Sparse recovery algorithms from exact and incomplete data

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Outline

1. Back to Compressed Sensing and beyond ℓ_1-minimization
   - Variational limits
   - Beyond ℓ_1-minimization
2. Free-discontinuity problems
   - The Mumford-Shah functional
   - Approximation by discrete functionals
   - A general discrete functional
   - Iterative thresholding algorithms for Mumford-Shah minimization
3. Adaptive frame methods for PDEs

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Back to our functional: how do the parameters influence minimizers?

In our previous lecture we considered the following functional:

$$J(u,v) = \underbrace{\sum_{j=1}^{N} \| \sum_{\ell=1}^{M} T_{\ell,j} u^{\ell} - g^{j} \|_{\mathcal{K}}^{2}}_{:=\mathcal{D}(u)} + \sum_{\lambda \in \Lambda} v_\lambda \left(\sum_{\ell=1}^{M} |u_\lambda^\ell|^q \right)^{1/q}$$

$$+ \sum_{\lambda \in \Lambda} \omega_\lambda \sum_{\ell=1}^{M} |u_\lambda^\ell|^2$$

$$+ \sum_{\lambda \in \Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2.$$

where $u = (u_\lambda^\ell)_{\lambda \in \Lambda, \ell=1,\dots,M}$, and $v = (v_\lambda)_\lambda \geq 0$.

Variational limits and De Giorgi's Γ-convergence

Given a sequence of functionals $(F_h)_{h\in\mathbb{N}}$ on a topological space X, we wonder whether there exists a target "limit" functional F such that

$$\underbrace{\lim_h \quad F_h}_{\text{the limit is understood in some sense to be clarified}} \qquad = F$$

with the property that if $(x_h)_{h\in\mathbb{N}}$ is a sequence of minimizers of $(F_h)_{h\in\mathbb{N}}$ then

$$\lim_h x_h = x \in \operatorname{argmin}_y F(y).$$

For "nice" lower-semicontinuous functionals this convergence is given by the so-called Γ-convergence introduced by De Giorgi (1975). We emphasize that in general Γ-convergence does NOT coincide with pointwise convergence!

Just recalling notations

The second variable of the minimizers do depend on the first by the following formula:

$$\left(V_{\theta,\rho}^{(q)}(u) \right)_\lambda := \begin{cases} \rho_\lambda - \frac{1}{2\theta_\lambda}\|u_\lambda\|_q, & \|u_\lambda\|_q < 2\theta_\lambda\rho_\lambda \\ 0, & \text{otherwise} . \end{cases}$$

Approaching again ℓ_1-minimization

$$J(u,v) = J^{(q)}_{\theta,\rho,\omega}(u,v)$$
$$= \quad \|Tu-g\|^2_{\mathcal{K}} + \sum_{\lambda\in\Lambda}\omega_\lambda\|u_\lambda\|_2^2 + \sum_{\lambda\in\Lambda} v_\lambda\|u_\lambda\|_q + \sum_{\lambda\in\Lambda}\theta_\lambda(\rho_\lambda - v_\lambda)^2$$

Theorem (Fornasier and Rauhut '07)

Let $u^\star_{\theta,\rho,\omega}$ be the first component of the minimizer of $J(u,v)$, we have $u^\star_{\theta,\rho,\omega} \overset{\theta\to\infty}{\underset{\omega\to 0}{\rightharpoonup}} u^\star$ which is a minimizer of $J_{1,\rho}$. In particular,
$\Gamma - \lim_{\tau\to 0}\mathcal{J}(u, V^{(q)}_{\theta,\rho}(u)) = J_{1,\rho}(u)$,

$$J_{1,\rho}(u) = \mathcal{D}(u) + \sum_{\lambda}\rho_\lambda\|u_\lambda\|_q.$$

Approaching sparse recovery

We define

$$\underbrace{\Phi^{(q)}_{\theta,\rho,\omega}(u,v) = \sum_{\lambda\in\Lambda} v_\lambda \|u_\lambda\|_q + \sum_{\lambda\in\Lambda} \omega_\lambda \|u_\lambda\|_2^2 + \sum_{\lambda\in\Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2}_{\text{our joint-sparsity measure}}$$

and

$$\mathcal{J}^1_\tau(u,v) := \|Tu - g\|^2_{\mathcal{K}} + \tau \Phi^{(q)}_{\theta,\rho,\omega}(u,v).$$

Theorem (Fornasier and Rauhut '07)

Let $u^\star_\tau$ be the first component of the minimizer of $\mathcal{J}^1_\tau(u,v)$, then

$$u^\star_\tau \overset{\tau\to 0}{\rightharpoonup} u^\star \in \operatorname{argmin}_{Tu=g} \Phi^{(q)}_{\theta,\rho,\omega}(u, V^{(q)}_{\theta,\rho}(u)).$$

In particular, $\Gamma - \lim_{\tau\to 0} \mathcal{J}^1_\tau(u, V^{(q)}_{\theta,\rho}(u)) = \|Tu - g\|^2$.

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CS and ℓ_1-minimization

We recall that for a matrix Φ which satisfies the RIP of order $2k$ for $\delta_{2k} \leq \delta < 1/3$. Then, for any $x \in \mathbb{R}^N$ with $\#(\text{supp}(x_0)) \leq k$ and $y = \Phi x$, the decoder

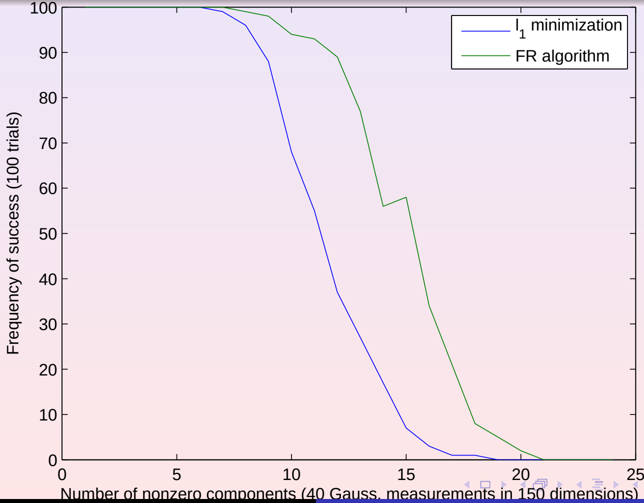
$$\Delta(y) := \text{argmin}_{\Phi z = y} \|z\|_{\ell_1^N},$$

reconstructs $x = \Delta(y)$ exactly. What happens if we define as a decoder

$$\Delta(y) := \text{argmin}_{\Phi z = y} \Phi^{(1)}_{\theta,\rho,\omega}(z, V^{(1)}_{\theta,\rho}(z)),$$

for some $M = 1$, and some parameters ρ, θ and

We surprisingly outperform the results of ℓ_1-minimization!



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Free-discontinuity problems

The terminology "free-discontinuity problems" was introduced by E. De Giorgi [1] to indicate a class of variational problems that consist in the minimization of a functional, involving both volume and a surface energy, depending on a closed set K, and a function u usually smooth outside of K.

- K is not fixed a priori and it is an unknown of the problem;
- K is not a boundary in general, but a free-surface inside the domain of the problem.

[1] E. De Giorgi, *Free-discontinuity problems in calculus of variations*, Frontiers in pure and applied mathematics, a collection of papers dedicated to J.-L. Lions on the occasion of his 60^{th} birthday, R. Dautray, ed. North Holland, 1991, 55-62.

A prominent example in visual analysis

The Mumford-Shah functional is defined by

$$J(u,K) := \int_{\Omega\setminus K} \left[|\nabla u|^2 + \alpha(u-g)^2 \right] dx + \beta \mathcal{H}^{d-1}(K \cap \Omega).$$

The set Ω is a bounded open subset of $\mathbb{R}^d$, $\alpha, \beta > 0$ are fixed parameters and $g \in L^\infty(\Omega)$.
We seek for a function $u \in W^{1,2}(\Omega \setminus K)$ which approximates the datum g, smooth out of the discontinuity set K.
In visual analysis the set K is used in order to *segment* the image into connected components.

The relevant unknown

If the set K were fixed, then the minimization of $J(u, K)$ with respect to u is a relatively easy problem whose solution is given by

$$\begin{aligned} \Delta u &= \alpha(u - g), && \text{in } \Omega \setminus K, \\ \frac{\partial u}{\partial \nu} &= 0, && \text{on } \partial\Omega \cup K. \end{aligned}$$

So, the relevant unknown in free-discontinuity problems is the set K!

SBV functions

The minimization of J in both u and K is not easy because there is NO topology on the closed sets which ensures

- compactness of minimizing sequences and
- lower semicontinuity of the Hausdorff measure.

However, we can think $u \in BV(\Omega) \cap W^{1,2}(\Omega \setminus K)$, with $K = \overline{S}_u$ (the discontinuity set of u), and minimize J in this space. Unfortunately this space is TOO large. It contains Cantor-like functions such that $\nabla u = 0$ a.e. and $\mathcal{H}^{d-1}(S_u) = 0$, dense in $L^2(\Omega)$, so that the problem is trivialized.

SBV functions

Instead it is possible to give a meaningful formulation of the functional J in $SBV(\Omega)$ (the special $BV(\Omega)$ functions, with vanishing Cantor part),

$$\mathcal{F}(u) = \int_{\Omega \setminus S_u} \left[|\nabla u|^2 + \alpha (u-g)^2 \right] dx + \beta \mathcal{H}^{d-1}(S_u).$$

Γ-convergence

Definition

Let (X, d) be a metric space and let $f, f_n : X \to [0, \infty]$ be functions. We say that $(f_n)_n$ Γ-converges to f if the following two conditions are satisfied:

i) for any sequence $(x_n)_n \subset X$ converging to x, the following holds:

$$\liminf_n f_n(x_n) \geq f(x);$$

ii) for any $x \in X$ there exists a sequence $(x_n)_n \subset X$ converging to x such that

$$\limsup_n f_n(x_n) \leq f(x);$$

Eventually the minimizers of f_n do converge to a minimizer of f.

Γ-approximation by Ambrosio and Tortorelli

Let us define

$$J_\varepsilon(u,v) := \int_\Omega \left[v|\nabla u|^2 + \alpha(u-g)^2 \right] + \frac{\beta}{2}\left(\varepsilon |\nabla v|^2 + \frac{(1-v)^2}{\varepsilon} \right) dx.$$

The function $0 \leq v \leq 1$ tends to indicate S_u for $\varepsilon \to 0$. We have

Theorem (Ambrosio-Tortorelli '92)

The functional $J_{\varepsilon_n}(u,v)$ (v depends on u in a suitable way) Γ-converges in $(L^2(\Omega))^2$ to $\mathcal{F}(u)$ for any infinitesimal sequence $(\varepsilon_n)_n$.

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The Blake-Zisserman's discrete functional

Let $d = 2$, $\Omega = [0,1]^2$, and $u_{i,j} = u(hi, hj)$ a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}^2$, for $h > 0$. Define $W_h(t) = \min\{t^2, \beta/h\}$ the truncated quadratic potential and

$$
\begin{aligned}
J_h^\delta(u) &= h^2 \sum_{(hi,hj)\in\Omega_h} W_h\left(\frac{u_{i+1,j}-u_{i,j}}{h}\right) \\
&+ h^2 \sum_{(hi,hj)\in\Omega_h} W_h\left(\frac{u_{i,j+1}-u_{i,j}}{h}\right) \\
&+ \alpha h^2 \sum_{(hi,hj)\in\Omega_h} (u_{i,j}-g_{i,j})^2.
\end{aligned}
$$

The Blake-Zisserman's discrete functional

Theorem (Chambolle '95)

The functional $J_h^\delta(u)$ Γ-converges in $\mathcal{B}(\Omega)$ to

$$\mathcal{E}(u) = \int_{\Omega \setminus S_u} \left[|\nabla u|^2 + \alpha(u-g)^2 \right] dx + \beta \mathcal{C}(S_u),$$

for $h \to 0$, where $\mathcal{C}$ is the so-called "cab-driver" measure.

Basically $\mathcal{C}$ measures the length of a curve only through its projections along horizontal and vertical axes; for a regular C^1 curve $c = \gamma([0,1])$, with $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \Omega$, we have

$$\mathcal{C}(c) = \int_0^1 |\gamma_1'(t)| + |\gamma_2'(t)| dt.$$

Outline

The derivation

For simplicity assume $d = 1$. We consider $\Omega = [0, 1]$, and $u_i = u(hi)$ a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}$, for $h > 0$. Define $W_h(t) = \min\{t^2, \frac{1}{2h}\}$ and

$$J_h^\delta(u) = h \sum_{(hi)\in\Omega_h} W_h\left(\frac{u_{i+1} - u_i}{h}\right) + h \sum_{(hi)\in\Omega_h} (u_i - g_i)^2.$$

Again, it is possible to show that this discrete functional Γ-converges to a suitable functional *a la* Mumford-Shah.

The derivation

Let us denote $S_u^\delta = \{hi : \left|\frac{u_{i+1}-u_i}{h}\right| \leq \frac{1}{\sqrt{h}}\}$ and

$$v(i) = \begin{cases} 1 & i \in \Omega_h \setminus S_u^\delta \\ 0 & i \in S_u^\delta \end{cases}$$

Hence, we have

$$\begin{aligned} J_h^\delta(u) &= h \sum_{(hi)\in\Omega_h} v(i) \left|\frac{u_{i+1}-u_i}{h}\right|^2 + \frac{1}{2} \sum_{(hi)\in\Omega_h} (1-v(i))^2 \\ &+ h \sum_{(hi)\in\Omega_h} (u_i - g_i)^2. \end{aligned}$$

The derivation

Inspired by Ambrosio-Tortorelli, we can even dare to consider v as an additional variable

$$
\begin{aligned}
J_h^\delta(u,v) &= h \sum_{(hi)\in\Omega_h} v(i) \left| \frac{u_{i+1}-u_i}{h} \right|^2 + \frac{1}{2} \sum_{(hi)\in\Omega_h} (1-v(i))^2 \\
&+ h \sum_{(hi)\in\Omega_h} (u_i - g_i)^2 .
\end{aligned}
$$

Indeed the miminization w.r.t. $v \geq 0$ for u fixed gives though

$$
v(i) = \begin{cases} 1 - h \left| \frac{u_{i+1}-u_i}{h} \right|^2, & i \in \Omega_h \setminus S_u^\delta \\ 0, & i \in S_u^\delta \end{cases}
$$

The derivation

Plugging this minimal $v \geq 0$ into the functional we obtain

$$J_h^\delta(u,v) = J_h^\delta(u) \underbrace{- \frac{h^2}{2} \sum_{(hi) \in \Omega_h \setminus S_u^\delta} \left| \frac{u_{i+1} - u_i}{h} \right|^4}_{\text{Additional term}}.$$

To make disappear the additional term we need to consider (at least)

$$\mathcal{J}_h^\delta(u,v) = J_h^\delta(u,v) + \frac{h^2}{2} \sum_{(hi) \in \Omega_h \setminus S_u^\delta} \left| \frac{u_{i+1} - u_i}{h} \right|^4.$$

Equivalence of minimizers

Theorem

The u-component of the minimizers (u, v) of the functional

$$
\begin{aligned}
\mathcal{J}_h^\delta(u,v) &= h \sum_{(hi)\in\Omega_h} v(i) \left| \frac{u_{i+1}-u_i}{h} \right|^2 + \frac{1}{2} \sum_{(hi)\in\Omega_h} (1-v(i))^2 \\
&+ \frac{h^2}{2} \sum_{(hi)\in\Omega_h \setminus S_u^\delta} \left| \frac{u_{i+1}-u_i}{h} \right|^4 \\
&+ h \sum_{(hi)\in\Omega_h} (u_i - g_i)^2,
\end{aligned}
$$

are minimizers of $J_h^\delta(u)$.

Discrete derivative

For $(u(i))_{hi \in \Omega_h}$ we define the discrete derivative as the matrix D_h which maps $(u(i))_{hi \in \Omega_h}$ into $\left(\frac{u(i+1)-u(i)}{h}\right)_i$, given by

$$D_h = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & \dots & -1 & 1 \end{pmatrix}.$$

We have

$$u = D_h^\dagger D_h u + c_u,$$

where $= D^\dagger$ is the pseudo-inverse matrix of D_h and c_u is constant vector. Observe that $\langle D_h^\dagger D_h u, c\rangle = 0$, for any constant c.

A functional of the derivative

We can reformulate the problem in terms of derivatives

$$
\begin{aligned}
\mathcal{J}_h^\delta(z,v) &= h\sum_i v(i)\,|z_i|^2 + \frac{1}{2}\sum_i (1-v(i))^2 \\
&+ \frac{h^2}{2}\sum_{(hi)\in\Omega_h\setminus S_u^\delta} |z_i|^4 \\
&+ h\sum_i (D_h^\dagger z_i - f_i)^2,
\end{aligned}
$$

where $z = D_h u$, $f = D_h^\dagger D_h g$, and $c_u = c_g$ is assumed at the minimizer.

A general discrete functional: surprise ...

On the basis of this derivation, we are interested to study the minimizers of the following general form of functionals

$$\begin{aligned} J(u,v) = \|Tu-g\|^2_{\mathcal{K}} &+ \sum_{\lambda\in\Lambda} v_\lambda |u_\lambda|^p \\ &+ \sum_{\lambda\in\Lambda} \omega_\lambda |u_\lambda|^{2p} \\ &+ \sum_{\lambda\in\Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2, \end{aligned}$$

where Λ is a countable index set, $u = (u_\lambda)_\lambda$ is a vector in $\ell^2(\Lambda)$, $T : \ell^2(\Lambda) \to \mathcal{K}$ is a bounded operator, $g \in \mathcal{K}$, and $1 \le p \le 2$.

Outline

Convexity

Lemma.

Assume $1 < p \leq 2$ and $\rho, \omega, \theta > 0$. If $(\sigma_{\min} + \omega_\lambda)\theta_\lambda \geq \frac{p^2}{4p(2p-1)}$, $\lambda \in \Lambda$, then $J(u, v)$ is convex, and strictly convex for $(\sigma_{\min} + \omega_\lambda)\theta_\lambda > \frac{p^2}{4p(2p-1)}$.

For some initial choice $v^{(0)}$, for example $v^{(0)} \equiv (\rho_\lambda)_{\lambda\in\Lambda}$, define

$$\begin{cases} u^{(n+1)} := \operatorname{argmin}_{u\in\ell_{2p}(\Lambda)} J(u, v^{(n)}) \\ v^{(n+1)} := \operatorname{argmin}_{0\le v_\lambda\le\rho_\lambda} J(u^{(n+1)}, v). \end{cases}$$

This algorithm was already studied for $p = 1$. The analysis for $1 < p \le 2$ is analogous but certain differences arise.

Theorem (Fornasier '07)

Let $1 < p \le \infty$, $\theta_\lambda(\sigma_\lambda + \omega_\lambda) > \frac{p^2}{4p(2p-1)}$, and $\omega_\lambda \ge \gamma > 0$, for all $\lambda \in \Lambda$. Then the sequence $(u^{(n)}, v^{(n)})$ generated by the algorithm converges to $(u^*, v^*) \in \ell_{2p}(\Lambda) \times \ell_{\infty,\rho^{-1}}(\Lambda)$, the unique minimizer of J. The convergence of $u^{(n)}$ is in the weak sense and the convergence of $v^{(n)}$ is componentwise.

Again, the minimization with respect to the second variable is explicit:

$$v_\lambda^{(n)} = \begin{cases} \rho_\lambda - \frac{|u_\lambda^{(n)}|^p}{2\theta_\lambda}, & |u_\lambda^{(n)}|^p \leq 2\theta_\lambda \rho_\lambda \\ 0, & \text{otherwise.} \end{cases}$$

For the minimization with respect to the first variable we have

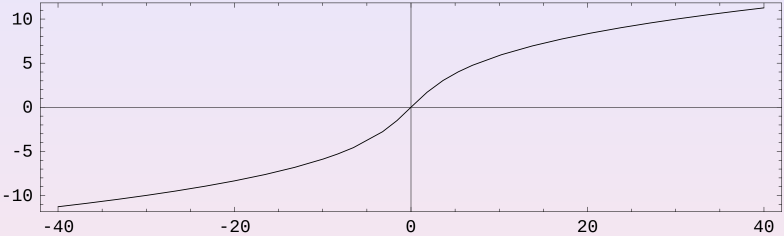
Theorem

If $\|T\| < 1$, $1 < p \leq 2$ and $\omega_\lambda \geq \gamma > 0$, for all $\lambda \in \Lambda$, then for any initial choice $u^{(0)} \in \ell_{2p}(\Lambda)$, the sequence

$$u^{(m+1)} := \mathbb{U}_{v,\omega}^{(p)}\left(u^{(m)} + T^*(g - Tu^{(m)})\right),$$

converges to the minimizer of $J(\cdot, v)$.

The thresholding operator $\mathbb{U}^{(p)}_{v,\omega}$



The operator $\mathbb{U}^{(p)}_{v,\omega}$ acts componentwise by applying curves $U^p_{v,\omega}$ as the one depicted above. It is defined as

$$\begin{aligned}\mathbb{U}^{(p)}_{v,\omega}(z) &:= \operatorname{argmin}_u \mathcal{J}^{surr}(u,v,z)\\ &:= \operatorname{argmin}_u J(u,v) + \|u-z\|^2 - \|T(u-y)\|^2.\end{aligned}$$

The single iteration algorithm

Instead of an alternating minimization we may want to consider:

$$(u^{(n+1)}, v^{(n+1)}) = \text{argmin}_{(u,v)\in\ell_{2p}(\Lambda)\times\ell_{\infty,\rho^{-1}}(\Lambda)} \mathcal{J}^{sur}(u, v, u^{(m)}).$$

Observe that in this iteration the variable v does not play any role.

Theorem (Fornasier '07)

The latter algorithm is equivalent to the following iterative thresholding algorithm:

$$u^{(n+1)} = \mathbb{H}^p_{\omega,\theta,\rho}\left(u^{(n)} + T^*(g - Tu^{(n)})\right).$$

The nonlinear thresholding operator $\mathbb{H}^p_{\omega,\theta,\rho} : \ell_2(\Lambda) \to \ell_2(\Lambda)$ *is defined componentwise* $\mathbb{H}^p_{\omega,\theta,\rho}(\xi) = (H^p_{\omega_\lambda,\theta_\lambda,\rho_\lambda}(\xi_\lambda))_{\lambda\in\Lambda}$, *where for* $z \in \mathbb{R}$,

The single iteration algorithm

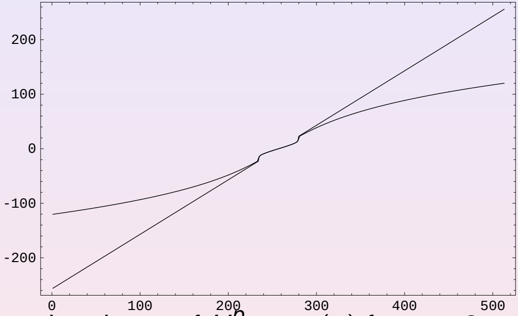
we have

$$H^p_{\omega_\lambda,\theta_\lambda,\rho_\lambda}(z) := \lim_{m\to\infty} z^{(m)},$$

and $z^{(m)}$ is defined by the algorithm:

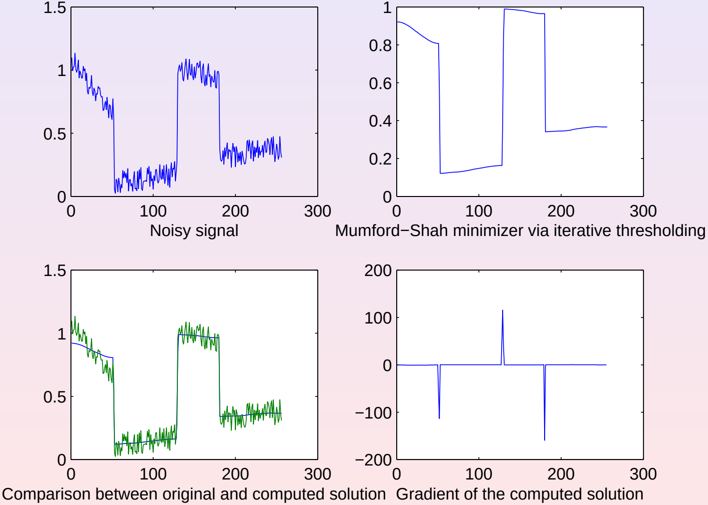
$$\begin{aligned} z^{(m+1)} &= U^p_{v^{(m)},\omega_\lambda}(z), \\ v^{(m+1)} &= \begin{cases} \rho_\lambda - \frac{|z^{(m+1)}|^p}{2\theta_\lambda}, & |z^{(m+1)}|^p \leq 2\theta_\lambda\rho_\lambda \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Typical shape of the function $H^p_{\omega_\lambda,\theta_\lambda,\rho_\lambda}(z)$



We illustrate the shape of $H^p_{\omega_\lambda,\theta_\lambda,\rho_\lambda}(z)$ for $p = 2$ and for different choice of the parameters.

Signal denoising via Mumford-Shah minimization



Signal segmentation via Mumford-Shah minimization

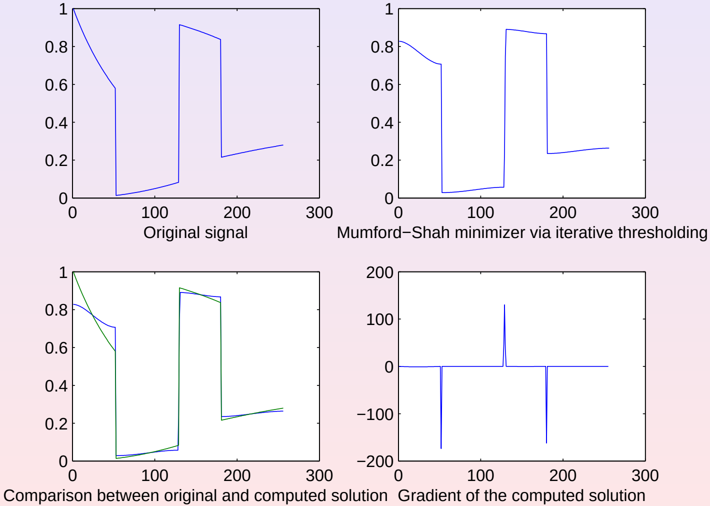


Image segmentation via Mumford-Shah minimization

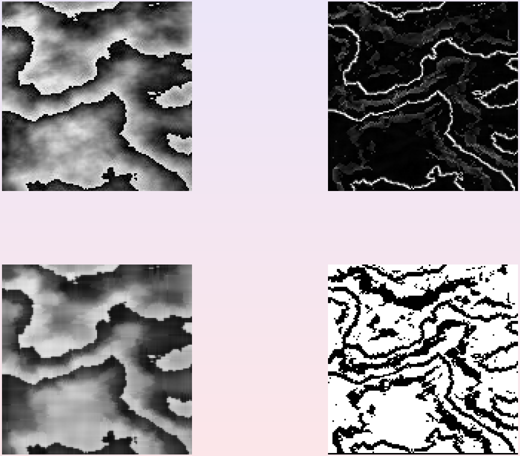


Image denoising via Mumford-Shah minimization

Original image

Noisy image

Mumford−Shah minimizer via iterative thresholding

Discrete discontinuity set

Image segmentation via Mumford-Shah minimization

Original image

The gradient of the computed minimizer

Mumford−Shah minimizer via iterative thresholding

Discrete discontinuity set

Elliptic equation: theoretical setting

- $\Omega \subset \mathbb{R}^d$ Lipschitz domain
- H Hilbert space, $H \subset L_2(\Omega) \subset H'$ Gelfand triple
- $\mathcal{L} : H \to H'$ linear, boundedly invertible

$$a(\cdot,\cdot) := \langle \mathcal{L}\cdot,\cdot\rangle : H \times H \to \mathbb{R}$$

  symmetric, continuous, H-elliptic: $a(v,v) \eqsim \|v\|_H^2$.
- task: for $f \in H'$, solve

$$\mathcal{L}u = f,$$

  i.e., find $u \in H$ s.t.

$$a(u,v) = \langle f,v\rangle, \quad v \in H.$$

Discretization via bases

Cohen, Dahmen, DeVore (1998):

- I choose wavelet basis $\Psi = \{\psi_\lambda\}_{\lambda \in \mathcal{J}} \subset L_2(\Omega)$ with

$$\|\mathbf{c}\|_{\ell_2} \eqsim \|\mathbf{c}^T D^{-1} \Psi\|_H.$$

- This implies

$$\mathcal{L}u = f \Rightarrow \mathbf{L}\mathbf{u} = \mathbf{f},$$

  where

$$\begin{aligned} \mathbf{L} &:= D^{-1}\langle \mathcal{L}\Psi, \Psi \rangle D^{-1}, \\ \mathbf{f} &:= D^{-1}\langle f, \Psi \rangle. \end{aligned}$$

- problem: construction of wavelet bases on general domains difficult/complicated!

Discretization via frames

Dahlke, Fornasier, Raasch, Stevenson (2003-2006)

- choose a frame $\Psi = \{\psi_\lambda\}_{\lambda \in \mathcal{J}} \subset L_2(\Omega)$ with

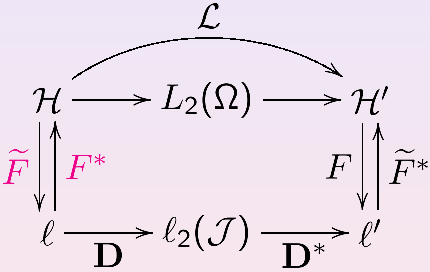
$$\|g\|_{L_2} \eqsim \|\langle g, \Psi \rangle\|_{\ell_2}.$$

- Ψ is a Gelfand frame for (H, L_2, H') if

$$F^* : \ell \to H : c \mapsto c^T \Psi, \quad \tilde{F} : H \to \ell : g \mapsto \langle g, \tilde{\Psi} \rangle$$

are bounded.

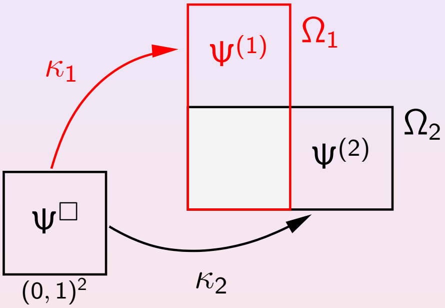
Mapping diagram



Construction of wavelet Gelfand frames (Dahlke, Fornasier, Raasch '04)

- $H = H_0^t(\Omega)$;
- reference Riesz basis $\Psi^\square \subset H_0^t(\square)$, $\square := (0,1)^d$;
- overlapping decomposition $\Omega = \sum_{i=1}^n \Omega_i$;
- $\kappa_i : \square \to \Omega_i$, C^m-diffeomorphisms, $m \geq t$;
- appropriate lifting yields Gelfand frames $\Psi = \bigcup_{i=1}^n \Psi_i$.

Gelfand frame construction



Solution of the discrete problem

The operator $\mathbf{L} : \ell_2(\mathcal{J}) \to \ell_2(\mathcal{J})$ is going to be bounded, symmetric, positive, but has nontrivial kernel. Nevertheless we have

$$\mathbf{L} : ran(L) \to ran(L)$$

is boundedly invertible;

- (ideal) damped Richardson iteration:

$$(R) \quad \mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \mathbf{r}^{(n)}, \quad \mathbf{r}^{(n)} = \mathbf{f} - \mathbf{L}\mathbf{u}^{(n)}$$

- (ideal) steepest descent method

$$(SD) \quad \mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \frac{\langle \mathbf{r}^{(n)}, \mathbf{r}^{(n)} \rangle}{\langle \mathbf{L}\mathbf{r}^{(n)}, \mathbf{r}^{(n)} \rangle} \mathbf{r}^{(n)}.$$

Basic discrete procedures

Assume that we have the following procedures at our disposal:

- **RHS**$[\varepsilon, \mathbf{g}] \to \mathbf{g}_\varepsilon$: determines for $\mathbf{g} \in \ell_2(\mathcal{J})$ a finitely supported $\mathbf{g}_\varepsilon \in \ell_2(\mathcal{J})$ such that
$$\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2(\mathcal{J})} \leq \varepsilon; \tag{1}$$
- **APPLY**$[\varepsilon, \mathbf{N}, \mathbf{v}] \to \mathbf{w}_\varepsilon$: determines for $\mathbf{N} \in B(\ell_2(\mathcal{J}))$ and for a finitely supported $\mathbf{v} \in \ell_2(\mathcal{J})$ a finitely supported $\mathbf{w}_\varepsilon$ such that
$$\|\mathbf{N}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2(\mathcal{J})} \leq \varepsilon; \tag{2}$$
- **COARSE**$[\varepsilon, \mathbf{v}] \to \mathbf{v}_\varepsilon$: determines for a finitely supported $\mathbf{v} \in \ell_2(\mathcal{J})$ a finitely supported $\mathbf{v}_\varepsilon \in \ell_2(\mathcal{J})$ with at most N significant coefficients, such that
$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell_2(\mathcal{J})} \leq \varepsilon. \tag{3}$$

Moreover, $N \lesssim N_{\min}$ holds, $N_{\min}$ being the minimal number

The algorithm

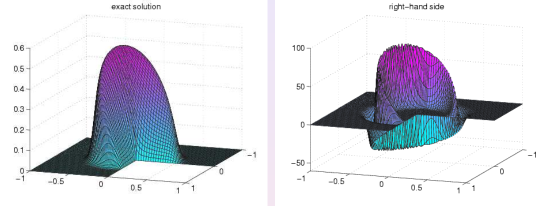
SOLVE$[\varepsilon, \mathbf{L}, \mathbf{f}] \to \mathbf{u}_\varepsilon$:
Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.
$i := 0,\ \mathbf{v}^{(0)} := 0,\ \varepsilon_0 := \|\mathbf{L}^{-1}_{|\operatorname{ran}(\mathbf{L})}\| \|\mathbf{f}\|_{\ell_2(\mathcal{J})}$
While $\varepsilon_i > \varepsilon$ do
$i := i + 1$
$\varepsilon_i := 3\rho^K \varepsilon_{i-1}/\theta$
$\mathbf{f}^{(i)} := \mathbf{RHS}[\frac{\theta\varepsilon_i}{6\alpha K}, \mathbf{f}]$
$\mathbf{v}^{(i,0)} := \mathbf{v}^{(i-1)}$
For $j = 1, ..., K$ do
$\mathbf{v}^{(i,j)} := \mathbf{v}^{(i,j-1)} - \alpha(\mathbf{APPLY}[\frac{\theta\varepsilon_i}{6\alpha K}, \mathbf{L}, \mathbf{v}^{(i,j-1)}] - \mathbf{f}^{(i)})$
od
$\mathbf{v}^{(i)} := \mathbf{COARSE}[(1-\theta)\varepsilon_i, \mathbf{v}^{(i,K)}]$
od
$\mathbf{u}_\varepsilon := \mathbf{v}^{(i)}$.

The proof of convergence and optimality

Convergence and optimal complexity:

- (R) Riesz basis case: Cohen, Dahmen, DeVore (2000)
- (R) Frame case: Stevenson (2003)
- (SD) Riesz basis case: Dahmen, Urban, Vorloeper (2002) Canuto, Urban (2003)
- (SD) Dahlke, Fornasier, Raasch, Stevenson, Werner (2005)

Numerical examples



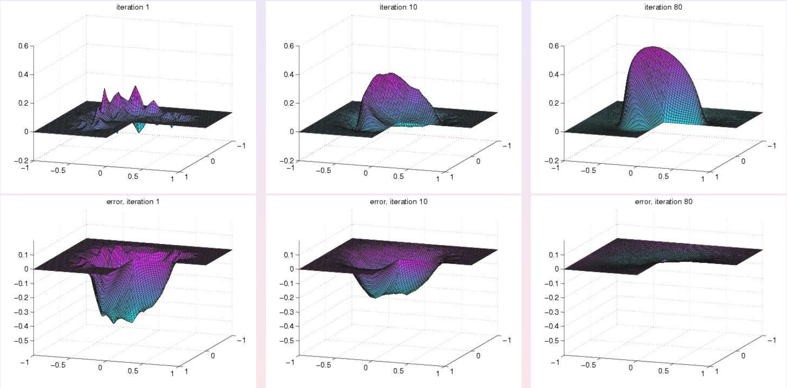
Let us consider the Poisson's equation on the *L*-shaped domain:

$$-\Delta u = f, \quad u|_{\partial\Omega} = 0,$$

and choose f with singularity at the re-entrant corner.

Numerical examples

some (SD) iterations:



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