{i+1} - u_i}{h} \right|^4}_{\text{Additional term}}.$$

To make disappear the additional term we need to consider (at least)

$$\mathcal{J}_h^\delta(u, v) = J_h^\delta(u, v) + \frac{h^2}{2} \sum_{(hi) \in \Omega_h \setminus S_u^\delta} \left| \frac{u_{i+1} - u_i}{h} \right|^4.$$

Equivalence of minimizers

Theorem

The u -component of the minimizers (u, v) of the functional

$$\begin{aligned} \mathcal{J}_h^\delta(u, v) &= h \sum_{(hi) \in \Omega_h} v(i) \left| \frac{u_{i+1} - u_i}{h} \right|^2 + \frac{1}{2} \sum_{(hi) \in \Omega_h} (1 - v(i))^2 \\ &+ \frac{h^2}{2} \sum_{(hi) \in \Omega_h \setminus S_u^\delta} \left| \frac{u_{i+1} - u_i}{h} \right|^4 \\ &+ h \sum_{(hi) \in \Omega_h} (u_i - g_i)^2, \end{aligned}$$

are minimizers of $J_h^\delta(u)$.

Discrete derivative

For $(u(i))_{hi \in \Omega_h}$ we define the discrete derivative as the matrix D_h which maps $(u(i))_{hi \in \Omega_h}$ into $\left(\frac{u(i+1)-u(i)}{h}\right)_i$, given by

$$D_h = \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & \dots & -1 & 1 \end{pmatrix}.$$

We have

$$u = D_h^\dagger D_h u + c_u,$$

where D_h^\dagger is the pseudo-inverse matrix of D_h and c_u is constant vector. Observe that $\langle D_h^\dagger D_h u, c \rangle = 0$, for any constant c .

A functional of the derivative

We can reformulate the problem in terms of derivatives

$$\begin{aligned}\mathcal{J}_h^\delta(z, v) &= h \sum_i v(i) |z_i|^2 + \frac{1}{2} \sum_i (1 - v(i))^2 \\ &+ \frac{h^2}{2} \sum_{(hi) \in \Omega_h \setminus S_u^\delta} |z_i|^4 \\ &+ h \sum_i (D_h^\dagger z_i - f_i)^2,\end{aligned}$$

where $z = D_h u$, $f = D_h^\dagger D_h g$, and $c_u = c_g$ is assumed at the minimizer.

A general discrete functional: surprise ...

On the basis of this derivation, we are interested to study the minimizers of the following general form of functionals

$$\begin{aligned} J(u, v) = & \|Tu - g\|_{\mathcal{K}}^2 + \sum_{\lambda \in \Lambda} v_{\lambda} |u_{\lambda}|^p \\ & + \sum_{\lambda \in \Lambda} \omega_{\lambda} |u_{\lambda}|^{2p} \\ & + \sum_{\lambda \in \Lambda} \theta_{\lambda} (\rho_{\lambda} - v_{\lambda})^2, \end{aligned}$$

where Λ is a countable index set, $u = (u_{\lambda})_{\lambda}$ is a vector in $\ell^2(\Lambda)$, $T : \ell^2(\Lambda) \rightarrow \mathcal{K}$ is a bounded operator, $g \in \mathcal{K}$, and $1 \leq p \leq 2$.

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Convexity

Lemma.

Assume $1 < p \leq 2$ and $\rho, \omega, \theta > 0$. If $(\sigma_{\min} + \omega_\lambda)\theta_\lambda \geq \frac{\rho^2}{4p(2p-1)}$,
 $\lambda \in \Lambda$, then $J(u, v)$ is convex, and strictly convex for
 $(\sigma_{\min} + \omega_\lambda)\theta_\lambda > \frac{\rho^2}{4p(2p-1)}$.

For some initial choice $v^{(0)}$, for example $v^{(0)} \equiv (\rho_\lambda)_{\lambda \in \Lambda}$, define

$$\begin{cases} u^{(n+1)} := \operatorname{argmin}_{u \in \ell_{2p}(\Lambda)} J(u, v^{(n)}) \\ v^{(n+1)} := \operatorname{argmin}_{0 \leq v_\lambda \leq \rho_\lambda} J(u^{(n+1)}, v). \end{cases}$$

This algorithm was already studied for $p = 1$. The analysis for $1 < p \leq 2$ is analogous but certain differences arise.

Theorem (Fornasier '07)

Let $1 < p \leq \infty$, $\theta_\lambda(\sigma_\lambda + \omega_\lambda) > \frac{p^2}{4p(2p-1)}$, and $\omega_\lambda \geq \gamma > 0$, for all $\lambda \in \Lambda$. Then the sequence $(u^{(n)}, v^{(n)})$ generated by the algorithm converges to $(u^, v^*) \in \ell_{2p}(\Lambda) \times \ell_{\infty, p-1}(\Lambda)$, the unique minimizer of J . The convergence of $u^{(n)}$ is in the weak sense and the convergence of $v^{(n)}$ is componentwise.*

Again, the minimization with respect to the second variable is explicit:

$$v_\lambda^{(n)} = \begin{cases} \rho_\lambda - \frac{|u_\lambda^{(n)}|^p}{2\theta_\lambda}, & |u_\lambda^{(n)}|^p \leq 2\theta_\lambda \rho_\lambda \\ 0, & \text{otherwise.} \end{cases}$$

For the minimization with respect to the first variable we have

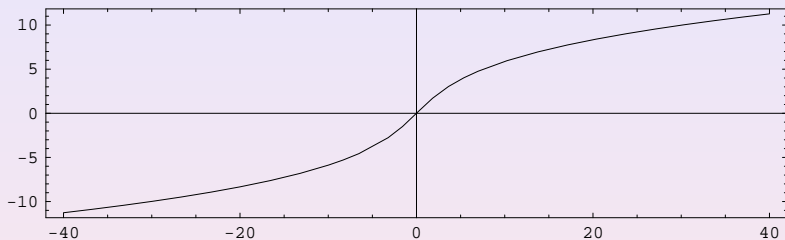
Theorem

If $\|T\| < 1$, $1 < p \leq 2$ and $\omega_\lambda \geq \gamma > 0$, for all $\lambda \in \Lambda$, then for any initial choice $u^{(0)} \in \ell_{2p}(\Lambda)$, the sequence

$$u^{(m+1)} := \mathbb{U}_{v,\omega}^{(p)} \left(u^{(m)} + T^*(g - Tu^{(m)}) \right),$$

converges to the minimizer of $J(\cdot, v)$.

The thresholding operator $\mathbb{U}_{v,\omega}^{(\rho)}$



The operator $\mathbb{U}_{v,\omega}^{(\rho)}$ acts componentwise by applying curves $U_{v,\omega}^\rho$ as the one depicted above. It is defined as

$$\begin{aligned}\mathbb{U}_{v,\omega}^{(\rho)}(z) &:= \operatorname{argmin}_u \mathcal{J}^{\text{surr}}(u, v, z) \\ &:= \operatorname{argmin}_u J(u, v) + \|u - z\|^2 - \|T(u - y)\|^2.\end{aligned}$$

The single iteration algorithm

Instead of an alternating minimization we may want to consider:

$$(u^{(n+1)}, v^{(n+1)}) = \operatorname{argmin}_{(u,v) \in \ell_{2\rho}(\Lambda) \times \ell_{\infty, \rho^{-1}}(\Lambda)} \mathcal{J}^{sur}(u, v, u^{(m)}).$$

Observe that in this iteration the variable v does not play any role.

Theorem (Fornasier '07)

The latter algorithm is equivalent to the following iterative thresholding algorithm:

$$u^{(n+1)} = \mathbb{H}_{\omega, \theta, \rho}^p \left(u^{(n)} + T^*(g - Tu^{(n)}) \right).$$

The nonlinear thresholding operator $\mathbb{H}_{\omega, \theta, \rho}^p : \ell_2(\Lambda) \rightarrow \ell_2(\Lambda)$ is defined componentwise $\mathbb{H}_{\omega, \theta, \rho}^p(\xi) = (H_{\omega_\lambda, \theta_\lambda, \rho_\lambda}^p(\xi_\lambda))_{\lambda \in \Lambda}$, where for $z \in \mathbb{R}$,

The single iteration algorithm

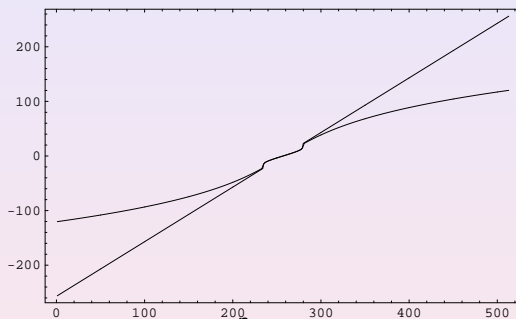
we have

$$H_{\omega_\lambda, \theta_\lambda, \rho_\lambda}^p(z) := \lim_{m \rightarrow \infty} z^{(m)},$$

and $z^{(m)}$ is defined by the algorithm:

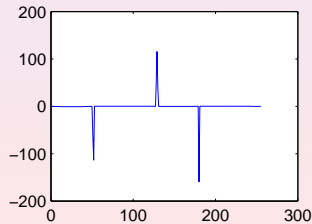
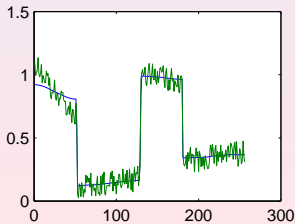
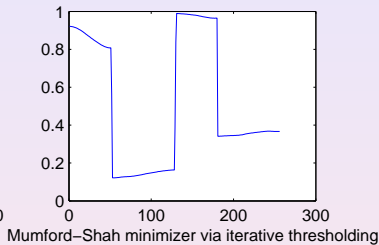
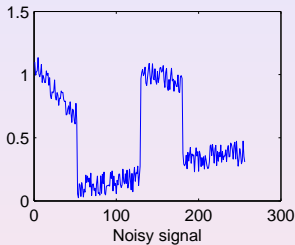
$$\begin{aligned} z^{(m+1)} &= U_{v^{(m)}, \omega_\lambda}^p(z), \\ v^{(m+1)} &= \begin{cases} \rho_\lambda - \frac{|z^{(m+1)}|^p}{2\theta_\lambda}, & |z^{(m+1)}|^p \leq 2\theta_\lambda \rho_\lambda \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Typical shape of the function $H_{\omega_\lambda, \theta_\lambda, \rho_\lambda}^p(z)$



We illustrate the shape of $H_{\omega_\lambda, \theta_\lambda, \rho_\lambda}^p(z)$ for $p = 2$ and for different choice of the parameters.

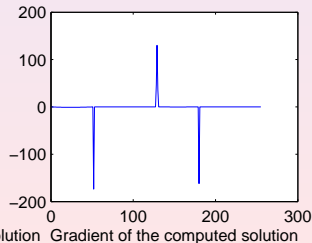
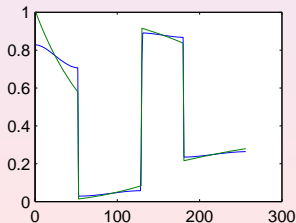
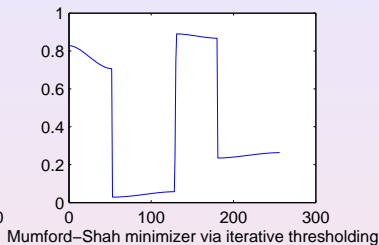
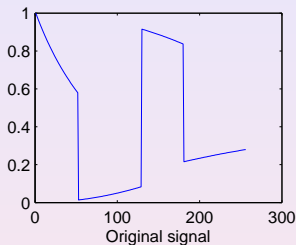
Signal denoising via Mumford-Shah minimization



Comparison between original and computed solution

Gradient of the computed solution

Signal segmentation via Mumford-Shah minimization



Comparison between original and computed solution Gradient of the computed solution

Image segmentation via Mumford-Shah minimization

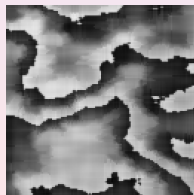
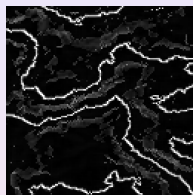
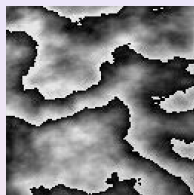


Image denoising via Mumford-Shah minimization



Original image



Noisy image



Mumford-Shah minimizer via iterative thresholding



Discrete discontinuity set

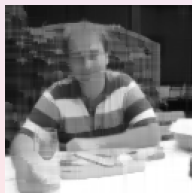
Image segmentation via Mumford-Shah minimization



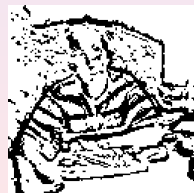
Original image



The gradient of the computed minimizer



Mumford-Shah minimizer via iterative thresholding



Discrete discontinuity set

Elliptic equation: theoretical setting

- $\Omega \subset \mathbb{R}^d$ Lipschitz domain
- H Hilbert space, $H \subset L_2(\Omega) \subset H'$ Gelfand triple
- $\mathcal{L} : H \rightarrow H'$ linear, boundedly invertible

$$a(\cdot, \cdot) := \langle \mathcal{L}\cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$$

symmetric, continuous, H -elliptic: $a(v, v) \approx \|v\|_H^2$.

- task: for $f \in H'$, solve

$$\mathcal{L}u = f,$$

i.e., find $u \in H$ s.t.

$$a(u, v) = \langle f, v \rangle, \quad v \in H.$$

Discretization via bases

Cohen, Dahmen, DeVore (1998):

- I choose wavelet basis $\Psi = \{\psi_\lambda\}_{\lambda \in \mathcal{J}} \subset L_2(\Omega)$ with

$$\|\mathbf{c}\|_{\ell_2} \approx \|\mathbf{c}^T D^{-1} \Psi\|_H.$$

- This implies

$$\mathcal{L}u = f \Rightarrow \mathbf{L}u = \mathbf{f},$$

where

$$\begin{aligned} \mathbf{L} &:= D^{-1} \langle \mathcal{L}\Psi, \Psi \rangle D^{-1}, \\ \mathbf{f} &:= D^{-1} \langle f, \Psi \rangle. \end{aligned}$$

- problem: construction of wavelet bases on general domains **difficult/complicated!**

Discretization via frames

Dahlke, Fornasier, Raasch, Stevenson (2003-2006)

- choose a frame $\Psi = \{\psi_\lambda\}_{\lambda \in \mathcal{J}} \subset L_2(\Omega)$ with

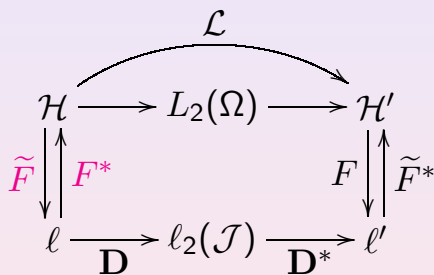
$$\|g\|_{L_2} \approx \|\langle g, \Psi \rangle\|_{\ell_2}.$$

- Ψ is a **Gelfand frame** for (H, L_2, H') if

$$F^* : \ell \rightarrow H : c \mapsto c^T \Psi, \quad \tilde{F} : H \rightarrow \ell : g \mapsto \langle g, \tilde{\Psi} \rangle$$

are bounded.

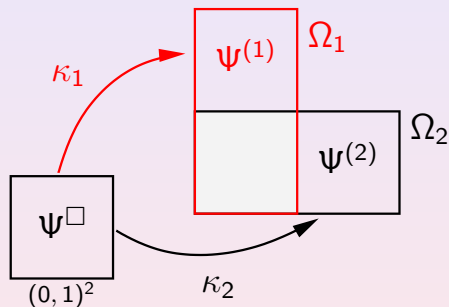
Mapping diagram



Construction of wavelet Gelfand frames (Dahlke, Fornasier, Raasch '04)

- $H = H_0^t(\Omega)$;
- reference Riesz basis $\Psi^\square \subset H_0^t(\square)$, $\square := (0, 1)^d$;
- overlapping decomposition $\Omega = \sum_{i=1}^n \Omega_i$;
- $\kappa_i : \square \rightarrow \Omega_i$, C^m -diffeomorphisms, $m \geq t$;
- appropriate lifting yields Gelfand frames $\Psi = \bigcup_{i=1}^n \Psi_i$.

Gelfand frame construction



Solution of the discrete problem

The operator $\mathbf{L} : \ell_2(\mathcal{J}) \rightarrow \ell_2(\mathcal{J})$ is going to be bounded, symmetric, positive, but has nontrivial kernel. Nevertheless we have

$$\mathbf{L} : \text{ran}(L) \rightarrow \text{ran}(L)$$

is boundedly invertible;

- (ideal) damped Richardson iteration:

$$(R) \quad \mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \mathbf{r}^{(n)}, \quad \mathbf{r}^{(n)} = \mathbf{f} - \mathbf{L}\mathbf{u}^{(n)}$$

- (ideal) steepest descent method

$$(SD) \quad \mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \frac{\langle \mathbf{r}^{(n)}, \mathbf{r}^{(n)} \rangle}{\langle \mathbf{L}\mathbf{r}^{(n)}, \mathbf{r}^{(n)} \rangle} \mathbf{r}^{(n)}.$$

Basic discrete procedures

Assume that we have the following procedures at our disposal:

- **RHS** $[\varepsilon, \mathbf{g}] \rightarrow \mathbf{g}_\varepsilon$: determines for $\mathbf{g} \in \ell_2(\mathcal{J})$ a finitely supported $\mathbf{g}_\varepsilon \in \ell_2(\mathcal{J})$ such that

$$\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2(\mathcal{J})} \leq \varepsilon; \quad (1)$$

- **APPLY** $[\varepsilon, \mathbf{N}, \mathbf{v}] \rightarrow \mathbf{w}_\varepsilon$: determines for $\mathbf{N} \in B(\ell_2(\mathcal{J}))$ and for a finitely supported $\mathbf{v} \in \ell_2(\mathcal{J})$ a finitely supported \mathbf{w}_ε such that

$$\|\mathbf{N}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2(\mathcal{J})} \leq \varepsilon; \quad (2)$$

- **COARSE** $[\varepsilon, \mathbf{v}] \rightarrow \mathbf{v}_\varepsilon$: determines for a finitely supported $\mathbf{v} \in \ell_2(\mathcal{J})$ a finitely supported $\mathbf{v}_\varepsilon \in \ell_2(\mathcal{J})$ with at most N significant coefficients, such that

$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell_2(\mathcal{J})} \leq \varepsilon.$$

Moreover, $N \lesssim N_{\min}$ holds, N_{\min} being the minimal number

The algorithm

SOLVE $[\varepsilon, \mathbf{L}, \mathbf{f}] \rightarrow \mathbf{u}_\varepsilon$:

Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.

$i := 0$, $\mathbf{v}^{(0)} := \mathbf{0}$, $\varepsilon_0 := \|\mathbf{L}^{-1}|_{\text{ran}(\mathbf{L})}\| \|\mathbf{f}\|_{\ell_2(\mathcal{J})}$

While $\varepsilon_i > \varepsilon$ do

$i := i + 1$

$\varepsilon_i := 3\rho^K \varepsilon_{i-1} / \theta$

$\mathbf{f}^{(i)} := \mathbf{RHS}[\frac{\theta \varepsilon_i}{6\alpha K}, \mathbf{f}]$

$\mathbf{v}^{(i,0)} := \mathbf{v}^{(i-1)}$

For $j = 1, \dots, K$ do

$\mathbf{v}^{(i,j)} := \mathbf{v}^{(i,j-1)} - \alpha(\mathbf{APPLY}[\frac{\theta \varepsilon_i}{6\alpha K}, \mathbf{L}, \mathbf{v}^{(i,j-1)}] - \mathbf{f}^{(i)})$

od

$\mathbf{v}^{(i)} := \mathbf{COARSE}[(1 - \theta)\varepsilon_i, \mathbf{v}^{(i,K)}]$

od

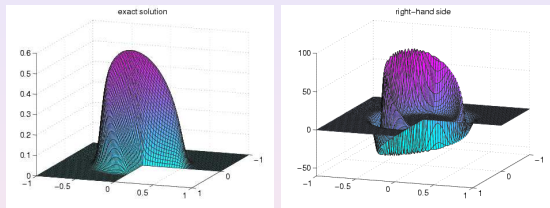
$\mathbf{u}_\varepsilon := \mathbf{v}^{(i)}$.

The proof of convergence and optimality

Convergence and optimal complexity:

- (R) Riesz basis case: Cohen, Dahmen, DeVore (2000)
- (R) Frame case: Stevenson (2003)
- (SD) Riesz basis case: Dahmen, Urban, Vorloeper (2002)
Canuto, Urban (2003)
- (SD) Dahlke, Fornasier, Raasch, Stevenson, Werner (2005)

Numerical examples



Let us consider the Poisson's equation on the L -shaped domain:

$$-\Delta u = f, \quad u|_{\partial\Omega} = 0,$$

and choose f with singularity at the re-entrant corner.

Numerical examples

some (SD) iterations:

