Iterative Thresholding meets Free-Discontinuity Problems

Massimo Fornasier* and Rachel Ward†

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Abstract

Free-discontinuity problems describe situations where the solution of interest is defined by a function and a lower dimensional set consisting of the discontinuities of the function. Hence, the derivative of the solution is assumed to be a 'small' function almost everywhere except on sets where it concentrates as a singular measure. This is the case, for instance, in crack detection from fracture mechanics or in certain digital image segmentation problems. If we discretize such situations for numerical purposes, the free-discontinuity problem in the discrete setting can be re-formulated as that of finding a derivative vector with small components at all but a few entries that exceed a certain threshold. This problem is similar to those encountered in the field of 'sparse recovery', where vectors with a small number of dominating components in absolute value are recovered from a few given linear measurements via the minimization of related energy functionals. Several iterative thresholding algorithms that intertwine gradient-type iterations with thresholding steps have been designed to recover sparse solutions in this setting. It is natural to wonder if and/or how such algorithms can be used towards solving discrete free-discontinuity problems. The current paper explores this connection, and, by establishing an iterative thresholding algorithm for discrete free-discontinuity problems, provides new insights on properties of minimizing solutions thereof.

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*Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040 Linz, Austria, email: massimo.fornasier@oeaw.ac.at.

†Mathematics Department, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, N.Y. 10012-1185, U.S.A., email: rward@cims.nyu.edu.

1 Introduction

1.1 Free-discontinuity problems: the Mumford-Shah functional

The terminology 'free-discontinuity problem' was introduced by De Giorgi [21] to indicate a class of variational problems that consist in the minimization of a functional, involving both volume and surface energies, depending on a closed set $K \subset \mathbb{R}^d$, and a function u on $\mathbb{R}^d$ usually smooth outside of K. In particular,

- K is not fixed a priori and is an unknown of the problem;
- K is not a boundary in general, but a free-surface inside the domain of the problem.

The best-known example of a free-discontinuity problem is the one modelled by the so-called Mumford-Shah functional [29], which is defined by

$$J(u, K) := \int_{\Omega \setminus K} \left[|\nabla u|^2 + \alpha(u-g)^2 \right] dx + \beta \mathcal{H}^{d-1}(K \cap \Omega).$$

The set Ω is a bounded open subset of $\mathbb{R}^d$, $\alpha, \beta > 0$ are fixed constants, and $g \in L^\infty(\Omega)$. Here $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure. Throughout this paper, the dimension of the underlying Euclidean space $\mathbb{R}^d$ will always be $d = 1$ or $d = 2$. In the context of visual analysis, g is a given noisy image that we want to approximate by the minimizing function $u \in W^{1,2}(\Omega \setminus K)$; the set K is simultaneously used in order to *segment* the image into connected components. For a broad overview on free-discontinuity problems, their analysis, and applications, we refer the reader to [4].

If the set K were fixed, then the minimization of J with respect to u would be a relatively simple problem, equivalent to solving the following system of equations:

$$\begin{aligned} \Delta u &= \alpha(u-g), && \text{in } \Omega \setminus K, \\ \frac{\partial u}{\partial \nu} &= 0, && \text{on } \partial\Omega \cup K, \end{aligned}$$

where ν is the outward-pointing normal vector at any $x \in \partial\Omega \cup K$. Therefore the relevant unknown in free-discontinuity problems is the set K. Ensuring the existence of minimizers (u, K) of J is a challenging problem because there is no topology on the closed sets that ensures

(a) compactness of minimizing sequences and

(b) lower semicontinuity of the Hausdorff measure.

Indeed, it is well-known, by the direct method of calculus of variations [19, Chapter 1], that the two previous conditions ensure the existence of minimizers. However, the problem becomes more manageable if we restrict our domain to functions $u \in BV(\Omega) \cap W^{1,2}(\Omega \setminus K)$, and make the identification $K \equiv \overline{S_u}$ where S_u is the well-defined discontinuity set of u. In this case, we need to work only with a topology on the space $BV(\Omega)$ of bounded variation, and no set topology is anymore required.

Unfortunately the space $BV(\Omega)$ is 'too large'; it contains Cantor-like functions whose approximate gradient vanishes, $\nabla u = 0$, almost everywhere, and whose discontinuity set has measure zero, $\mathcal{H}^{d-1}(S_u) = 0$. As these functions are dense in $L^2(\Omega)$, the problem is trivialized; see [4] for details.

Nevertheless, it is possible to give a meaningful formulation of the functional J if we exclude such functions and restrict J to the space $SBV(\Omega)$ constituted of BV-functions with vanishing Cantor part. If we assume again $K \equiv \overline{S_u}$, the solution can be recasted as the minimization of

$$\mathcal{J}(u) = \int_{\Omega \setminus S_u} \left[|\nabla u|^2 + \alpha(u-g)^2 \right] dx + \beta \mathcal{H}^{d-1}(S_u). \tag{1}$$

The existence of minimizers in SBV for the functional (1) was established by Ambrosio on the basis of his fundamental compactness theorem in [3], see also [4, Theorem 4.7 and Theorem 4.8].

1.2 Γ-convergence approximation to free-discontinuity problems

The discontinuity set S_u of a SBV-function u is not an object that can be easily handled, especially numerically. This difficulty gave rise to the development of approximation methods for the Mumford-Shah functional and its minimizers where *sets* are no longer involved, and instead substituted by suitable *indicator functions*. In order to understand the theoretical basis for these approximations, we need to introduce the notion of Γ-convergence, which is today considered one of the most successful notions of 'variational convergence'; we state only the definition of Γ-convergence below, but refer the reader to [19, 11] for a broad introduction.

Definition 1.1. *Let (X, d) be a metric space*[1] *and let $f, f_n : X \to [0, \infty]$ be functions for $n \in \mathbb{N}$. We say that $(f_n)_{n\in\mathbb{N}}$ Γ-converges to f if the following two conditions are satisfied:*

i) for any sequence $(x_n)_n \subset X$ converging to x,

$$\liminf_n f_n(x_n) \geq f(x);$$

ii) for any $x \in X$, there exists a sequence $(x_n)_n \subset X$ converging to x such that

$$\limsup_n f_n(x_n) \leq f(x).$$

One important consequence of Definition 1.1 is that subject to lower semicontinuity and coercivity conditions, if a sequence of functionals f_n Γ-converges to target functional f, then the corresponding minimizers of f_n also converge to minimizers of f, see [19, Corollary 7.20], see also Proposition 2.5 for an example of such variational

[1] Observe that by [19, Proposition 8.7] suitable bounded sets X endowed with the weak topology induced by a larger Banach space are indeed metrizable, so this condition is not that restrictive.

limits.

We define now

$$F_\varepsilon(u,v) := \int_\Omega \left[v^2|\nabla u|^2 + \alpha(u-g)^2\right] + \frac{\beta}{2}\left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{\varepsilon}\right) dx \tag{2}$$

over the domain $(L^2(\Omega))^2$, along with the related functional

$$\mathcal{J}_\varepsilon(u,v) := \begin{cases} F_\varepsilon(u,v) & \text{, if } v \in W^{1,2}(\Omega),\ uv \in W^{1,2}(\Omega), \text{ and } 0 \le v \le 1, \\ \infty & \text{, else.} \end{cases} \tag{3}$$

Note that at the minimizer (u,v) of $\mathcal{J}_\varepsilon$, the function $0 \le v \le 1$ tends to indicate the discontinuity set S_u of the functional (1) as $\varepsilon \to 0$. In [5] Ambrosio and Tortorelli proved the following Γ-approximation result:

Theorem 1.2 (Ambrosio-Tortorelli '90)**.** *For any infinitesimal sequence $(\varepsilon_n)_n$, the functional $\mathcal{J}_{\varepsilon_n}(u,v)$ Γ-converges in $(L^2(\Omega))^2$ to the functional*

$$\mathcal{J}(u,v) := \begin{cases} \mathcal{J}(u) & \text{, if } v \equiv 1, \\ \infty & \text{, otherwise.} \end{cases} \tag{4}$$

1.3 Discrete approximation

In fact, the Mumford-Shah functional is the continuous version of a previous discrete formulation of the image segmentation problem proposed by Geman and Geman in [26]; see also the work of Blake and Zisserman in [8]. Let us recall this discrete approach. Let $d = 2$ (as for image processing problems), $\Omega = [0,1]^2$, and let $u_{i,j} = u(hi, hj)$, $(i,j) \in \mathbb{Z}^2$ be a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}^2$, for $h > 0$. Define $W_h(t) = \min\left\{t^2, \frac{\beta}{h}\right\}$ to be the truncated quadratic potential, and

$$\begin{aligned} \mathcal{J}_h(u) \quad &:= \quad h^2 \sum_{(hi,hj)\in\Omega_h} W_h\left(\frac{u_{i+1,j} - u_{i,j}}{h}\right) \\ &+ \quad h^2 \sum_{(hi,hj)\in\Omega_h} W_h\left(\frac{u_{i,j+1} - u_{i,j}}{h}\right) \\ &+ \quad \alpha h^2 \sum_{(hi,hj)\in\Omega_h} (u_{i,j} - g_{i,j})^2. \end{aligned} \tag{5}$$

Chambolle [14, 15] gave formal clarification as to how the discrete functional $\mathcal{J}_h$ approximates the continuous functional $\mathcal{J}$ of Ambrosio: discrete sequences can be interpolated by piecewise linear functions in such a way as to allow for discontinuities when the discrete finite differences of the sampling values are large enough. On the basis of this identification of discrete functions on Ω_h and functions defined on the 'continuous domain' Ω, we have the following result:

Theorem 1.3 (Chambolle '95). *The functional* $\mathcal{J}_h$ Γ*-converges in* $\mathcal{B}(\Omega)$ *(the space of Borel-measurable functions, which is metrizable, see [15] for details) to*

$$\mathcal{J}^{cab}(u) = \int_{\Omega \setminus S_u} \left[|\nabla u|^2 + \alpha (u-g)^2 \right] dx + \beta \mathcal{C}(S_u),$$

as $h \to 0$*, where* $\mathcal{C}$ *is the so-called 'cab-driver' measure defined below.*

Basically $\mathcal{C}$ measures the length of a curve only through its projections along horizontal and vertical axes; for a regular C^1 curve $c = \gamma([0,1])$, with $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \Omega$, we have

$$\mathcal{C}(c) = \int_0^1 \left(|\gamma_1'(t)| + |\gamma_2'(t)| \right) dt.$$

The reason this anisotropic (or, direction dependent) measure appears, in place of the Hausdorff measure in the Mumford-Shah functional, is because of the approximation of derivatives by finite differences defined on a 'rigid' squared geometry. A discretization of derivatives based on meshes *adapted* to the morphology of the discontinuity indeed leads to precise approximations of the Mumford-Shah functional [16, 10].

1.4 Free-discontinuity problems and discrete derivatives

In the literature, several methods have been proposed to numerically approximate minimizers of the Mumford-Shah functional [7, 10, 14, 15, 28]. In particular, a relaxation algorithm, based essentially on alternated minimization of a finite element approximation to the Ambrosio and Tortorelli functional (3), leads to iterated solutions of suitable elliptic PDEs, where the differential part includes the auxiliary variable v which encodes and indicates information about the discontinuity set. These implementations are basically finite dimensional approximations to the following algorithm: *Starting with* $v^{(0)} \equiv 1$*, iterate*

$$\begin{cases} u^{(n+1)} := \arg\min_{u \in W^{1,2}(\Omega)} \mathcal{J}_\varepsilon(u, v^{(n)}) \\ v^{(n+1)} := \arg\min_{v \in W^{1,2}(\Omega)} \mathcal{J}_\varepsilon(u^{(n+1)}, v). \end{cases}$$

However, neither has a proof of convergence of this iterative process to its stationary points been explicitly provided in the literature, nor have the properties of such stationary points been investigated, especially in case of genuine inverse problems (see the discussion in Subsection 1.4.3).

In this paper, we take a different approach and investigate how minimization of the Γ-approximating discrete functionals (5) can be implemented efficiently by iterative thresholding on the discrete derivatives. Unlike the aforementioned approach, we will be able to provide a rigorous proof of convergence to stationary points, which coincide with local minimizers of the discrete Mumford-Shah functional. Moreover, we are able to characterize stability properties of such stationary points, and demonstrate the stability of global minimizers of the discrete Mumford Shah functional.

Let us recall: the solutions u of a free-discontinuity problem are supposed to be smooth out of a minimal hypersurface K. This means that the distributional derivative of u is a 'small function' everywhere except on K where it coincides with a singular measure. In the discrete approximation (5), the vector of finite differences $(w_j) = (\frac{u_{i,j+1}-u_{i,j}}{h}, \frac{u_{i+1,j}-u_{i,j}}{h})$ corresponds to a piecewise constant function that is small everywhere except for a few locations, corresponding to $|w_j| \geq \sqrt{\beta/h}$, that approximate the discontinuity set K. So, in terms of derivatives, solutions of (5) are vectors having only few large entries. In the next section, we clarify how we can indeed work with just derivatives and forget the primal problem.

1.4.1 The 1D case

Let us assume presently that the dimension $d = 1$, and the domain $\Omega = [0, 1]$. Denote by $u_i = u(hi)$ a discrete function defined on $hi \in \Omega_h := \Omega \cap h\mathbb{Z}$, for $h > 0$; note that the vector $(u_i) \in \mathbb{R}^n$ for $n = \lfloor 1/h \rfloor$. In this setting, the discrete functional (5) reduces to

$$\begin{aligned} \mathcal{J}_h(u) &= h \sum_{(hi)\in\Omega_h} \min\left\{ \left(\frac{u_{i+1}-u_i}{h}\right)^2, \frac{\beta}{h} \right\} \\ &+ h\alpha \sum_{(hi)\in\Omega_h} (u_i - g_i)^2. \end{aligned}$$

Since no geometrical anisotropy is now involved ($d = 1$), it is possible to show that this discrete functional Γ-converges precisely to the corresponding Mumford-Shah functional on intervals [14]. Moreover, the solution to the discrete functional $\mathcal{J}_h(u)$ may be found in time polynomial in $n = \lfloor 1/h \rfloor$ using dynamic programming [15].

For $(u_i)_{hi\in\Omega_h}$ we define the discrete derivative as the matrix $D_h : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that maps $(u_i)_{hi\in\Omega_h}$ into $\left(\frac{u_{i+1}-u_i}{h}\right)_i$, given by

$$D_h = \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & \dots & -1 & 1 \end{pmatrix}. \tag{6}$$

It is not too difficult to show that

$$u = D_h^\dagger D_h u + c,$$

where $D_h^\dagger$ is the pseudo-inverse matrix of D_h (in the Moore-Penrose sense; note that $D_h^\dagger$ maps $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ and is an injective operator) and c is a constant vector which depends on u, and the values of its entries coincide with the mean value $h \sum_{hi\in\Omega_h} u_i$ of u. Therefore, any vector u is uniquely identified by the pair $(D_h u, c)$.

Since constant vectors comprise the null space of D_h, the orthogonality relation $\langle D_h^\dagger D_h u, c\rangle_{\ell_2^n} = 0$ holds for any vector u and any constant vector c. Here the scalar product $\langle \cdot, \cdot \rangle_{\ell_2^n} = \sum_{i=1}^n u_i v_i$ is the standard Euclidean scalar product on $\mathbb{R}^n$, which induces the Euclidean norm $\|u\|_{\ell_2^n} := \left(\sum_i u_i^2\right)^{1/2}$. We denote ℓ_2^n the Euclidean space $\mathbb{R}^n$ endowed with this norm. Using this orthogonality property, we have that

$$\begin{aligned} \|u-g\|_{\ell_2^n}^2 &= \|D_h^\dagger D_h u - D_h^\dagger D_h g + (c - c_g)\|_{\ell_2^n}^2 \\ &= \|D_h^\dagger D_h u - D_h^\dagger D_h g\|_{\ell_2^n}^2 + \|c - c_g\|_{\ell_2^n}^2 \end{aligned}$$

Hence, with a slight abuse of notation, we can reformulate the original problem in terms of derivatives, and mean values, by

$$\mathcal{J}_h(z,c) \quad = \quad h\alpha \|D_h^\dagger z - f\|_{\ell_2^n}^2 + h\alpha\|c - c_g\|_{\ell_2^n}^2 + h\sum_i \min\left\{|z_i|^2, \frac{\beta}{h}\right\}$$

where $z = D_h u$ and $f = D_h^\dagger D_h g$. Of course at the minimizer u we have $c = c_g$, since this term in $\mathcal{J}_h$ does not depend on z. Therefore, $\|c - c_g\|_2^2$ does not play any role in the minimization and can be neglected. Once the minimal derivative vector z is computed, we can assemble the minimal u by incorporating the mean value of g as follows:

$$u = D_h^\dagger z + c_g.$$

1.4.2 The 2D case, discrete Schwarz conditions, and constrained optimization

Let us assume now $d = 2$, and $\Omega = [0,1]^2$. Denote $u_{i,j} = u(hi, hj)$, $(i,j) \in \mathbb{Z}^2$, a discrete function defined on $\Omega_h := \Omega \cap h\mathbb{Z}^2$, $n = \lfloor 1/h \rfloor$, $W_h(t) = \min\{t^2, \frac{\beta}{h}\}$, and

$$\begin{aligned} \mathcal{J}_h(u) \quad := \quad & h^2 \sum_{(hi,hj)\in\Omega_h} W_h\left(\frac{u_{i+1,j} - u_{i,j}}{h}\right) \\ + \quad & h^2 \sum_{(hi,hj)\in\Omega_h} W_h\left(\frac{u_{i,j+1} - u_{i,j}}{h}\right) \\ + \quad & h^2\alpha \sum_{(hi,hj)\in\Omega_h} (u_{i,j} - g_{i,j})^2. \end{aligned}$$

In two dimensions, we have to consider the derivative matrix $D_h : \mathbb{R}^{n^2} \to \mathbb{R}^{2n(n-1)}$ that maps the vector $(u_{j+(i-1)n}) := (u_{i,j})$ to the vector composed of the finite differences in the horizontal and vertical directions u_x and u_y respectively, given by

$$D_h u := \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad \begin{cases} (u_x)_{j+n(i-1)} := (u_x)_{i,j} := \frac{u_{i+1,j} - u_{i,j}}{h}, i = 1, \dots, n-1, j = 1, \dots, n \\ (u_y)_{j+(n-1)(i-1)} := (u_y)_{i,j} := \frac{u_{i,j+1} - u_{i,j}}{h}, i = 1, \dots, n, j = 1, \dots, n-1 \end{cases}.$$

Note that its range $R(D_h) \subset \mathbb{R}^{2n(n-1)}$ is a (n^2-1)-dimensional subspace because $D_h c = 0$ for constant vectors $c \in \mathbb{R}^{n^2}$. Again, we have the differentiation-integration formula, given by

$$u = D_h^\dagger D_h u + c,$$

where $D_h^\dagger$ is the pseudo-inverse matrix of D_h (in the Moore-Penrose sense); note that $D_h^\dagger$ maps $R(D_h)$ injectively into $\mathbb{R}^{n^2}$. Also, c is a constant vector that depends on u, and the values of its entries coincide with the mean value $h^2 \sum_{(hi,hj)\in\Omega_h} u_{i,j}$ of u.

Proceeding as before and again with a slight abuse of notation, we can reformulate the original discrete functional (5) in terms of derivatives, and mean values, by

$$\mathcal{J}_h(z,c) \quad = \quad h^2\big[\alpha\|D_h^\dagger z - f\|^2_{\ell_2^{n^2}} + \alpha\|c - c_g\|^2_{\ell_2^{n^2}} + \sum_{i,j} \min\left\{|z_{i,j}|^2, \frac{\beta}{h}\right\}\big].$$

where $z = D_h u \in \mathbb{R}^{2n(n-1)}$, and $f = D_h^\dagger D_h g \in \mathbb{R}^{n^2}$. Of course $c = c_g$ is again assumed at the minimizer u, since this latter term in $\mathcal{J}_h$ does not depend on z. However, in order to minimize only over vectors in $\mathbb{R}^{2n(n-1)}$ that are derivatives of vectors in $\mathbb{R}^{n^2}$, we must minimize $\mathcal{J}_h(z,c)$ subject to the constraint $D_h D_h^\dagger z = z$.

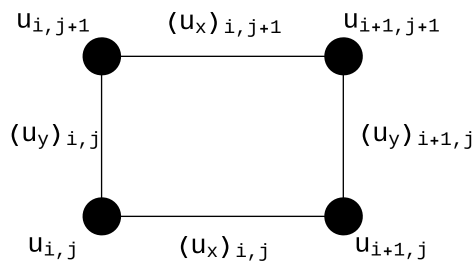


Figure 1: Compatibility conditions of derivatives in 2D.

The $2n(n-1)$ linearly independent constraints $D_h D_h^\dagger z = z$ are equivalent to the *discrete Schwarz constraints*[2],

$$(u_y)_{i,j} + (u_x)_{i,j+1} = (u_y)_{i+1,j} + (u_x)_{i,j}, \tag{7}$$

that establish the equivalence of the length of the paths from $u_{i,j}$ to $u_{i+1,j+1}$, whether one moves in vertical first and then in horizontal direction or in horizontal first and then in vertical direction (see Figure 1).

In short, we arrive at the following constrained optimization problem:

[2]These discrete conditions correspond to the well-known Schwarz mixed derivative theorem for which $\partial_{xy} u = \partial_{yx} u$ for any $u \in C^2(\Omega)$.

$$\begin{cases} \text{Minimize} & \mathcal{J}_h(z) = h^2 \left[\alpha \|Tz - f\|^2_{\ell_2^{n^2}} + \sum_{i,j} \min \left\{ |z_{i,j}|^2, \frac{\beta}{h} \right\} \right]. \\ \\ \text{subject to} & \mathcal{Q}z = 0, \end{cases} \tag{8}$$

for $T = D_h^\dagger$ and $\mathcal{Q} = \mathcal{I} - D_h D_h^\dagger$. Once the minimal derivative vector z is computed, we can assemble the minimal u by incorporating the mean value of g as follows:

$$u = D_h^\dagger z + c_g.$$

1.4.3 Regularization of inverse problems by means of the Mumford-Shah constraint

The Mumford-Shah regularization term

$$MS(u) = \int_{\Omega \setminus S_u} |\nabla u|^2 + \beta \mathcal{H}^{d-1}(S_u), \tag{9}$$

has been used frequently in inverse problems for image processing [22, 32], such as in-painting and tomographic inversion. Despite the successful numerical results observed in the aforementioned papers for the minimization of functionals of the type

$$\mathcal{J}(u) = \alpha \|Ku - g\|_{L^2(\Omega)} + MS(u), \tag{10}$$

where $K : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator which is not boundedly invertible, and $g = K\bar{u} + n$ (n is a stochastic noise), no rigorous results on existence of minimizers are currently available in the literature. Indeed, the Ambrosio compactness theorem [3] used for the proof of the case $K = I$ does not apply in general. A few attempts towards using the regularization MS for inverse problems in fracture detection appear in the work of Rondi [33, 34, 35], although restrictive technical assumptions on the admissible discontinuities of the solutions are required.

As one of the contributions to this paper, we show that discretizations of regularized functionals of the type (10) *always* have minimizers (see Theorem 2.3). More precisely, these discretizations correspond to functionals of the form,

$$\mathcal{J}_h(u) := \alpha h^2 \|Ku - g\|^2_{\ell_2} + h^2 \sum_{(hi,hj) \in \Omega_h} \left[W_h \left(\frac{u_{i+1,j} - u_{i,j}}{h} \right) + W_h \left(\frac{u_{i,j+1} - u_{i,j}}{h} \right) \right], \tag{11}$$

with $W_h(t) = \min \left\{ t^2, \frac{\beta}{h} \right\}$, and we prove that such functionals admit minimizers. Note that the discrete Mumford-Shah approximation (5) can be written in this form. In the one-dimensional situation, we are able to show that such minimizers, along with fixed points of our iterative algorithm, will have components that fall either below one threshold in absolute value, or fall above a second, *strictly* larger threshold, which depend only on the parameters α and β, and not on the data (K, g), see Theorem 5.1. Any a priori information concerning these thresholds may thus be incorporated towards selection of the parameters (α, β). As a consequence of these achievements we can prove that global minimizers are always isolated, although not necessarily unique, whereas local minimizers may constitute a continuum of unstable equilibria. Hence, our analysis will shed light on fundamental properties, virtues, and limitations, of regularization by means of the Mumford-Shah functional MS, and provide a rigorous justification of the numerical results appearing in the literature. The model (11) was considered in the work of Nikolova [30], where she also investigated localization properties of minimizers, assuming their existence, but without addressing any algorithmic issues or means for approximating such minimizers.

It is useful to show how the discrete functional (11) can be still expressed in terms of the sole derivatives for general K. As done before in the case $K = \mathcal{I}$, and with the now usual identification $u = (D_h u, c)$, we can rewrite the functional in terms of derivatives and mean value as follows:

$$\mathcal{J}_h(z, c) \quad = \quad h^2 \alpha \|K D_h^\dagger z - (g - Kc)\|_2^2 + h^2 \sum_{i,j} \min \left\{ |z_{i,j}|^2, \frac{\beta}{h} \right\}, \tag{12}$$

Note that in general we cannot anymore split orthogonally the discrepancy $\|K D_h^\dagger z - (g - Kc)\|_2^2$ into a sum of two terms which depend only on derivatives z and mean value c respectively. Nevertheless, for fixed z, it is straightforward to show that $\bar{c} = \arg\min_c \mathcal{J}_h(z, c)$ depends on z via an affine map. Indeed we can compute

$$\bar{c} = \left(\frac{\langle K\mathbf{1}, g - K D_h^\dagger z \rangle}{\|K\mathbf{1}\|_{\ell^2}^2} \right) \mathbf{1},$$

where $\mathbf{1}$ is the constant vector with entries identically 1. Here we assume that $\mathbf{1} \notin \ker K$, which is necessary in order to identify the mean value of minimizers (a similar condition is required anytime we deal with regularization functionals which depend on the sole derivatives, see, e.g., [17, 37]). By substituting this expression for $\bar{c}$ into (12), it is clear that the minimization of functionals (11) can be reformulated, in terms of the sole derivatives, as constrained minimization problems of the form (8). Otherwise, if $\mathbf{1} \in \ker K$, then $\mathcal{J}_h(z, c) = \mathcal{J}_h(z)$ becomes a function of the derivative vector z only.

2 Existence of minimizers for a class of discrete free-discontinuity problems

2.1 A generalization of the Frank-Wolfe theorem and existence results for inverse free-discontinuity problems

Here and later we use specific symbols associated to our problem.

Notation 2.1. We denote $\mathcal{I}, \mathcal{K}, \mathcal{K}'$ countable index sets, $\ell_2(\mathcal{I})$ is the set of square summable sequences on $\mathcal{I}$, and $\ell_2(\mathcal{K})$, $\ell_2(\mathcal{K}')$ are defined likewise. For us $T : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{K})$ is usually a bounded linear operator. For $r > 0$ and $u \in \ell_2(\mathcal{I})$ fixed we denote $\mathcal{I}_0 := \{i \in \mathcal{I} : |u_i| \leq r\}$ and $\mathcal{I}_1 = \mathcal{I} \backslash \mathcal{I}_0$. Consider the unique decomposition $u = u_0 + u_1$ into a vector u_0 supported on $\mathcal{I}_0$ and another u_1 supported on $\mathcal{I}_1$, i.e., the vectors $u_0 \in \ell_2^{\mathcal{I}_0}(\mathcal{I}) := \{u \in \ell_2(\mathcal{I}) : u_i = 0, \quad i \in \mathcal{I}_1\}$ and $u_1 \in \ell_2^{\mathcal{I}_1}(\mathcal{I}) := \{u \in \ell_2(\mathcal{I}) : u_i = 0, \quad i \in \mathcal{I}_0\}$. Let $\mathcal{P}$ and $\mathcal{P}^{\perp} = \mathcal{I} - \mathcal{P}$ denote the orthogonal projections onto the subspaces $\ell_2^{\mathcal{I}_1}(\mathcal{I})$ and $\ell_2^{\mathcal{I}_0}(\mathcal{I})$, respectively. We fix the notation for the operators $T_0 = T\mathcal{P}^{\perp}$ and $T_1 = T\mathcal{P}$; note that clearly $T = T_0 + T_1$ is satisfied. In the following $T_1^{\dagger}$ will denote the pseudo-inverse operator of T_1 in the Moore-Penrose sense. It is well-known that $\mathcal{P}_1 = T_1 T_1^{\dagger}$ is the orthogonal projection onto the range of T_1.

For this section $\mathcal{I} = \{1, \dots, N\}$, $\mathcal{K} = \{1, \dots, M\}$, and $\mathcal{K} = \{1, \dots, M'\}$, so that $\ell_2(\mathcal{I}) = \ell_2^N$, $\ell_2(\mathcal{K}) = \ell_2^M$, and $\ell_2(\mathcal{K}') = \ell_2^{M'}$. In light of the observations of the previous sections, we can transform the problem of the minimization of functionals of the type (10), by means of discretization first and then reduction to sole derivatives, into the (possibly, but not necessarily) constrained minimization problem:

$$\begin{cases} \text{Minimize} & \mathcal{J}(u) = \left[\|Tu - g\|_{\ell_2(\mathcal{K})}^2 + \gamma \sum_{i \in \mathcal{I}} \min\left\{|u_i|^2, r^2\right\}\right]. \\ \text{subject to} & \mathcal{Q}u = 0. \end{cases} \tag{13}$$

Our first result ensures the existence of minimizers for the constrained optimization problem (13):

Proposition 2.2. *Assume $r, \gamma > 0$, and fix linear operators $T : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{K})$ and $\mathcal{Q} : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{K}')$, which are identified in the following with their matrices with respect to the canonical bases. We also fix $g \in \ell_2(\mathcal{K})$. The constrained minimization problem*

$$\begin{cases} \textit{Minimize} & \mathcal{J}(u) = \left[\|Tu - g\|_{\ell_2(\mathcal{K})}^2 + \gamma \sum_{i \in \mathcal{I}} \min\left\{|u_i|^2, r^2\right\}\right] \\ \textit{subject to} & \mathcal{Q}u = 0. \end{cases} \tag{14}$$

has minimizers u^*.

Proof. We begin by noting that $\inf_{\mathcal{Q}u=0} \mathcal{J}(u)$ is well-defined and finite, since $\mathcal{J} \geq 0$ is bounded from below. It remains to show that there exists a vector u^* that satisfies $\mathcal{J}(u^*) = \inf_{\mathcal{Q}u=0} \mathcal{J}(u)$. Towards this goal, consider the following partition $\mathscr{P} = \{\mathcal{U}_{\mathcal{I}_0^j}\}_{j=1}^{2^N}$ of $\mathbb{R}^N$ indexed by the subsets $\mathcal{I}_0^j$ of the index set $\mathcal{I} = \{1, 2, ..., N\}$, as follows:

$$\mathcal{U}_{\mathcal{I}_0^j} := \{u \in \mathbb{R}^N : |u_i| \leq r, i \in \mathcal{I}_0^j, |u_i| > r, i \in \mathcal{I} \setminus \mathcal{I}_0^j\}. \tag{15}$$

The minimization of $\mathcal{J}$ subject to $\mathcal{Q}u = 0$ and constrained to the closure of the subset $\mathcal{U}_{\mathcal{I}_0^j}$ can be reformulated as a quadratic optimization problem, for which the classical Frank-Wolfe theorem [6] guarantees the existence of a minimizer $u(\mathcal{I}_0^j)$. Now, since $\mathbb{R}^N = \cup_j \mathcal{U}_{\mathcal{I}_0^j}$, the minimal value of $\mathcal{J}$ subject to $\mathcal{Q}u = 0$ and over all of $\mathbb{R}^N$ is just the minimal value from the finite set $\{\mathcal{J}(u(\mathcal{I}_0^j)) : \quad j = 1, \dots, 2^N\}$; that is,

$$\min_{\mathcal{Q}u=0} \mathcal{J}(u) \quad = \quad \min_{\mathcal{I}_0^j \subset \mathcal{I}} \mathcal{J}(u(\mathcal{I}_0^j))$$

and $u^* = \arg\min_{\mathcal{Q}u=0} \mathcal{J}(u) = u\big(\arg\min_{\mathcal{I}_0^j \subset \mathcal{I}} \mathcal{J}(u(\mathcal{I}_0^j))\big)$. $\square$

In fact, Proposition 2.2 extends to a much larger class of free-discontinuity type minimization problems; by the same reasoning as before, we arrive at the more general result:

Theorem 2.3. *The constrained minimization problem*

$$\begin{cases} \textit{Minimize} & \mathcal{J}_p(u) = \left[\|Tu - g\|^2_{\ell_2(\mathcal{K})} + \gamma \sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\} \right] \\ \textit{subject to} & \mathcal{Q}u = 0. \end{cases} \tag{16}$$

has minimizers u^* for any real-valued parameter $p \geq 1$.

The Frank-Wolfe theorem, which guarantees the existence of minimizers for quadratic programs with bounded objective function, does not apply to the general case $p \geq 1$ where the objective function $\mathcal{J}_p$ is not necessarily quadratic. Nevertheless, with the following generalization for the Frank-Wolfe theorem, Theorem 2.3 follows from an argument similar to that used for Proposition 2.2.

Proposition 2.4. *Suppose A is an $N \times N$ positive semidefinite matrix, and suppose b and c are $N \times 1$ vectors. Suppose also that X is a nonempty convex polyhedral subset of $\mathbb{R}^N$. The convex optimization problem*

$$\begin{cases} \textit{minimize} & u^t A u + b^t u + \sum_{1 \leq j \leq N} c_j |u_j|^p \\ \textit{subject to} & u \in X. \end{cases} \tag{17}$$

admits minimizers for any real parameter $p \geq 1$, as long as the objective function is bounded from below.

For ease of presentation, we reserve the proof of Proposition 2.4 to the Appendix.

From the proof of Proposition 2.2, one could in principle obtain a minimizer for $\mathcal{J}$ by computing a minimizer $u(\mathcal{I}_0^j)$ for each subset $\mathcal{I}_0^j \subset \mathcal{I}$ using a quadratic program solver [6], and then minimizing $\mathcal{J}_p$ over the finite set of points $\{u(\mathcal{I}_0^j)\}$. Unfortunately, this algorithm is computationally infeasible as the number of subsets of the index set $\{1, 2, ..., N\}$ grows exponentially with the dimension N of the underlying space. Indeed, the minimization problem (16) was recently shown to be NP-hard, as the known NP-complete problem SUBSET-SUM can be reduced to this problem. This stands in contrast to the case $T = D_h^\dagger$ corresponding to the one-dimensional discrete Mumford-Shah functional, where minimizers can be found in polynomial time using dynamic programming algorithms [15]. A complete discussion about the NP-hardness of (16) can be found in [2].

2.2 A connection to sparse recovery

In Theorem 5.1, we will show that the truncated quadratic regularization term

$$\sum_{i \in \mathcal{I}} \min\{|u_i|^2, r^2\}$$

promotes segmentation of solutions, in the sense that minimizers of the functional $\mathcal{J}$ without constraints (i.e., $\mathcal{Q} = 0$ in (14)) have components u_i, which either are smaller than a first threshold $|u_i| \le (1+\gamma)^{-1/2} r$, for $i \in \mathcal{I}_0$, or larger than a second threshold $|u_i| \ge (1+\gamma)^{1/2} r$, for $i \in \mathcal{I}_1$. If γ and r are chosen such that the lower threshold tends to zero, while the upper threshold stays fixed, then one recovers solutions that tend to be *sparse*, or have only a small number of nonzero coordinates. We will say that a vector $u \in \ell_2(\mathcal{I})$ is K*-sparse* if it has at most K nonzero coordinates. We use the counting measure $\|u\|_{\ell_0(\mathcal{I})} := \sum_{i\in\mathcal{I}} |u_i|_0$, defined component-wise by

$$|u_i|_0 = \begin{cases} 0, & \text{if } u_i = 0 \\ 1, & \text{otherwise} \end{cases}$$

to refer to the number of nonzero coordinates of u. Then we have

Proposition 2.5. *Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive scalars satisfying $\lim_{n\to\infty} r_n = 0$, and fix $\rho > 0$. Then the sequence of functionals*

$$\mathcal{J}_p^n(u) = \|Tu - g\|^2_{\ell_2(\mathcal{K})} + \left(\frac{\rho}{r_n^p}\right) \sum_{i\in\mathcal{I}} \min\{|u_i|^p, r_n^p\}, \tag{18}$$

Γ-converges in $\ell_2(\mathcal{I})$ (see Definition 1.1) to the sparsity-promoting functional

$$\mathcal{J}_0(u) = \|Tu - g\|^2_{\ell_2(\mathcal{K})} + \rho \|u\|_{\ell_0(\mathcal{I})}. \tag{19}$$

Furthermore, for a fixed $K \in \mathbb{N}$, assume that there exists $\sigma > 0$ for which

$$\sigma \|u\|_{\ell_2(\mathcal{I})} \le \|Tu\|_{\ell_2(\mathcal{K})}, \text{ for all } u \text{ such that } \|u\|_{\ell_0(\mathcal{I})} \le K. \tag{20}$$

If $\rho \ge \frac{\|g\|^2_{\ell_2(\mathcal{K})}}{K}$ then there exists a sequence $(u^n)_{n\in\mathbb{N}}$ of minimizers u^n of $\mathcal{J}_n^p$ whose accumulation points are minimizers of $\mathcal{J}_0$.

Proof. First of all note that $\frac{1}{r_n^p} \min\{|t|^p, r_n^p\} = \min\left\{\frac{|t|^p}{r_n^p}, 1\right\}$ is an increasing sequence monotonically converging pointwise to $|x|_0$ for all $t \in \mathbb{R}$. Hence, the sequence of functionals $\mathcal{J}_p^n$ is also increasing and monotonically converging pointwise to $\mathcal{J}_0$. The Γ-convergence of $\mathcal{J}_p^n$ to $\mathcal{J}_0$ follows directly from Definition 1.1 by noting that in finite dimensions, $|\mathcal{I}| < \infty$ the functionals are continuous. We address now the second part of the theorem. In order to search for minimizers of $\mathcal{J}_p^n$ we can restrict the problem to $X = \{u \in \ell_2(\mathcal{I}) : \mathcal{J}_p^n(u) \le \mathcal{J}_0(0) = \|g\|^2_{\ell_2(\mathcal{K})}\}$. Note that at a minimizer $u^n \in X$ the functional $\mathcal{J}_p^n$ can be re-written according to

$$\mathcal{J}_p^n({u_0}^n + {u_1}^n) = \|T_0 u_0^n + T_1 u_1^n - g\|^2_{\ell_2(\mathcal{K})} + \left(\frac{\rho}{r_n^p}\right) \|u_0^n\|^p_{\ell_p^{\mathcal{I}_0}(\mathcal{I})} + \rho|\mathcal{I}_1|, \tag{21}$$

where $\mathcal{I}_0, \mathcal{I}_0, T_0, T_1$ are defined associated to $r_n > 0$ and u^n as in Notation 2.1. From (21) and (20) we deduce

(a) $|\mathcal{I}_1| \leq \frac{\|g\|^2_{\ell_2(\mathcal{K})}}{\rho} \leq K$;

(b) $u_1^n = T_1^\dagger(g - T_0 u_0^n) = \left(T_1^* T_1\right)^{-1} T_1^*(g - T_0 u_0^n)$ and $\|u_1^n\|_{\ell_2(\mathcal{I})} \leq \sigma^{-1}\|T\|(\|g\|_{\ell_2(\mathcal{K})} + \|T\|\|u_0^n\|_{\ell_2(\mathcal{I})})$;

(c) $\|u_0^n\|^2_{\ell_2(\mathcal{I})} \leq C\|u_0^n\|^p_{\ell_p^{\mathcal{I}_0}(\mathcal{I})} \leq \frac{\|g\|^2_{\ell_2(\mathcal{K})} r_n^p}{\rho}$, for a constant $C > 0$.

Hence $u_n = u_0^n + u_1^n$ is uniformly bounded with $u_0^n \to 0$ for $n \to \infty$. We can extract subsequences of $(u_n)_n$ which converge to accumulation points. For Γ-convergence of $\mathcal{J}_p^n$ to $\mathcal{J}_0$, the accumulation points are necessarily minimizers of $\mathcal{J}_0$, see also [19, Corollary 7.20]. Note that the minimizers of $\mathcal{J}_0$ so constructed are K-sparse. □

In this light, the free-discontinuity functional $\mathcal{J}_p$ can be viewed as a relaxation of the sparsity-promoting functional $\mathcal{J}_0$. Note that condition (20) is related to the concept of *restricted isometry property* introduced in the context of recovery methods for sparse vectors from a few linear measurements [13].

3 The minimization problem in a general Hilbert space

The problem we address in the following section is the minimization of the real-valued functional $\mathcal{J}^p : \ell_2(\mathcal{I}) \to \mathbb{R}$ having the form

$$\mathcal{J}_p(u) = \|Tu - g\|^2_{\ell_2(\mathcal{K})} + \gamma \sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\}, \tag{22}$$

subject to the conditions:

- $\mathcal{I}$ and $\mathcal{K}$ are countable sets of indices, and $T : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{K})$ is a bounded linear operator, which is in the following identified with its matrix associated to the canonical basis or any Hilbert basis;
- As in [20], the operator T has spectral norm $\|T\| < 1$. Note that this requirement is easily met by an appropriate scaling for the functional, i.e., we may have to consider instead

  $$\mathcal{J}_p(u) = \tau\|Tu - g\|^2_{\ell_2(\mathcal{K})} + \tau\gamma \sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\}, \quad \tau \leq 1.$$

  Throughout this paper we assume, without loss of generality, that $\tau = 1$;
- the parameter p is in the range $1 \leq p \leq 2$. In case the index set $\mathcal{I}$ is *finite*, only the restriction $p \geq 1$ is necessary.

The scaled 1D discrete Mumford-Shah functional $\frac{1}{h}\mathcal{J}_h$ is clearly a functional of the form (22) having $p = 2$, $r = \sqrt{\beta/h}$, $\gamma = 1/\alpha$, index set $\mathcal{I} = \{1, \ldots, \lfloor r^2 \rfloor\}$, and operator $T = D^\dagger_{1/r^2} : \mathbb{R}^{\lfloor r^2 \rfloor - 1} \to \mathbb{R}^{\lfloor r^2 \rfloor}$. As shown in the Appendix, the operators $D^\dagger_{1/r^2}$ satisfy the uniform bound $\|D^\dagger_{1/r^2}\| \leq 1/2$, independent of dimension, so a scaling factor τ is not needed in this case. However, we would like to stress a few motivations for choosing a Hilbert space setting, with countable sets of indexes $\mathcal{I}$, $\mathcal{K}$, which adds a level of difficulty to some of our results. On the one hand, we are required to produce an analysis which is dimension-independent; as mentioned above, the finite dimensional problem under consideration is NP-hard and therefore it is a very challenging issue in high-dimension. On the other hand, the countable Hilbert space setting allows us to use the functional (22) in situations beyond a finite element discretization of the Mumford-Shah regularization; it can be applied, for instance, as a model for input $u \in \ell_2(\mathcal{I})$ representing the coefficient vector of a functional solution with respect to a countable Hilbert basis, for instance, with respect to wavelets. In fact, while ℓ_1-minimization of wavelet coefficients in inverse problems, as analyzed in [20], necessarily promotes a purely sparse solution, with finitely many nonzero coefficients in u, the truncated polynomial constraint $\sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\}$ tends to promote solutions which are only well-approximated by sparse vectors. In practice the latter regularization property is much more realistic, and therefore it is of interest to study its properties and its associated numerical methods.

Remark 1. Although we proved the main existence theorem, Theorem 2.2, in the finite-dimensional setting, we can provide certain examples where free-discontinuity problems are also guaranteed to have minimizers in the infinite-dimensional setting. The simplest case is reported as follows. Assume $\mathcal{K} = \mathcal{I}$ and T is a bounded diagonal operator with respect to the canonical basis; that is, $Tu = (a_i u_i)_{i \in \mathcal{I}}$ and $a = (a_i)_{i \in \mathcal{I}} \in \ell_\infty(\mathcal{I})$. In this case, minimization of $\mathcal{J}_p$ decouples in the variables u_i, so that the computation of the minimizer $u^* = (u_i^*)_{i \in \mathcal{I}}$ can be solved component-wise by an application of Proposition 4.3 to the scalar problems

$$u_i^* = \arg\min_{t \in \mathbb{R}} \Big((a_i t - g_i)^2 + \gamma \min\{|t|^p, r^p\} \Big), \quad i \in \mathcal{I},$$

and without much trouble one verifies that u^* so defined belongs to $\ell_2(\mathcal{I})$. Hence legitimately we will assume later that global minimizers of $\mathcal{J}_p$ can exist also for $\mathcal{I}$ being a countable set.

4 An iterative thresholding algorithm for 1D free-discontinuity inverse problems

In the following, we will not minimize $\mathcal{J}_p$ directly. Instead, we propose a *forward-backward* or *majorization-minimization* algorithm of Douglas-Rachford type [27] for finding solutions to $\mathcal{J}_p$, motivated by the recent application of such algorithms for minimizing energy functionals arising in sparse signal recovery and image denoising [9, 20]. More precisely, consider the following *surrogate* objective function,

$$\mathcal{J}_p^{surr}(u, a) := \mathcal{J}_p(u) - \|Tu - Ta\|^2_{\ell_2(\mathcal{K})} + \|u - a\|^2_{\ell_2(\mathcal{I})}, \quad u, a \in \ell_2(\mathcal{I}). \tag{23}$$

The surrogate functional $\mathcal{J}_p^{surr}$ satisfies $\mathcal{J}_p^{surr}(u,a) \geq \mathcal{J}_p(u)$ everywhere, with equality if and only if $u = a$, and is such that the sequence

$$u^{n+1} = \arg\min_u \mathcal{J}_p^{surr}(u, u^n) \tag{24}$$

obtained by successive minimizations of $\mathcal{J}_p^{surr}(u,a)$ in u for fixed a results in a nonincreasing sequence of the original functional $\mathcal{J}_p(u^n)$ (see Lemmas 4.1 and 4.2). Moreover, the functional $\mathcal{J}_p^{surr}(u,a)$ decouples the variables u_j, so that $u^{n+1} = \arg\min_u \mathcal{J}_p^{surr}(u, u^n)$ reduces to component-wise thresholding.

We will study the implementation and convergence properties of the iteration (24) as follows:

- in Section 4.1 , we review the standard properties of forward-backward iterations,
- in Section 4.2, we compute u-global minimizers of the surrogate functional $\mathcal{J}_p^{surr}(u,a)$, for a fixed,
- in Section 4.3, Section 4.4, and Section 4.5 we show that the sequence $(u^n)_{n\in\mathbb{N}}$ defined by (24) will converge to a stationary value $\bar{u} = \arg\min_u \mathcal{J}_p^{surr}(u,\bar{u})$, starting from any initial value u^0 for which $\mathcal{J}_p(u^0) < \infty$,
- in Section 5, we show that such stationary values $\bar{u}$ are also local minimizers of the original functional $\mathcal{J}_p$ that satisfy a certain fixed point condition, and it is shown that any global minimizer of $\mathcal{J}_p$ is among the set of possible fixed points $\bar{u}$ of the iteration (24). By means of the thresholding properties, we also show that global minimizers of the functional $\mathcal{J}_p$ are isolated, and moreover possess a certain segmentation property that is also shared by fixed points of the algorithm.

4.1 Preliminary lemmas

The lemmas in this section are standard when using surrogate functionals (see [20] and [9]), and concern general real-valued surrogate functionals of the form

$$\mathcal{F}^{surr}(u,a) = \mathcal{F}(u) - \|Tu - Ta\|^2_{\ell_2(\mathcal{K})} + \|u - a\|^2_{\ell_2(\mathcal{I})}. \tag{25}$$

The lemmas in this section hold independent of the specific form of the functional $\mathcal{F} : \ell_2(\mathcal{I}) \to \mathbb{R}^+$, but do rely on the restriction that $\|T\| < 1$.

Lemma 4.1. *If the real-valued functionals $\mathcal{F}(u)$ and $\mathcal{F}^{surr}(u,a)$ satisfy the relation (25) and the sequence $(u^n)_{n\in\mathbb{N}}$ defined by $u^{n+1} = \arg\min_{u\in\ell_2(\mathcal{I})} \mathcal{F}^{surr}(u,u^n)$ is initialized in such a way that $\mathcal{F}(u^0) < \infty$, then the sequences $\mathcal{F}(u^n)$ and $\mathcal{F}^{surr}(u^{n+1},u^n)$ are non-increasing as long as $\|T\| < 1$.*

Proof. Since $\|T\| < 1$, also $\|T^*T\| < 1$, and so the operator $L = \sqrt{I - T^*T}$ is a well-defined positive operator whose spectrum is contained within a closed interval

$[c, 1]$ that is bounded away from zero $c > 0$. We can then rewrite $\mathcal{F}^{surr}(u^{n+1}, u^n)$ as $\mathcal{F}^{surr}(u^{n+1}, u^n) = \mathcal{F}(u^{n+1}) + \|L(u^{n+1} - u^n)\|^2_{\ell_2(\mathcal{I})}$, from which it follows that

$$\begin{aligned}
\mathcal{F}(u^{n+1}) &\leq \mathcal{F}(u^{n+1}) + \|L(u^{n+1} - u^n)\|^2_{\ell_2(\mathcal{I})} \\
&= \mathcal{F}^{surr}(u^{n+1}, u^n) \\
&\leq \mathcal{F}^{surr}(u^n, u^n) \\
&= \mathcal{F}(u^n) \\
&\leq \mathcal{F}(u^n) + \|L(u^n - u^{n-1})\|^2_{\ell_2(\mathcal{I})} \\
&= \mathcal{F}^{surr}(u^n, u^{n-1}),
\end{aligned} \tag{26}$$

where the second inequality follows from u^{n+1} being a minimizer of $\mathcal{F}^{surr}(u, u^n)$. □

From Lemma 4.1 we obtain the following corollary:

Lemma 4.2. *As long as the conditions of Lemma* 4.1 *are satisfied, one can choose* $N \in \mathbb{N}$ *sufficiently large such that for all* $n \geq N$, $\|u^{n+1} - u^n\|_{\ell_2(\mathcal{I})} \leq \epsilon$, *i.e.,*

$$\lim_{n \to \infty} \|u^{n+1} - u^n\|_{\ell_2(\mathcal{I})} = 0.$$

Proof. From Lemma 4.1, it follows that $\mathcal{F}(u^n) \geq 0$ is a nonincreasing sequence, therefore it converges, and $\mathcal{F}(u^n) - \mathcal{F}(u^{n+1}) \to 0$ for $n \to \infty$. The lemma follows from (26), and the estimates

$$\mathcal{F}(u^n) - \mathcal{F}(u^{n+1}) \geq \|L(u^{n+1} - u^n)\|^2_{\ell_2(\mathcal{I})} \geq (1 - \|T\|^2)\|u^{n+1} - u^n\|^2_{\ell_2(\mathcal{I})}.$$

□

4.2 The surrogate functional $\mathcal{J}_p^{surr}$ and a new thresholding operator

We now develop an explicit representation for the iterative thresholding algorithm,

$$u^{n+1} = \arg\min_u \mathcal{J}_p^{surr}(u, u^n),$$

with surrogate functional

$$\mathcal{J}_p^{surr}(u, a) = \mathcal{J}_p(u) - \|Tu - Ta\|^2_{\ell_2(\mathcal{K})} + \|u - a\|^2_{\ell_2(\mathcal{I})}, \quad u, a \in \ell_2(\mathcal{I}),$$

and

$$\mathcal{J}_p(u) = \|Tu - g\|^2_{\ell_2(\mathcal{K})} + \gamma \sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\}.$$

Expanding the squared terms on the right hand side of the expression for $\mathcal{J}_p^{surr}$,

$$\begin{aligned}
\mathcal{J}_p^{surr}(u, a) &= \|u - (I - T^*T)a + T^*g\|^2_{\ell_2(\mathcal{I})} + \gamma \sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\} + C \\
&= \sum_{i \in \mathcal{I}} \Big[(u_i - [a - T^*Ta + T^*g]_i)^2 + \gamma \min\{|u_i|^p, r^p\}\Big] + C,
\end{aligned}$$

where the term $C = C(T, a, g)$ depends only on T, a and g. It is now clear that the surrogate functional $\mathcal{J}_p^{surr}$ decouples in the variables u_i, due to the cancellation of terms involving $\|Tu\|^2_{\ell_2(\mathcal{K})}$. Because of this decoupling, global u-minimizers of $\mathcal{J}_p^{surr}(u, a)$, for a fixed, can be computed *component-wise* according to

$$\bar{u}_i = \arg\min_{t \in \mathbb{R}} \left[(t - [a - T^*Ta + T^*g]_i)^2 + \gamma \min\{|t|^p, r^p\} \right], \quad i \in \mathcal{I}. \tag{27}$$

One can solve (27) explicitly when, e.g., $p = 2$, $p = 3/2$, and $p = 1$; in the general case $p \geq 1$, we have the following result:

Proposition 4.3 (Minimizers of $\mathcal{J}_p^{surr}(u, a)$ for a fixed)**.**

1. *If $p > 1$, the minimization problem $\bar{u} = \arg\min_{u \in \ell_2(\mathcal{I})} \mathcal{J}_p^{surr}(u, a)$ can be solved component-wise by*

$$\bar{u}_i = H_p([a - T^*Ta + T^*g]_i), \quad i \in \mathcal{I}, \tag{28}$$

where $H_p : \mathbb{R} \to \mathbb{R}$ is the 'thresholding function',

$$H_p(\lambda) = \begin{cases} F_p^{-1}(\lambda), & |\lambda| \leq \lambda'(r, \gamma, p) \\ \lambda, & \textit{else} \end{cases}$$

Here, $F_p^{-1}(\lambda)$ is the inverse of the function $F_p(t) = t + \frac{\gamma p}{2} \operatorname{sgn} t |t|^{p-1}$, and $\lambda' \in (r, r + \frac{\gamma p}{2} r^{p-1})$ is the unique positive value at which

$$(F_p^{-1}(\lambda') - \lambda')^2 + \gamma |F_p^{-1}(\lambda')|^p = \gamma r^p. \tag{29}$$

2. *When $p = 1$, the general form* (28) *still holds, but we have to consider two cases:*

   (a) If $r > \gamma/4$, the thresholding function $H_1 : \mathbb{R} \to \mathbb{R}$ satisfies

$$H_1(\lambda) = \begin{cases} 0, & |\lambda| \leq \gamma/2 \\ (|\lambda| - \gamma/2) \operatorname{sgn} \lambda, & \gamma/2 < |\lambda| \leq r + \gamma/4 = \lambda' \\ \lambda, & |\lambda| > r + \gamma/4 \end{cases} \tag{30}$$

   (b) If, on the other hand, $r \leq \gamma/4$, the function H_1 satisfies

$$H_1(\lambda) = \begin{cases} 0, & |\lambda| \leq \sqrt{\gamma r} = \lambda' \\ \lambda, & |\lambda| > \sqrt{\gamma r} \end{cases} \tag{31}$$

In all cases, the thresholding function is continuous except at $\lambda'(r, \gamma, p)$, where it has a jump-discontinuity of size $\delta = |\lambda' - H_p(\lambda')| > 0$ if $r, \gamma > 0$. In particular, $\lambda' > r$ while $H_p(\lambda') < r$.

We leave the proof of Proposition 4.3 to the Appendix. It will be useful later to extend the function F_p^{-1} at the value $p = 1$ by the identification $F_1^{-1} = S_\gamma$, where

$$S_\gamma(\lambda) = \begin{cases} 0, & |\lambda| \leq \gamma/2 \\ (|\lambda| - \gamma/2) \operatorname{sgn} \lambda, & |\lambda| > \gamma/2, \end{cases} \tag{32}$$

is the so-called *soft-thresholding* function [20].

Remark 2. In the particular case $p = 2$ corresponding to Mumford-Shah regularization (13), the thresholding function $H_2 : \mathbb{R} \to \mathbb{R}$ has a particularly simple explicit form:

$$H_2(\lambda) = \begin{cases} \frac{\lambda}{1+\gamma}, & |\lambda| \leq (1+\gamma)^{1/2} r \\ \lambda, & else \end{cases} \tag{33}$$

In addition to H_2 and H_1, the thresholding operator $H_{3/2}$ corresponding to $p = 3/2$ can also be computed explicitly, by solving for the positive root of a suitable polynomial of third degree. In Figure 2 below, we plot $H_2, H_{3/2}$, and H_1 with parameters $r = 1$, $\gamma = 1$. For general values of p, H_p cannot be solved in closed form. However, recall the following general properties of H_p:

- H_p is an odd function,
- $H_p(0) = 0$, and
- $H_p(\lambda) = \lambda$ once $|\lambda| > r + \frac{\gamma p}{2} r^{p-1}$.

In fact, we can effectively *pre*compute H_p by numerically solving for the value of $H_p(\lambda_j)$ on a discrete set $\{\lambda_j\}$ of points $\lambda_j \in (0, \frac{\gamma p}{2} r^{p-1} + r]$. At λ_j, one just needs to solve the real equation

$$h_j + \frac{\gamma p}{2} h_j^{p-1} - \lambda_j = 0 \tag{34}$$

which can be computed effortlessly via a root-finding procedure such as Newton's method: while h_j satisfies $(h_j - \lambda_j)^2 + \gamma (h_j)^p \leq \gamma r^p$, set $H_p(\lambda_j) = h_j$; once this constraint is violated, set $H_p(\lambda_j) = \lambda_j$.

To summarize, the iterative algorithm,

$$u^{n+1} \quad = \quad \arg\min_u \mathcal{J}_p^{surr}(u, u^n) \tag{35}$$

can be recast in terms of a component-wise thresholding algorithm,

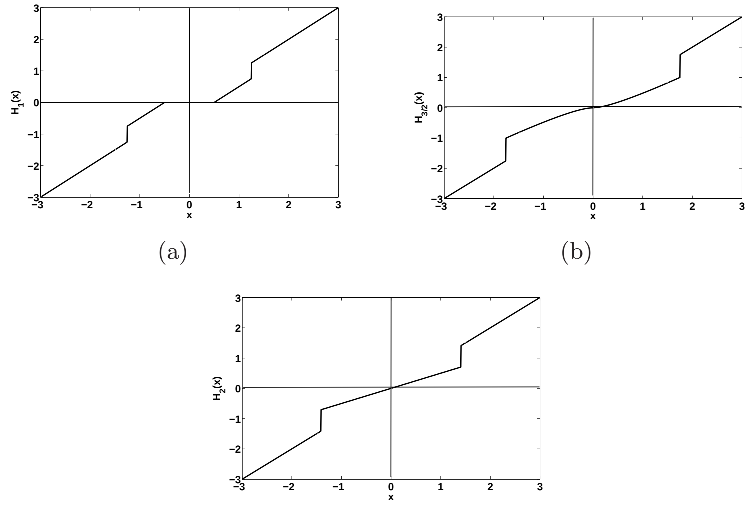
$$u_i^{n+1} \quad = \quad H_p([u^n - T^*Tu^n + T^*g]_i), \tag{36}$$

which, for $p = 2$, reduces to

$$u_i^{n+1} = \begin{cases} (1+\gamma)^{-1}([u^n - T^*Tu^n + T^*g]_i), & |[u^n - T^*Tu^n + T^*g]_i| \leq (1+\gamma)^{1/2} r \\ [u^n - T^*Tu^n + T^*g]_i, & \text{else} \end{cases}$$

Remark 3. The thresholding function H_1 corresponding to $p = 1$ and $r \leq \gamma/4$ is known as *hard thresholding* $\tilde{H}$ in the area of sparse recovery, and is equivalently derived as the thresholding function corresponding to the ℓ_0-regularized functional $\mathcal{J}_0$ with $\rho = \sqrt{r\gamma}$. More generally, one verifies the pointwise convergence of $H_p = H_p(r_n, \gamma_n) = H_p(r_n, \rho r_n^{-p})$ to $\widetilde{H}$ as $r_n \to 0$, reflecting the Γ-convergence of their associated functionals, described in Proposition 2.5.

As convergence of the iterative thresholding algorithm (36) with hard thresholding $\tilde{H}$ was already studied in [9] (albeit in finite dimension), we omit this case in the sequel.



(a) (b) (c)

Figure 2: The discontinuous thresholding functions H_1, $H_{3/2}$, and H_2, with parameters $p = 1, 3/2$, and 2, respectively, and $r = 1$, $\gamma = 1$.

4.3 Fixation of the discontinuity set

To ease notation, we define the operator $\mathbb{H} : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{I})$ by its component-wise action,

$$[\mathbb{H}(u)]_i := H_p([u - T^*Tu + T^*g]_i); \tag{37}$$

the iteration (36) can then be written more concisely in operator notation as

$$u^{n+1} = \mathbb{H}(u^n). \tag{38}$$

We omit the dependence of $\mathbb{H}$ on the parameters p, r, γ, and function g for continuity of presentation. Furthermore, we assume in the sequel that $\|T\| < 1$.

At the core of the convergence proof is the fixation of the 'discontinuity set', indicated below by $\mathcal{I}_1^n$, during the iteration (36), at which point the 'free-discontinuity' problem is transformed into a simpler 'fixed-discontinuity' problem.

Lemma 4.4 (Fixation of the index set $\mathcal{I}_1$)**.** *Fix $p \geq 1, r, \gamma \in \mathbb{R}^+$, and $g \in \ell_2(\mathcal{K})$. Consider the iteration*

$$u^{n+1} = \mathbb{H}(u^n) \tag{39}$$

and the time-dependent partition of the index set $\mathcal{I}$ into 'small' set

$$\mathcal{I}_0^n = \{i \in \mathcal{I} : |u_i^n| \leq \lambda' - \delta\} \tag{40}$$

and 'large' set

$$\mathcal{I}_1^n = \{i \in \mathcal{I} : |u_i^n| > \lambda'\} \tag{41}$$

where $\lambda' = \lambda'(r, \gamma, p)$ is the position of the jump-discontinuity of the thresholding function and δ is the size of its jump-discontinuity, as defined in Proposition 4.3. *For $N \in \mathbb{N}$ sufficiently large, this partition fixes during the iteration $u^{n+1} = \mathbb{H}(u^n)$; that is, there exists a set $\mathcal{I}_0$ such that for all $n \geq N$, $\mathcal{I}_0^n = \mathcal{I}_0$ and $\mathcal{I}_1^n = \mathcal{I}_1 := \mathcal{I} \setminus \mathcal{I}_0$.*

Proof. By *discontinuity* of the thresholding operator $H_p(\lambda)$, each sequence component

$$u_i^n = H_p([u^{n-1} - T^*Tu^{n-1} + T^*g]_i) \tag{42}$$

satisfies

(a) $|u_i^n| \leq \lambda' - \delta < \lambda'(r, p)$, if $i \in \mathcal{I}_0^n$, or

(b) $|u_i^n| > \lambda'$, if $i \in \mathcal{I}_1^n$.

Thus, $|u_i^{n+1} - u_i^n| \geq \delta$ if $i \in \mathcal{I}_0^{n+1} \bigcap \mathcal{I}_1^n$, or if $i \in \mathcal{I}_0^n \bigcap \mathcal{I}_1^{n+1}$. At the same time, Lemma 4.2 implies

$$|u_i^{n+1} - u_i^n| \leq \|u^{n+1} - u^n\|_{\ell_2(\mathcal{I})} \leq \epsilon, \tag{43}$$

once $n \geq N(\epsilon)$, and $\epsilon > 0$ can be taken arbitrarily small. In particular, (43) implies that $\mathcal{I}_0$ and $\mathcal{I}_1$ must be fixed once $n \geq N(\epsilon)$ and $\epsilon < \delta$. □

After fixation of the index set $\mathcal{I}_0 = \{i \in \mathcal{I} : |u_i^n| \leq \lambda'\}$, $\mathbb{H}(u^n) = \mathbb{U}_{\mathcal{I}_0}(u^n)$ and $\mathbb{U}_{\mathcal{I}_0}$ is an operator having component-wise action

$$[\mathbb{U}_{\mathcal{I}_0} u]_i = \begin{cases} F_p^{-1}([(I - T^*T)u + T^*g]_i), & \text{if } i \in \mathcal{I}_0 \\ \left((I - T^*T)u + T^*g\right)_i, & \text{if } i \in \mathcal{I}_1 \end{cases} \tag{44}$$

Here, the function F_p^{-1} is as in Proposition 4.3, with the mentioned extension to soft-thresholding for $p = 1$. Again, for ease of presentation, we omit the dependence of $\mathbb{U}_{\mathcal{I}_0}$ on the parameters p, r, γ, and g. One easily verifies the equivalence

$$\mathbb{U}_{\mathcal{I}_0}(v) = \arg \min_{u \in \ell_2(\mathcal{I})} \mathcal{J}_p^{\mathcal{I}_0, surr}(u, v) \tag{45}$$

where $\mathcal{J}_p^{\mathcal{I}_0, surr}(u, v)$ is a surrogate for the *convex* functional,

$$\mathcal{J}_p^{\mathcal{I}_0}(u) := \|Tu - g\|_{\ell_2(\mathcal{K})}^2 + \gamma \sum_{i \in \mathcal{I}_0} |u_i|^p. \tag{46}$$

That is, fixation of the index set $\mathcal{I}_0$ implies that the sequence $(u^n)_{n \in \mathbb{N}}$ has become constrained to a subset of $\ell_2(\mathcal{I})$ on which the map $\mathbb{H}$ agrees with a map $\mathbb{U}_{\mathcal{I}_0}$, associated to the convex functional $\mathcal{J}_p^{\mathcal{I}_0}$. As we will see, this implies that the nonconvex functional $\mathcal{J}_p$ behaves *locally* like a convex functional in neighborhoods of fixed points $u = \mathbb{H}(u)$, which includes the set of global minimizers of $\mathcal{J}_p$.

4.4 On the nonexpansiveness and convergence for T injective

Given that $\mathbb{H}(u^n) = \mathbb{U}_{\mathcal{I}_0}(u^n)$ after a finite number of iterations, we can use tools from convex analysis to prove that the sequence $(u^n)_{n\in\mathbb{N}}$ converges. If the operator $T^*T : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{I})$ is boundedly invertible (such as the discrete pseudoinverse $D_h^\dagger$ in the 1D Mumford-Shah approximation) then the mapping $\mathbb{U}_{\mathcal{I}_0}$ has the nice property of being a contraction mapping, so that a direct application of the Banach fixed point theorem ensures exponential convergence of the sequence $(u^n)_{n\in\mathbb{N}}$ after fixation of the index sets.

Proposition 4.5. *Suppose $T : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{K})$ is a boundedly invertible operator on its image. In particular, let $\sigma > 0$ be a lower bound on the spectrum of T^*T. Then the sequence*

$$u^{n+1} = \mathbb{H}(u^n), \tag{47}$$

as defined in (38)*, is guaranteed to converge in norm. In particular, after a finite number of iterations $N \in \mathbb{N}$, this mapping takes the form*

$$u^{N+m} = \mathbb{U}^m_{\mathcal{I}_0}(u^N), \quad m \in \mathbb{N}\setminus\{0\}, \tag{48}$$

and the sequence $(u^n)_{n\in\mathbb{N}}$ converges to the unique fixed point $\bar{u}$ of the map $\mathbb{U}_{\mathcal{I}_0}$. Moreover, after fixation of the index set $\mathcal{I}_0$, the rate of convergence becomes exponential:

$$\|u^{N+m} - \bar{u}\|_{\ell_2(\mathcal{I})} = \|\mathbb{U}^m_{\mathcal{I}_0}(u^N) - \mathbb{U}^m_{\mathcal{I}_0}(\bar{u})\|_{\ell_2(\mathcal{I})} \le (1-\sigma)^m \|u^N - \bar{u}\|_{\ell_2(\mathcal{I})}, \quad m \in \mathbb{N}\setminus\{0\}. \tag{49}$$

Proof. The proof is straightforward and it is based on the observation that $\|I - T^*T\| \le 1 - \sigma < 1$, F_p^{-1} is a nonexpansive map, and eventually the map $\mathbb{U}_{\mathcal{I}_0}$ with componentwise action

$$[\mathbb{U}_{\mathcal{I}_0} u]_i \quad = \quad \begin{cases} F_p^{-1}([(I - T^*T)u + T^*g]_i), & \text{if } i \in \mathcal{I}_0 \\ \left((I - T^*T)u + T^*g\right)_i, & \text{if } i \in \mathcal{I}_1 \end{cases} \tag{50}$$

is a contraction, with Lipschitz constant $(1-\sigma)$. □

4.5 Convergence for general operators T

Unfortunately, if T^*T is not boundedly invertible (that is, if $\sigma = 0$), then the map $\mathbb{U}_{\mathcal{I}_0}$ is not necessarily a contraction, and we can no longer apply the Banach fixed point theorem to prove convergence of the sequence $(u^n)_{n\in\mathbb{N}}$. However, as long as $\|T\| < 1$, we observe by a more detailed inspection of the proof of Proposition 4.5 that $\mathbb{U}_{\mathcal{I}_0}$ is still *non-expansive*, meaning that for all $v, v' \in \ell_2(\mathcal{I})$, $\|\mathbb{U}_{\mathcal{I}_0}(v) - \mathbb{U}_{\mathcal{I}_0}(v')\|_{\ell_2(\mathcal{I})} \le \|v - v'\|_{\ell_2(\mathcal{I})}$. The following Opial's theorem [31], here reported adjusted to our notations and context, gives sufficient conditions under which non-expansive maps admit convergent successive iterations:

Theorem 4.6 (Opial's Theorem)**.** *Let the mapping $\mathbb{A}$ from $\ell_2(\mathcal{I})$ to $\ell_2(\mathcal{I})$ satisfy the following conditions:*

1. $\mathbb{A}$ is asymptotically regular: for all $v \in \ell_2(\mathcal{I})$, $\|\mathbb{A}^{n+1}(v) - \mathbb{A}^n(v)\|_{\ell_2(\mathcal{I})} \to 0$ for $n \to \infty$;

2. $\mathbb{A}$ is non-expansive: for all $v, v' \in \ell_2(\mathcal{I})$, $\|\mathbb{A}(v) - \mathbb{A}(v')\|_{\ell_2(\mathcal{I})} \leq \|v - v'\|_{\ell_2(\mathcal{I})}$;

3. the set $\mathrm{Fix}(\mathbb{A})$ of the fixed points of $\mathbb{A}$ in $\ell_2(\mathcal{I})$ is not empty.

Then, for all $v \in \ell_2(\mathcal{I})$, the sequence $(\mathbb{A}^n(v))_{n\in\mathbb{N}}$ converges weakly to a fixed point in $\mathrm{Fix}(\mathbb{A})$.

In fact, we already know that $\mathbb{U}_{\mathcal{I}_0}$ is asymptotically regular, in addition to being non-expansive - this follows by application of Lemma 4.1 and Lemma 4.2 to the functional $\mathcal{J}^p_{\mathcal{I}_0}$. Thus, in order to apply Opial's theorem, it remains only to show that $\mathbb{U}_{\mathcal{I}_0}$ has a fixed point; that is, that there exists a point $\bar{u} \in \ell_2(\mathcal{I})$ for which

$$\bar{u} = \mathbb{U}_{\mathcal{I}_0}(\bar{u}).$$

In more detail, we must prove the existence of a vector $\bar{u} \in \ell_2(\mathcal{I})$ satisfying the conditions

$$\bar{u}_i = \begin{cases} F_p^{-1}([(I - T^*T)\bar{u} + T^*g]_i), & \text{if } i \in \mathcal{I}_0 \\ \big((I - T^*T)\bar{u} + T^*g\big)_i, & \text{if } i \in \mathcal{I}_1, \end{cases} \tag{51}$$

corresponding to the Euler-Lagrange equations for a minimizer of the convex problem (46). We remind the reader that until now, all of the results of Section 4 remain valid in the infinite-dimensional setting $|\mathcal{I}| = \infty$. From this point on, however, certain results will only hold in finite dimensions; for clarity, we will account each such situation explicitly.

Proposition 4.7. *If $1 \leq p \leq 2$, there exist global minimizers of the convex functional,*

$$\mathcal{J}^p_{\mathcal{I}_0}(u) = \|Tu - g\|^2_{\ell_2(\mathcal{K})} + \gamma \sum_{i\in\mathcal{I}_0} |u_i|^p, \tag{52}$$

and any minimizer $\bar{u}$ of $\mathcal{J}^p_{\mathcal{I}_0}$ satisfies the fixed point relation $\bar{u} = \mathbb{U}_{\mathcal{I}_0}(\bar{u})$. In finite dimensions $|\mathcal{I}| < \infty$, the result also holds for $p > 2$.

Proof. In the finite-dimensional setting, minimizers necessarily exist for all $p \geq 1$ according to Proposition 2.4, which generalizes the Frank-Wolfe theorem. We now consider the general case. Consider the unique decomposition $u = u_0 + u_1$ into a vector u_0 supported on $\mathcal{I}_0$ and another u_1 supported on $\mathcal{I}_1$, according to Notation 2.1. The functional (52) can be re-written with this decomposition according to

$$\mathcal{J}^p_{\mathcal{I}_0}(u_0 + u_1) = \|T_0u_0 + T_1u_1 - g\|^2_{\ell_2(\mathcal{K})} + \gamma\|u_0\|^p_{\ell_p^{\mathcal{I}_0}(\mathcal{I})} \tag{53}$$

where $\|z\|_{\ell_p^{\mathcal{I}_0}(\mathcal{I})} := \big(\sum_{i\in\mathcal{I}_0} |z_i|^p\big)^{1/p}$ is the ℓ_p-norm on vectors supported on $\mathcal{I}_0$.

Note that we can compute $u_1 = T_1^\dagger(g - T_0u_0^n)$ and $\mathcal{P}_1 = T_1T_1^\dagger$ is the the orthogonal projection onto the range of T_1 in $\ell_2(\mathcal{K})$ (not to be confused with $\mathcal{P}$, which operates on the space $\ell_2(\mathcal{I})$). Let $\mathcal{P}_1^\perp = \mathcal{I} - \mathcal{P}_1$ be the orthogonal projection in $\ell_2(\mathcal{K})$ onto the orthogonal complement of the range of T_1. Minimizers of the functional $\mathcal{F} : \ell_2^{\mathcal{I}_0}(\mathcal{I}) \to \mathbb{R}^+$ defined by

$$\begin{aligned} \mathcal{F}(v) &= \|T_0 v + \mathcal{P}_1(g - T_0 v) - g\|^2_{\ell_2(\mathcal{K})} + \gamma \|v\|^p_{\ell_p^{\mathcal{I}_0}(\mathcal{I})} \\ &= \|Kv - y\|^2_{\ell_2(\mathcal{K})} + \gamma \|v\|^p_{\ell_p^{\mathcal{I}_0}(\mathcal{I})} \end{aligned} \quad (54)$$

with $K := \mathcal{P}_1^\perp T_0$, and $y := \mathcal{P}_1^\perp g$, will yield minimizers of $\mathcal{J}^p_{\mathcal{I}_0}$. Functionals of the form (54) were studied in [20]; there, it is shown that as long as $1 \le p \le 2$, $\mathcal{F}$ has minimizers, and any minimizer $\bar{v}$ can be characterized by the fixed point relation

$$\bar{v}_i = F_p^{-1}([(I - K^*K)\bar{v} + K^*y]_i), \quad i \in \mathcal{I}_0. \quad (55)$$

(Recall that F_p^{-1} is the inverse of the function $F_p(t) = t + \frac{\gamma p}{2} \operatorname{sgn} t |t|^{p-1}$ for $p > 1$ and it coincides with the soft-thresholding S_γ (32) for p=1.)
In the finite-dimensional setting $|\mathcal{I}| < \infty$, the Euler-Lagrange equations corresponding to minimizers of the convex functional $\mathcal{F}$ as in [20] imply the same fixed point relation (55) also, for all $p \ge 1$. Making the identification $\bar{u}_0 = \bar{v}$ and $\bar{u}_1 = T_1^\dagger(g - T_0\bar{v})$, and rewriting $K = \mathcal{P}_1^\perp T_0$, and $y = \mathcal{P}_1^\perp g$, the relations (55) imply the full fixed point characterization (51). □

Remark 4. The necessary restriction $p \le 2$ in the infinite dimensional setting $|\mathcal{I}| = \infty$ was only used in the proof of Proposition 4.7, where it was needed in [20] to prove the existence of minimizers of functionals $\mathcal{F}$ of the form (54). Given additional requirements on T, this restriction can be relaxed. For instance, if T is also a bounded operator from $\ell_q(\mathcal{I})$ to $\ell_2(\mathcal{I})$ for some $q > 2$ then the existence of minimizers is guaranteed also for $1 \le p \le q$. In this case we could consider a minimizing sequence (v^k) for $\mathcal{F}$ which is bounded in $\ell_p^{\mathcal{I}_0}(\mathcal{I})$. Therefore, there exists a subsequence (v^{k_h}) which weakly converges in $\ell_p^{\mathcal{I}_0}(\mathcal{I})$ to a point v^*, implying weak convergence of the sequence Kv^{k_h} in $\ell_2(\mathcal{K})$; note that $\langle Kv^{k_h}, w\rangle_{\ell_2 \times \ell_2} = \langle v^{k_h}, K^*w\rangle_{\ell_p^{\mathcal{I}_0} \times \ell_{p'}^{\mathcal{I}_0}}$, for $1/p + 1/p' = 1$. By Fatou's lemma we obtain $\mathcal{F}(v^*) \le \liminf_h \mathcal{F}(v^{k_h})$ and v^* is a minimizer of $\mathcal{F}$.

Combining the results from this section, we obtain:

Theorem 4.8. *Suppose* $1 \le p \le 2$. *Starting from any* u^0 *satisfying* $\mathcal{J}_p(u^0) < \infty$, *the sequence* $(u^n)_{n\in\mathbb{N}}$ *defined by* $u^{n+1} = \mathbb{H}^n(u^0)$ *as in* (38) *will converge weakly to a vector* $\bar{u} \in \ell_2(\mathcal{I})$ *that satisfies the fixed point condition,*

1. $|\bar{u}_i| \ge \lambda'(r, \gamma, p)$, *if* $i \in \mathcal{I}_1 = \{j \in \mathcal{I} : |\bar{u}_j| > r\}$
2. $|\bar{u}_i| \le F_p^{-1}(\lambda') \le \lambda' - \delta$, *for* $p > 1$, *if* $i \in \mathcal{I}_0 = \{j \in \mathcal{I} : |\bar{u}_j| \le r\}$, *and*
3. *(a) If* $p > 1$:

$$\big[T^*(g - T\bar{u})\big]_i = \begin{cases} 0, & \text{if } |\bar{u}_i| \ge \lambda' \\ F_p(\bar{u}_i) - \bar{u}_i, & \text{if } |\bar{u}_i| \le \lambda' - \delta \end{cases} \quad (56)$$

(b) If $p = 1$ and $r \geq 1/4$:

$$\begin{cases} \left[T^*(g - T\bar{u})\right]_i \in [-\gamma/2, \gamma/2], & |\bar{u}_i| \leq \gamma/2 \\ \left[T^*(g - T\bar{u})\right]_i = \gamma/2 \operatorname{sgn} \bar{u}_i, & \gamma/2 < |\bar{u}_i| \leq r + \gamma/4. \\ \left[T^*(g - T\bar{u})\right]_i = 0, & |\bar{u}_i| > r + \gamma/4. \end{cases} \tag{57}$$

If the index set $|\mathcal{I}| < \infty$ is finite dimensional, the theorem holds for all $p \geq 1$.

Proof. By Lemma 4.4, the iteration step $u^{n+1} = \mathbb{H}(u^n)$ becomes equivalent to a step of the form $u^{n+1} = \mathbb{U}_{\mathcal{I}_0}(u^n)$ after a finite number of iterations $N \in \mathbb{N}$. By Lemma 4.4 and Proposition 4.3, the subset $\mathcal{I}_0 \subset \mathcal{I}$ separates $\mathcal{I}$ in the sense that, for all $n \geq N$,

- $|u_i^n| \leq F_p^{-1}(\lambda') \leq \lambda' - \delta$, if $i \in \mathcal{I}_0$,
- $|u_i^n| > \lambda'$, if $i \in \mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0$.

That the sequence $(u^n)_{n\in\mathbb{N}}$ converges to a fixed point of the map $\mathbb{U}_{\mathcal{I}_0}$ follows from Opial's theorem applied to the map $\mathbb{U}_{\mathcal{I}_0}$:

1. the asymptotic regularity of $\mathbb{U}_{\mathcal{I}_0}$ is a consequence of Lemmas 4.1 and 4.2;
2. the nonexpansiveness of $\mathbb{U}_{\mathcal{I}_0}$ follows from the proof of Proposition 4.5, and
3. Theorem 4.7 guarantees that the set of fixed points of $\mathbb{U}_{\mathcal{I}_0}$ in $\ell_2(\mathcal{I})$ is nonempty.

The limit $\bar{u}$ of the sequence (u^n) will satisfy the fixed point conditions (51). Since weak convergence implies component-wise convergence, it follows for all $i \in \mathcal{I}_0$ that

$$\begin{aligned} |\bar{u}_i| &= \lim_{n\to\infty} |u_i^n| \\ &\leq \lambda' - \delta \end{aligned} \tag{58}$$

and, analogously for $i \in \mathcal{I}_1$, the lower bound $|\bar{u}_i| \geq \lambda'$ holds. □

Remark 5. When $p = 2$, we recover the fixed point condition

$$\text{either } |\bar{u}_i| \geq (1+\gamma)^{1/2} r, \text{ or else } |\bar{u}_i| \leq (1+\gamma)^{-1/2} r.$$

Thus, any prior knowledge about these threshold levels can be incorporated in selecting the parameters γ and r.

5 On Minimizers of $\mathcal{J}_p$

We are now in a position to explore the relationship between limit vectors $\bar{u}$ of the iterative thresholding algorithm (38) and minimizers of the free-discontinuity functional $\mathcal{J}_r^p$ (22). First of all let us make a few considerations. Due to the separation of the entries of any fixed point $\bar{u}$, such that $\bar{u}_i \leq \lambda' - \delta < r < \lambda' \leq \bar{u}_j$ for $i \in \mathcal{I}_0$ and $j \in \mathcal{I}_1$ (see Proposition 4.3), we have also $\mathcal{I}_0 \equiv \{i \in \mathcal{I} : |u_i| \leq r\}$ and $\mathcal{I}_1 \equiv \{j \in \mathcal{I} : |u_j| > r\}$ for all $u \in B(\bar{u}, \varepsilon)$, where $B(\bar{u}, \varepsilon)$ is a ball around an equilibrium point $\bar{u}$ of radius $\varepsilon > 0$

sufficiently small. On this neighborhood $B(\bar{u}, \varepsilon)$ of $\bar{u}$, the functional $\mathcal{J}_p$ is convex. Since $\bar{u}$ is obtained as the limit of a sequence (u^n) in $B(\bar{u}, \varepsilon)$ for which the sequence $\mathcal{J}_p(u^n)$ is nonincreasing, one would expect that $\bar{u}$ minimizes $\mathcal{J}_p(u^n)$ within this neighborhood, hence it is a local minimizer. Moreover, following the proof of Theorem 5.1 below, we see that $\mathcal{J}_p$ is in fact *strictly* convex for $p > 1$ whenever $\bar{u} = u^*$ is a global minimizer, since the restriction of T to the subspace $\ell_2^{\mathcal{I}_1}(\mathcal{I}) \subset \ell_2(\mathcal{I})$ of vectors with support in $\mathcal{I}_1$ must be an injective operator in this case. Hence a global minimizer is necessarily isolated. More precisely we have the following result.

Theorem 5.1. *Let us denote* $\mathrm{Fix}(\mathbb{H})$ *the set of fixed points of* $\mathbb{H}$*, or satisfying the fixed point relation of Theorem* 4.8*,* $\mathscr{G}$ *the set of* global minimizers *of* $\mathcal{J}_p$*, and* $\mathscr{L}$ *the set of* local minimizers *of* $\mathcal{J}_p$*. Then we have the following set inclusions*

$$\mathscr{G} \subset \mathrm{Fix}(\mathbb{H}) \subset \mathscr{L}. \tag{59}$$

Moreover, a global minimizer $u^* \in \mathscr{G}$ *is isolated as soon as one of the following conditions is satisfied*

- $p > 1$*;*
- $p = 1$ *and the operator* $\mathcal{P}_1^{\perp} T_0$ *is injective, where* $\mathcal{P}_1$ *and* T_0 *are defined depending on* u^* *as in Notation 2.1.*

The proof of Theorem 5.1 is rather long and we defer it to the Appendix. We illustrate some of the findings in Figure 3, which shows how different starting points u^0 of the iteration $u^{n+1} = \mathbb{H}(u^n)$ may converge to different fixed points or local minimizers of $\mathcal{J}_p$. The sets of initial points converging to the same local minimizer, shown in different colors in the figure, can be disconnected, i.e., they may be formed by the union of disjoint 'islands'. As it is shown in the bottom-right box of Figure 3 we cannot ensure that local minimizers are isolated if T has a nontrivial null-space; in this case, local minimizers may form continuous sets, whose main directions are determined by the eigenvectors (indicated in the figure with arrows) which span the kernel of T.

6 2D free-discontinuity inverse problems and a projected gradient method

As presented in Subsection 1.4.2, the minimization of the discrete functionals for 2D free-discontinuity inverse problems has the general form

$$\begin{cases} \text{Minimize} & \mathcal{J}_p(u) := \|Tu - g\|^2_{\ell_2(\mathcal{K})} + \gamma \sum_{i \in \mathcal{I}} \min\{|u_i|^p, r^p\} \\ \text{subject to} & \mathcal{Q}u = 0, \end{cases} \tag{60}$$

where $\mathcal{Q} : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{K}')$ is a suitable bounded linear operator.

We can not directly generalize the analysis of the previous sections to (60), as the introduction of surrogate functionals does not decouple the constraint $\mathcal{Q}u = 0$. However, when the index set $\mathcal{I}$ is finite dimensional, we can still say something. For ease of presentation, we will assume $p = 2$ throughout this section.

First, recall that the partition argument of Theorem 2.3 guarantees that the constrained minimization problem (60) has a minimizer. Again, one could in theory obtain such a minimizer by computing a minimizer $u(\mathcal{I}_0)$ for each subset $\mathcal{I}_0 \subset \mathcal{I} = \{1, 2, ..., N\}$. Of course, such an algorithm is computationally infeasible as the number of subsets of the index set $\{1, 2, ..., N\}$ grows exponentially with the dimension N of the underlying space.

We propose instead the following more practical projected gradient algorithm: for any initial u^0, iterate

$$u^{n+1} = \mathcal{P}_{\ker(\mathcal{Q})} \left[H_2(u^n + T^*(g - Tu^n)) \right], \tag{61}$$

where $\mathcal{P}_{\ker(\mathcal{Q})}$ is the orthogonal projection onto the null-space of $\mathcal{Q}$. This projection can be easily computed explicitly by

$$\begin{aligned} \mathcal{P}_{\ker \mathcal{Q}} &= I - \mathcal{Q}^\dagger \mathcal{Q} \\ &= I - \mathcal{Q}^*(\mathcal{Q}\mathcal{Q}^*)^{-1}\mathcal{Q}, \end{aligned}$$

where the latter equality holds whenever $\mathcal{Q}$ is a full-rank matrix, as the one associated to the Schwarz conditions (7). The analysis of the algorithm (61) is beyond the scope of this paper; nevertheless, note that locally around any minimizer, the functional $\mathcal{J}_2$ is convex, and that projected gradient iterations are well-known methods for constrained minimization of (non-smooth) convex functionals, see for instance [1].

7 Numerical Experiments

7.1 Dynamical systems, stability, and equilibria

Iterative thresholding algorithms have a natural interpretation as discrete-time dynamical systems with nonsmooth right-hand-side, and can be associated to continuous dynamical systems of the type:

$$\begin{aligned} \dot{u}(t) &= F(u(t), t) \\ &= \tau \left(H_p(u(t) + T^*(g - Tu(t))) - u(t) \right), \quad t \geq t_0, \quad \tau > 0. \end{aligned}$$

The study of the existence, uniqueness, stability, and long-time behavior of these ODE's is of fundamental interest in order to clarify also the stability properties of iterative thresholding algorithms. Indeed, other than soft-thresholding iterations [20], the corresponding right-hand-side is not Lipschitz continuous and can even be discontinuous, as is the case of free-discontinuity problems. In [12, 23] conditions are established for the existence, uniqueness, and continuous dependence on the initial data (at finite time) of solutions of dynamical systems with discontinuous right-hand-side. However, very little is known about long-time properties of such dynamical systems and about the nature of their equilibrium points.

For several continuous thresholding functions, such as the ones introduced in [20, 25, 24], one can easily show, for instance by means of Γ-convergence arguments, that equilibrium points depend continuously on the parameters of the thresholding, see, e.g., Proposition 2.5 and also [24, Theorem 5.1]. Nevertheless, for discontinuous thresholding functions H_p such as those studied in this paper, sudden bifurcation phenomena and instabilities do appear in general. Figure 3 shows that multiple equilibrium points can exist for these thresholding operators and their number may depend discontinuously on the thresholding shape parameters. Moreover, as established in Theorem 5.1, global minimizers of $\mathcal{J}_p$ are always stable equilibria and isolated points, while local minimizers can be unstable equilibria and form a continuous set, as shown in the bottom-right box of Figure 3.

7.2 Denoising and segmentation of 1D signals and digital images

In this subsection, we are concerned with numerical experiments in the use of an iterative thresholding algorithm for the minimization of

$$\mathcal{J}_2(u) := \|D_h^\dagger u - g\|_{\ell_2}^2 + \gamma \sum_{i=1}^N \min\{u_i^2, r^2\}, \tag{62}$$

modelling problems of denoising and segmentation.

In Figure 4 and Figure 6 we show the results of applying the iterative thresholding algorithm (36) and the projected gradient algorithm (61) respectively. In Figure 5 we show a comparison of the use of the thresholding H_2 and the *soft-thresholding* S_γ (32) associated to the minimization of functionals with ℓ_1-constraints,

$$\mathcal{J}_{TV}(u) := \|D_h^\dagger u - g\|_{\ell_2}^2 + \gamma \sum_{i=1}^N |u_i|. \tag{63}$$

In this case the iterative thresholding algorithm, as analyzed in [20], reads as follows:

$$u^{n+1} = \mathbb{S}_\gamma(u^n + T^*(g - Tu^n)),$$

where $\mathbb{S}_\gamma$ is defined component-wise $\mathbb{S}_\gamma(v) = (S_\gamma(v_i))_{i\in\mathcal{I}}$. While $\mathcal{J}_2$ promotes the minimization of the Mumford-Shah constraint MS and piecewise smooth solutions, $\mathcal{J}_{TV}$ promotes the minimization of a *total variation* constraint [36], which is also well-known to produce (almost) piecewise constant solutions with a perhaps unwanted 'staircase effect'; see also [17, Section 4] for details.

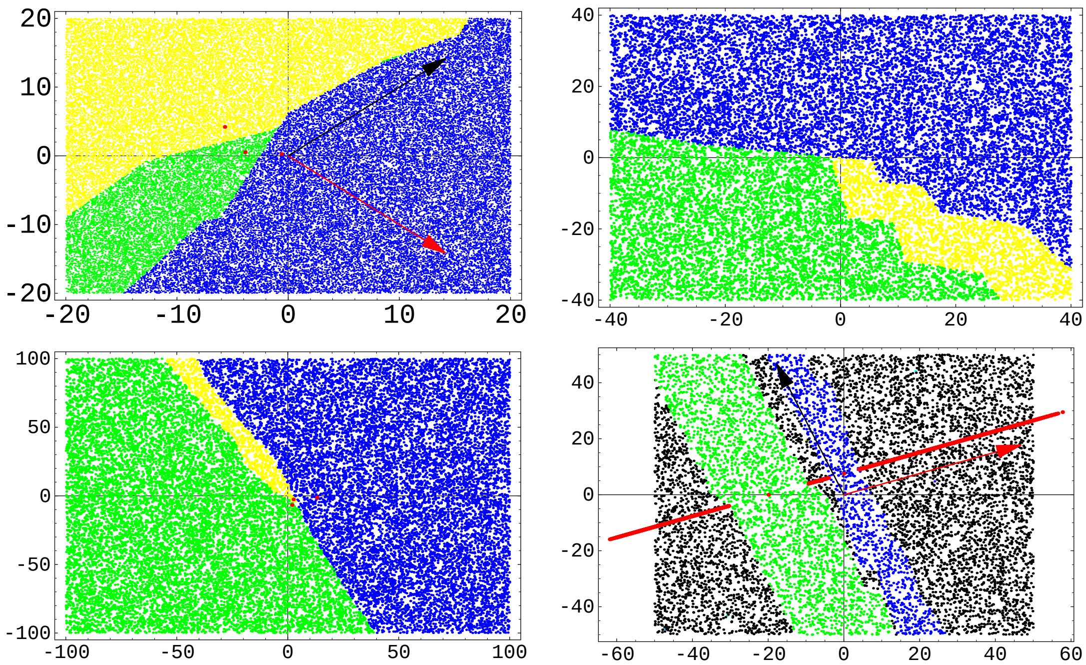


Figure 3: We show patterns in $\mathbb{R}^2$ formed by initial points u^0 colored according to the corresponding equilibria computed as limits of the iterative thresholding algorithm (36). For invertible 2×2 squared matrices T, the equilibria are isolated and the region of initial points for which (36) converges to a given equilibrium point do partition the space into sets which might be disconnected. Structures of the partition generated by different matrices T are exemplified in the top boxes and in the bottom-left one. In the bottom-right box we show the pattern related to iterations where the 2×2 squared matrix T has nontrivial null-space. We can see again that global minimizer are isolated and correspond to the points on the axes, whereas local minimizers are continuously distributed along an affine space generated by the kernel of T. It is not difficult to show that this structure always occurs for such matrices.

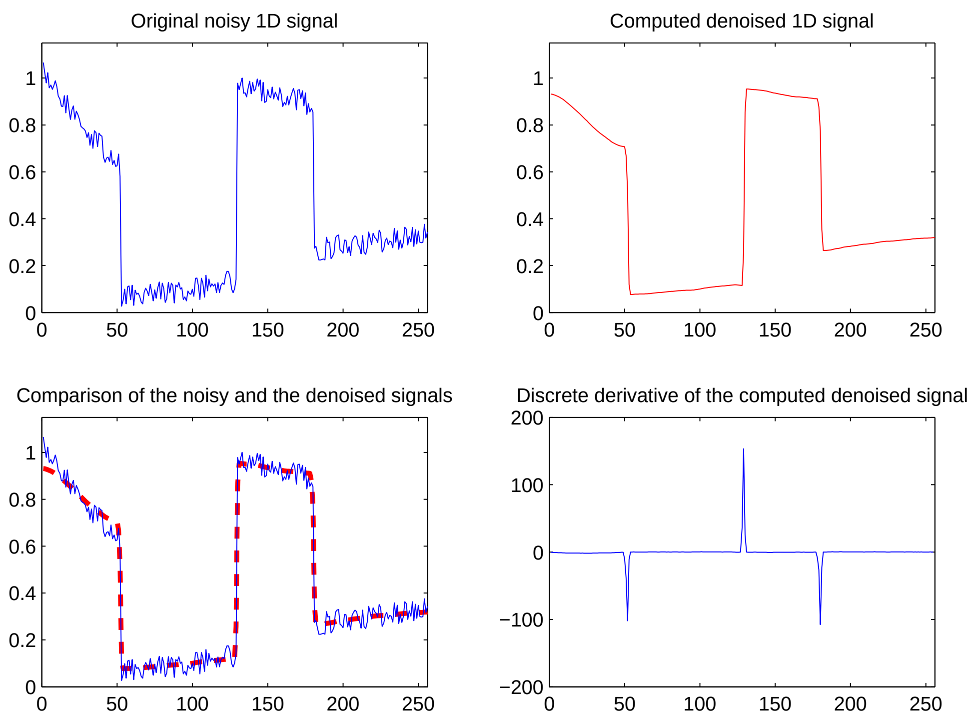


Figure 4: We show the application of the iterative thresholding algorithm (36) for the classical denoising problem of 1D signals where $K = I$ in (11), and hence $T = D_h^\dagger$. The thresholding parameters used for the numerics are $r = 2.2$ and $\gamma = 0.002$.

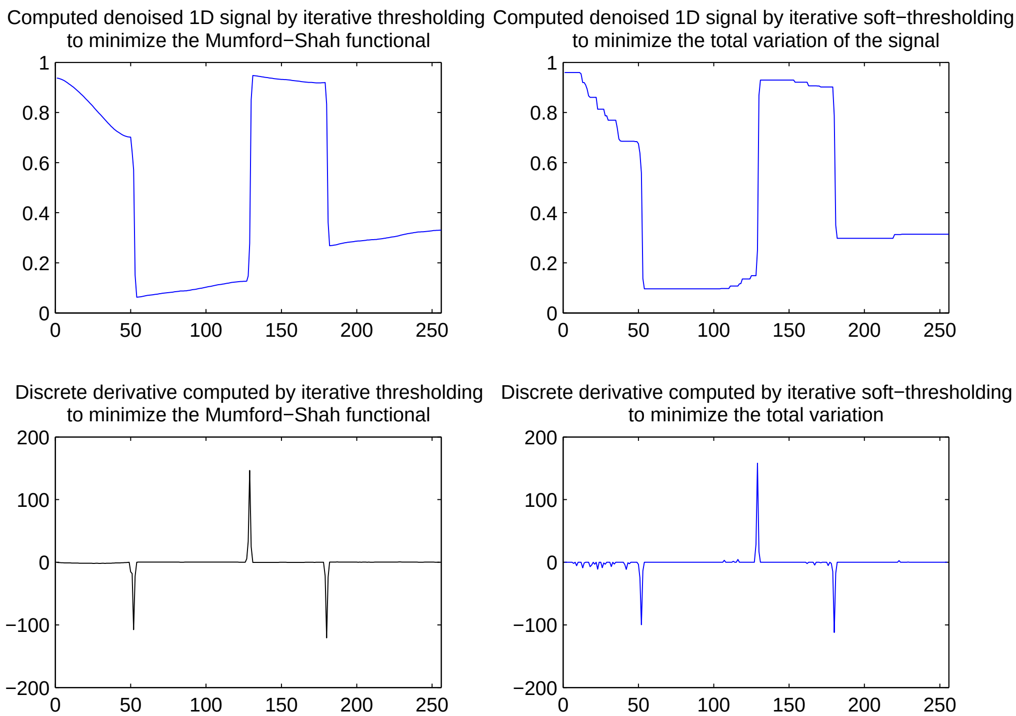


Figure 5: A comparison of the denoising of the signal in Figure 4 by means of the algorithm (36) and by iterative soft-thresholding [20] applied to discrete derivatives. We can appreciate how the algorithm (36) promotes piecewise smooth solutions, whereas the iterative soft-thresholding promotes the total variation minimization with the introduction of a 'staircase effect'. The thresholding parameters used for the numerics are $r = 2.2$ and $\gamma = 0.002$ for (31), and $\gamma = 0.002$ for the soft-thresholding (32).

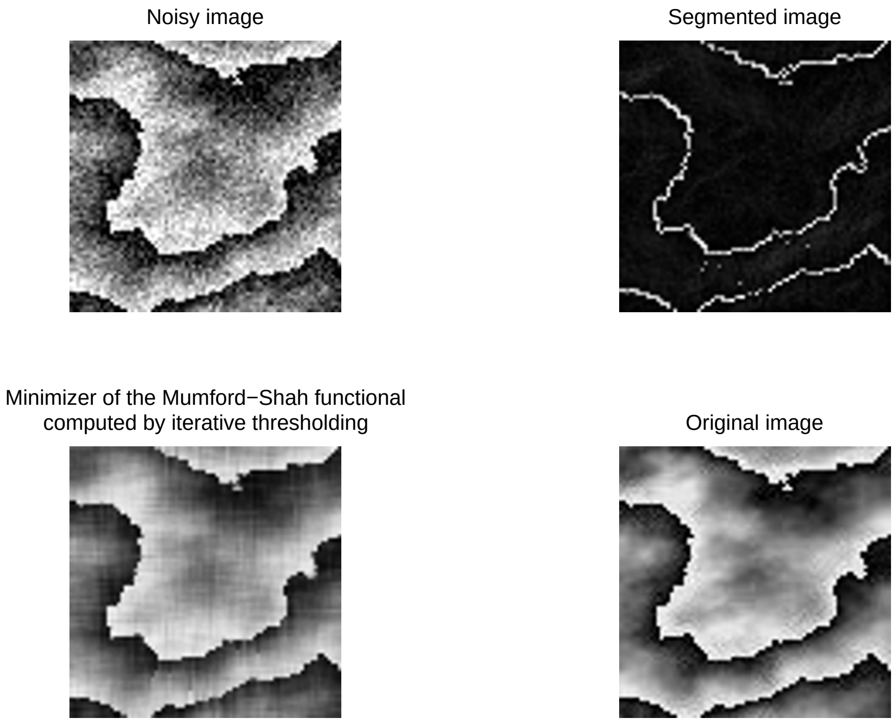


Figure 6: We show the application of the projected gradient algorithm (61) for the classical denoising problem of digital images where $K = I$ in (11), and hence $T = D_h^\dagger$. The thresholding parameters used for the numerics are $r = 5$ and $\gamma = 0.005$, and the image size is 80×80. The anisotropic effects of (8) are clearly visible, suggesting that for more effective image denoising, iterative thresholding on an isotropic (or direction-independent) variant of the 2D Mumford-Shah functional should be studied; see [16, 10]

7.3 Inverse problems

As already mentioned in Subsection 1.4.3 the Mumford-Shah term $MS(u) = \int_{\Omega\setminus S_u} |\nabla u|^2 + \beta\mathcal{H}^{d-1}(S_u)$ is also used for regularizing inverse problems involving operators T which are not boundedly invertible. In this section we present two numerical experiments on the use of algorithms (36) and (61) for 1D interpolation (Figure 7) and for 2D inpainting (Figure 8) respectively. In this case the operator T is a multiplier by a characteristic function of a subdomain, i.e., $Tu := \chi_D \cdot u$, for $D \subset \Omega$; see [22] for other numerical examples previously obtained with the Mumford-Shah regularization.

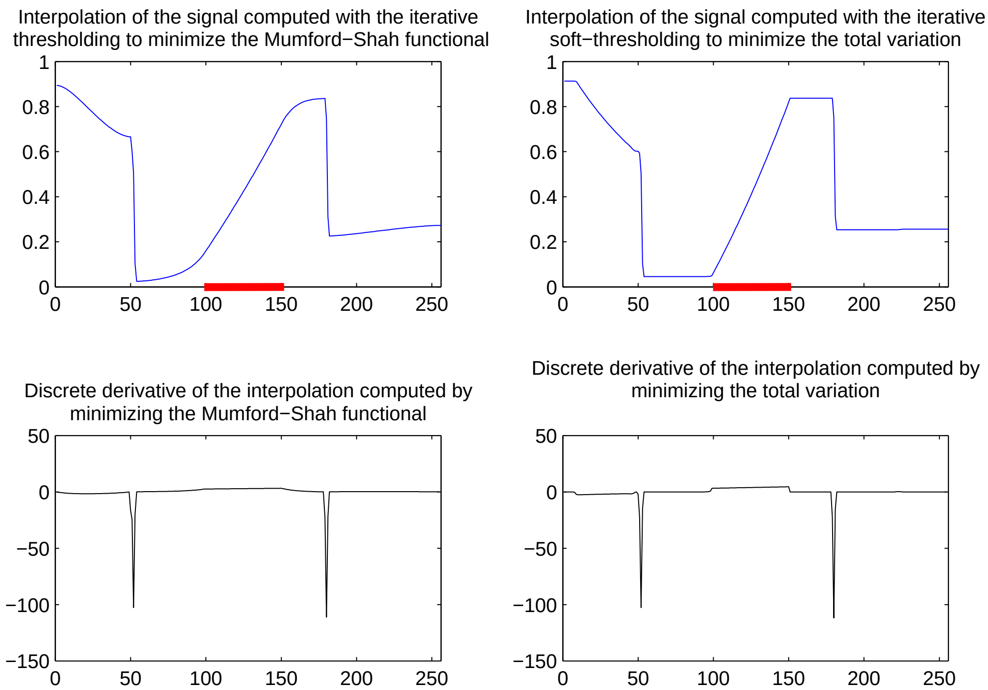


Figure 7: Interpolation of an incomplete signal by means of the Mumford-Shah regularization and the total variation minimization provided by respective iterative thresholding algorithms. The red interval is the region where no information on the original signal is provided. The thresholding parameters used for the numerics are $r = 2.2$ and $\gamma = 0.002$ for (31), and $\gamma = 0.002$ for the soft-thresholding (32).

In Figure 7 we show the reconstruction of the noiseless signal of Figure 4 provided information only out of the interval $[100, 150]$ which has to be restored. On the left boxes we show the results due to algorithm (36) and on the left ones the solution computed by iterative soft-thresholding. In the former the solution is again piecewise smooth and in the latter a (almost) piecewise constant solution is instead produced.

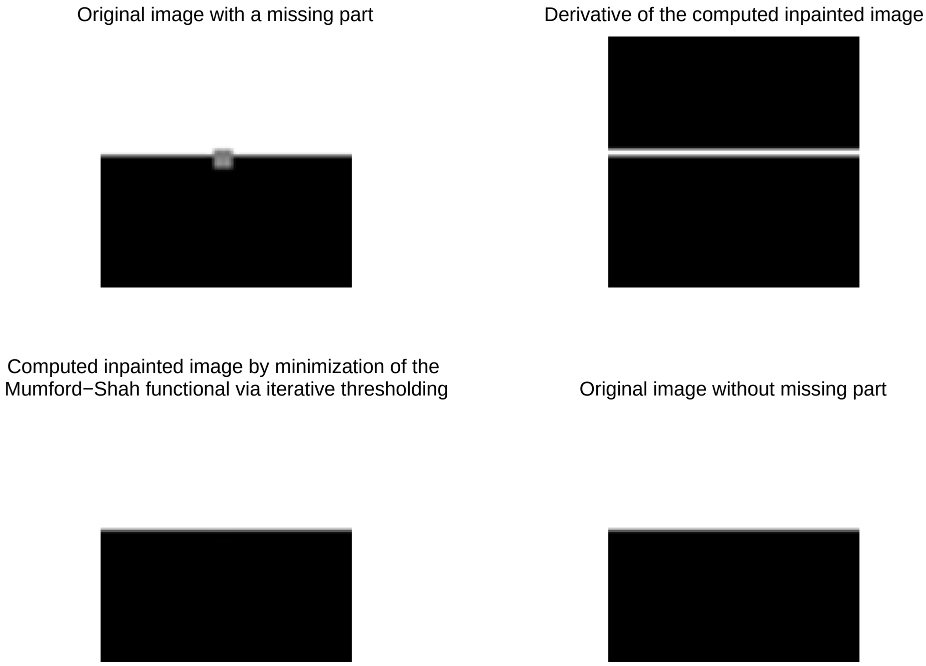


Figure 8: Inpainting of a binary image by means of algorithm (61). The occluded discontinuity is correctly recovered as already observed in [22]. The thresholding parameters used for the numerics are $r = 8$ and $\gamma = 0.0001$, and the image size is 40×40.

In Figure 8 we show the *inpainting* of a binary image with a missing information right at its center which is occluding precisely a discontinuity. As already shown in [22] the inpainting process produces minimal length connections of the discontinuity set as long as the inpainting region, i.e., the missing part, is not too large.

8 Appendix

8.1 Proof of Proposition 2.4

First, we recall Weierstrass' Theorem, which is used in the proof of Proposition 2.4 below, see also [19, Theorem 1.15].

Theorem 8.1 (Weierstrass' Theorem)**.** *The set of minima of a convex function f over a subset $X \subset \mathbb{R}^N$ is nonempty and compact if X is closed, f is lower semicontinuous* *over X, and the function $\tilde{f}$, given by*

$$\tilde{f} = \begin{cases} f(x) & , \textit{if } x \in X, \\ \infty & \textit{otherwise}, \end{cases} \tag{64}$$

is coercive, *i.e., for every sequence $(x_k) \subset X$ s.t. $\|x_k\| \to \infty$, we have $\lim_{k\to\infty} f(x_k) = \infty$.*

The following two lemmas will be helpful in the proof of Proposition 2.4.

Lemma 8.2. *Let $F(u)$ be a convex function defined on $\mathbb{R}^N$ having the general form $F(u) = \left[u^t A u + b^t u + \sum_{1\le j\le N} c_j |u_j|^p\right]$, for some $p \ge 1$, where A is a $N \times N$ matrix, $b \in \mathbb{R}^N$, and the c_j's are scalars. Fix x and d in $\mathbb{R}^N$. If F is bounded above and below on the ray $\{x + td, t \ge 0\}$, then F is constant on the line $x + td$.*

Proof. Let $\mu(t) = F(x+td)$, and note that μ is convex because F is convex. Moreover, μ has the general form $\mu(t) = P(t) + \sum_{1\le j\le N} c_j |x_j + td_j|^p$ where $P(t)$ is a polynomial in t of order at most 2. Without loss of generality, suppose $0 \le \mu(t) \le 1$ for all values of $t \in \mathbb{R}^+$. Then there exists a sequence of points $(t_n)_{n\in\mathbb{N}}$, $t_n \to \infty$ for $n \to \infty$, for which $\mu(t_n)$ is a convergent sequence; let us denote the limit of this sequence by γ.

1. **Case 1:** $1 \le p \le 2$. To repeat,

$$\lim_{n\to\infty} \mu(t_n) = \lim_{n\to\infty} P(t_n) + \sum_{1\le j\le N} c_j |x_j + t_n d_j|^p = \gamma. \tag{65}$$

   Since $0 = \lim_{n\to\infty} \mu(t_n)/t_n^2$, it follows that all coefficients in $\mu(t)$ of degree 2 must vanish. In turn, then, $0 = \lim_{n\to\infty} \mu(t_n)/t_n^p$, has the implication that for each j, one of the coefficients c_j or d_j must vanish as well. Following in the same manner, we conclude that all linear coefficients in $\mu(t)$ also vanish, leaving only the possibility that $\mu(t) \equiv \gamma$ is a constant function.

2. **Case 2:** $p > 2$: The proof in this case is identical to that of the previous case, and as such we leave the details to the reader.

□

Lemma 8.3. *Suppose F is a convex function defined on $\mathbb{R}^N$ that is bounded from below, and has the property that if F is bounded above on a ray $\{x + td, t \in \mathbb{R}^+\}$, then F is constant on the line $x + td$. Then if F is constant on the line $x + td$, F is also constant on any parallel line $y + td$.*

Proof. Let $\mu(t) = F(x + td)$ which by assumption is a constant function $\mu(t) = \gamma$, and let $v(t) = F(y + td)$. Fix $t \in \mathbb{R}^+$, and let z be the point $z = x + 2(y - x)$, i.e. $y = \frac{1}{2}x + \frac{1}{2}z$. By convexity of F, we have that

$$F(y + td) = F\Big(\frac{1}{2}z + \frac{1}{2}(x + 2td)\Big) \le \frac{1}{2}F(z) + \frac{1}{2}\mu(2t) = \alpha, \tag{66}$$

for a constant α. It follows that F is bounded above by α on the ray $\{y + td : t \in \mathbb{R}^+\}$, from which it follows, by assumption, that F is constant on the line $y + td$. □

We now prove Proposition 2.4. Choosing $x_0 \in X$, we define the (nonempty) set

$$M := X \cap \{x \in \mathbb{R}^N : F(x) \leq F(x_0)\}. \tag{67}$$

Obviously, the set M is convex and closed. By assumption, F is bounded from below on X and hence on M. Therefore, if M is bounded, then Weierstrass' Theorem yields the desired result.

We assume now that M is unbounded. As a convex, unbounded set, it follows that M must contain a ray $r = \{z + td : t \in \mathbb{R}^+\}$, see [18]. Denote by $r_1, r_2, ..., r_J$ a set of J rays in M corresponding to linearly independent directions $d_1, ..., d_J$. Without loss, we assume J sufficiently large that the direction d associated to any ray r in M can be expressed as a linear combination of the $d_1, ..., d_J$. Applying Lemma 8.2 to F which is bounded on M by construction, it follows that F is constant on each of the the rays $r_j = z_j + td_j$. From Lemma 8.3, we have also that F is constant along any line parallel to these rays; that is, for any $x \in \mathbb{R}^N$, F is constant on the line $\{x + td_j,\ t \in \mathbb{R}\}$. We then deduce by convexity that F is bounded above along any line $x+td$ for $d \in Y = \text{span}\{d_1, ..., d_J\}$, which implies by Lemma 8.2 that F is constant on such a line. We thus project X onto the subspace of $\mathbb{R}^N$ that is orthogonal to Y; call this subspace $\widetilde{X}$.

From the foregoing arguments, we have

$$\inf_{u \in \widetilde{X}} F(u) = \inf_{u \in X} F(u) \tag{68}$$

As $\widetilde{X}$ is still a convex polyhedral set, and by construction $\widetilde{M} = \widetilde{X} \cap \{x \in \mathbb{R}^N : F(x) \leq F(x_0)\}$ contains no rays, Weierstrass' Theorem yields the desired result.

8.2 On uniform boundedness of $\|D_h^\dagger\|$

The aim of the second part of the appendix is to prove the uniform bound $\|D_h^\dagger\| \leq 1/2$ eluded to in Section 3. Again, $\|A\|$ denotes the spectral norm of the matrix A, and $D_h^\dagger : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is the pseudo-inverse of the discrete derivative matrix D_h as given by (6), with the identification $n = \lfloor 1/h \rfloor$. From the expression for D_h, and the knowledge that $D_h D_h^\dagger = I$ is the identity operator and $D_h^\dagger D_h = (D_h^\dagger D_h)^*$ is self-adjoint, the $n \times (n-1)$ matrix $D_h^\dagger$ is identified as follows:

$$D_h^\dagger = \frac{1}{n^2} \begin{pmatrix} -(n-1) & -(n-2) & -(n-3) & \dots & \dots & -1 \\ 1 & -(n-2) & -(n-3) & \dots & \dots & -1 \\ 1 & 2 & -(n-3) & \dots & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & \dots & n-1 \end{pmatrix}. \tag{69}$$

It is well-known that the spectral norm of an $m \times n$ matrix can be bounded by the more manageable entry-wise Frobenius norm, according to

$$\|A\| \leq \|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2}. \tag{70}$$

As such, we need only to bound the sum of the squares of the entries of $D_h^\dagger$. The sum $\mathscr{S}_n^1 = \sum_{j=1}^{n-1} |d_{1,j}|^2$ over entries in the first row of $D_h^\dagger$ is given by $\mathscr{S}_n^1 = (n-1)(2n-1)/(6n^3)$, using the familiar formula $\sum_{j=1}^{N} j^2 = \frac{1}{6}N(N+1)(2N+1)$. The analogous sum over entries in the j^{th} row of $D_h^\dagger$ is seen inductively to satisfy $\mathscr{S}_n^j = \mathscr{S}_n^1 - \frac{(j-1)}{n^2} + \frac{j(j-1)}{n^3}$. The total sum $\mathscr{S}_n = \sum_{j=1}^{n} \mathscr{S}_n^j$ is then $\mathscr{S}_n = \frac{1}{6} - \frac{1}{6n^2}$, and we arrive at the desired uniform bound:

$$\|D_h^\dagger\| \leq \sqrt{\mathscr{S}_n} \leq \frac{1}{\sqrt{6}} < 1/2. \tag{71}$$

8.3 Proof of Proposition 4.3

In order to help the reading of the current proof, as well as the proof of Theorem 4.8, we report in Table 1 the notation of the functions used in the proof of Proposition 4.3 for the definition of H_p.

$L_p(t,\lambda)$	$= (t-\lambda)^2 + \gamma \min\{\|t\|^p, r^p\}$
$G_p(t,\lambda)$	$= (t-\lambda)^2 + \gamma \|t\|^p$
$F_p(t)$	$= t + \frac{\gamma p}{2} \operatorname{sgn} t \|t\|^{p-1}$, $p > 1$
$S_p(\lambda)$	$= G_p(F_p^{-1}(\lambda), \lambda) = (F_p^{-1}(\lambda) - \lambda)^2 + \gamma \|F_p^{-1}(\lambda)\|^p$, $p > 1$
$H_p(\lambda)$	$= \arg\min_{t \geq 0} L_p(t,\lambda)$ for general $\lambda \geq 0$, $p > 1$
	$= \arg\min_{0 \leq t \leq r} G_p(t,\lambda) = F_p^{-1}(\lambda)$ for $0 \leq \lambda \leq r$
	$= \begin{cases} F_p^{-1}(\lambda), & \text{if } G_p(F_p^{-1}(\lambda),\lambda) \leq \gamma r^p \\ \lambda, & \text{else} \end{cases}$ for $\lambda > r$.

Table 1: Notation of the functions involved in the definition of H_p as in the proof of Proposition 4.3.

Consider the functions

$$L_p(t,\lambda) = (t-\lambda)^2 + \gamma \min\{|t|^p, r^p\}, \tag{72}$$

and

$$G_p(t,\lambda) = (t-\lambda)^2 + \gamma |t|^p. \tag{73}$$

The proof reduces to solving for

$$H_p(\lambda) = \arg\min_{t \in \mathbb{R}} L_p(t,\lambda) \tag{74}$$

as a function of $\lambda \in \mathbb{R}$. Since $L_p(t,\lambda) = L_p(-t,-\lambda)$, the function $H_p(\lambda)$ will be odd, and since also $H_p(0) = 0$, we can, without loss of generality, restrict the domain of interest to $\lambda > 0$. On this domain, $H_p(\lambda) = \arg\min_{t\in\mathbb{R}} L_p(t,\lambda)$ is nonnegative, since $L_p(t,\lambda) \leq L_p(-t,\lambda)$ when $t \geq 0$ and $\lambda \geq 0$. Hence, we can restrict the minimization of $L_p(t,\lambda)$ to $t \geq 0$.

It will be convenient to split the proof into two cases: $1 < p$ and $p = 1$.

1. We first analyze the case $1 < p$.
   Note that

$$\begin{aligned} \arg\min_{t\geq r} L_p(t,\lambda) &= \arg\min_{t\geq r}(t-\lambda)^2 \\ &= \max\{\lambda, r\}, \end{aligned} \tag{75}$$

   so that the minimization (74) naturally splits into the following two cases:

   (a) If $\lambda \leq r$, the minimizer has to be searched in $[0,r]$, hence

$$H_p(\lambda) = \arg\min_{0\leq t\leq r} G_p(t,\lambda) = F_p^{-1}(\lambda) \leq \lambda \tag{76}$$

   where $F_p^{-1}(\lambda)$ is the functional inverse of the increasing, and continuous function

$$F_p(t) = t + \frac{\gamma p}{2}\operatorname{sgn} t|t|^{p-1}. \tag{77}$$

   (b) On the other hand, if $\lambda > r$, the minimizer has to be searched in $[0,\lambda]$, hence

$$H_p(\lambda) = \begin{cases} F_p^{-1}(\lambda), & \text{if } G_p(F_p^{-1}(\lambda),\lambda) \leq r^p \\ \lambda, & \text{else} \end{cases}.$$

   By implicit differentiation of the functional relation $F_p(F_p^{-1}(\lambda)) = \lambda$, it is clear that the functions $F_p^{-1}(\lambda)$ and $S_p(\lambda) := G_p(F_p^{-1}(\lambda),\lambda)$ are strictly increasing functions in λ. Indeed, we have the bounds

$$0 < \frac{d}{d\lambda}F_p^{-1}(\lambda) = \left(F_p'(F_p^{-1}(\lambda))\right)^{-1} = \left(1 + \frac{\gamma p(p-1)}{2}(F_p^{-1}(\lambda))^{p-2}\right)^{-1} \leq 1,$$

   and

$$\begin{aligned} \frac{d}{d\lambda}S_p(\lambda) &= \frac{\partial}{\partial t}G_p(F_p^{-1}(\lambda),\lambda)\frac{d}{d\lambda}F_p^{-1}(\lambda) + \frac{\partial}{\partial\lambda}G_p(F_p^{-1}(\lambda),\lambda) \\ &= (2(F_p^{-1}(\lambda)-\lambda) + p\gamma(F_p^{-1}(\lambda))^{p-1})\frac{d}{d\lambda}F_p^{-1}(\lambda) - 2(F_p^{-1}(\lambda)-\lambda) \\ &= 2\left(1 - \frac{d}{d\lambda}F_p^{-1}(\lambda)\right)(\lambda - F_p^{-1}(\lambda)) + p\gamma\frac{d}{d\lambda}F_p^{-1}(\lambda)(F_p^{-1}(\lambda))^{p-1} \geq 0, \end{aligned}$$

since $0 \leq \frac{d}{d\lambda} F_p^{-1}(\lambda) \leq 1$, and

$$0 \leq F_p^{-1}(\lambda) \leq \lambda. \tag{78}$$

Also observe that $F_p^{-1}(r + \frac{\gamma p}{2} r^{p-1}) = r$, and $S_p(r + \frac{\gamma p}{2} r^{p-1}) = \gamma r^p + \frac{(\gamma p)^2}{4} r^{2p-2} > \gamma r^p$. This leads us to immediately conclude that

(i) If $\lambda \leq r$, then $H_p(\lambda) = F_p^{-1}(\lambda)$ (from (76)).

(ii) If $\lambda \geq r + \frac{\gamma p}{2} r^{p-1}$, then $S_p(\lambda) = G_p(F_p^{-1}(\lambda), \lambda) > \gamma r^p$, so that $H_p(\lambda) = \lambda$.

(iii) Since $S_p(r) < \gamma r^p$ while $S_p(r + \frac{\gamma p}{2} r^{p-1})) > \gamma r^p$, the intermediate value theorem implies the existence of a unique value $\lambda' = \lambda'(r, \gamma, p)$ lying *strictly within* the interval $\big(r, r^{p-1}(\frac{\gamma p}{2} + r^{2-p})\big)$ at which

$$S_p(\lambda') = \gamma r^p, \tag{79}$$

and

$$H_p(\lambda) \quad = \quad \begin{cases} F_p^{-1}(\lambda) & \lambda < \lambda' \\ \lambda & \lambda > \lambda' \end{cases} . \tag{80}$$

At λ', $H_p(\lambda') = \arg\min_{t \geq 0} L_p(t, \lambda')$ is not uniquely defined and is realized at $F_p^{-1}(\lambda')$ and at λ'. In this case, we identify $H_p(\lambda') = F_p^{-1}(\lambda)$ for the sequel; as will be made clear, this will not cause problems in the ensuing analysis. Finally, note that

(iv) At λ', the function H_p has a discontinuity $\delta = \lambda' - H_p(\lambda')$ that is strictly positive, as long as $r, \gamma > 0$. Indeed, on the one hand, we know that $\lambda' > r$, on the other hand, $H_p(\lambda') < r$. This follows because $H_p(\lambda') = F_p^{-1}(\lambda')$, and

$$(F_p^{-1}(\lambda'))^p < (F_p^{-1}(\lambda') - \lambda')^2 + \gamma |F_p^{-1}(\lambda')|^p = S_p(\lambda') = \gamma r^p.$$

2. The analysis of the case $p = 1$ is left to the reader since it follows a similar argument as for $p > 1$.

8.4 Proof of Theorem 5.1

The proof will be much simplified by the following lemma which characterizes vectors such as $\bar{u}$ that satisfy the fixed point relations (56) or (57):

Lemma 8.4. *If u and v are such that*

$$\mathcal{J}_p^{surr}(u + v, u) - \|v\|^2_{\ell_2(\mathcal{I})} \geq \mathcal{J}_p^{surr}(u, u) = \mathcal{J}^p(u), \tag{81}$$

then $\mathcal{J}_p(u + v) \geq \mathcal{J}_p(u)$.

Proof. For any u and v, the following holds because $\|L\| \leq 1$:

$$\mathcal{J}_p(u + v) = \mathcal{J}_p^{surr}(u + v, u) - \|Lv\|^2_{\ell_2(\mathcal{I})} \geq \mathcal{J}_p^{surr}(u + v, u) - \|v\|^2_{\ell_2(\mathcal{I})}. \tag{82}$$

If in addition u and v satisfy (81), then the desired result is achieved by virtue of the equality $\mathcal{J}_p^{surr}(u, u) = \mathcal{J}_p(u)$. □

Let us show now the proof of Theorem 5.1. We address first the inclusion $\mathrm{Fix}(\mathbb{H}) \subset \mathscr{L}$. By Lemma 8.4, it suffices to show that at a fixed point $\bar{u}$ defined by (56) or (57), any perturbation $\delta h \in \ell_2(\mathcal{I})$ with norm $\|\delta h\|_{\ell_2(\mathcal{I})} \leq \min\{[\lambda'(r,p) - r], [r - H_p(\lambda')]\}$ will satisfy

$$\mathcal{J}_p^{surr}(\bar{u} + \delta h, \bar{u}) - \mathcal{J}_p^{surr}(\bar{u}, \bar{u}) \geq \|\delta h\|^2_{\ell_2(\mathcal{I})}. \tag{83}$$

After expanding the left-hand-side above, the inequality (83) is seen to be equivalent to

$$2 \sum_{i \in \mathcal{I}} \delta h_i [T^*(T\bar{u} - g)]_i + \gamma \sum_{i \in \mathcal{I}} \Big[\min\{|\bar{u}_i + \delta h_i|^p, r^p\} - \min\{|\bar{u}_i|^p, r^p\} \Big] \geq 0. \tag{84}$$

At this point, it is convenient to consider the summation over $i \in \mathcal{I}_0$ and $i \in \mathcal{I}_1$ separately.
By Lemma 4.4, the first summand above vanishes over $\mathcal{I}_1$ and

1. if $1 < p$, then $\sum_{i \in \mathcal{I}} \delta h_i [T^*(T\bar{u} - g)]_i = -\sum_{i \in \mathcal{I}_0} \delta h_i \operatorname{sgn} \bar{u}_i \frac{p\gamma}{2} |\bar{u}_i|^{p-1}$;
2. if $p = 1$, then $\sum_{i \in \mathcal{I}} \delta h_i [T^*(T\bar{u} - g)]_i = -\gamma/2 \sum_{i \in \mathcal{I}_0^b} \delta h_i \operatorname{sgn} \bar{u}_i + \sum_{i \in \mathcal{I}_0^a} \delta h_i [T^*(T\bar{u} - g)]_i$.

With respect to the second summation, observe from Proposition 4.3 that for all $1 \leq p$, $|\bar{u}_i| \geq \lambda'(r,p) > r$ for $i \in \mathcal{I}_1$, so that this summation vanishes over $\mathcal{I}_1$ for any perturbation δh satisfying the component-wise inequality $|\delta h_i| \leq \lambda'(r,p) - r$. Similarly, $|\bar{u}_i| \leq H_p(\lambda') < r$ for $i \in \mathcal{I}_0$, so that for any perturbation δh satisfying component-wise $|\delta h_i| \leq \min\{[\lambda'(r,p) - r], [r - H_p(\lambda')]\}$, we have that

$$\sum_{i \in \mathcal{I}} \Big[\min\{|\bar{u}_i + \delta h_i|^p, r^p\} - \min\{|\bar{u}_i|^p, r^p\} \Big] = \sum_{i \in \mathcal{I}_0} |\bar{u}_i + \delta h_i|^p - |\bar{u}_i|^p. \tag{85}$$

The desired result follows if we can show that

1. $1 < p \leq 2$: $\big[|\bar{u}_i + \delta h_i|^p - |\bar{u}_i|^p - \delta h_i p [\operatorname{sgn} \bar{u}_i] |\bar{u}_i|^{p-1}\big] \geq 0$, for all $i \in \mathcal{I}_0$
2. $p = 1$:
   (a) $|\delta h_i + \bar{u}_i| - |\bar{u}_i| - \delta h_i [\operatorname{sgn} \bar{u}_i]\big] \geq 0$ for all $i \in \mathcal{I}_0^b$, and
   (b) $\delta h_i [T^*(T\bar{u} - g)]_i + |\delta h_i| \geq 0$, for all $i \in \mathcal{I}_0^a$.

The inequality in $2(b)$ follows directly from Lemma 4.4; by symmetry, 1 and $2(a)$ follow if, for any $u \geq 0$,

$$\min_{v \in \mathbb{R}} \big[f(v) := |u + v|^p - u^p - p u^{p-1} v \big] = \min_{v \geq -u} (u+v)^p - u^p - p u^{p-1} v \geq 0. \tag{86}$$

When $p = 1$, the right-hand-side is identically zero and the result holds. When $1 < p \leq 2$, differentiating the right-hand-side gives that $f(v)$ has a local minimum at $v = 0$, at which $f(0) = 0$, and, at the endpoint, $f(-u) = (p-1)u^{p-1} \geq 0$. We address now the inclusion $\mathscr{G} \subset \mathrm{Fix}(\mathbb{H})$. Assume $u^* \in \mathscr{G}$, then

$$\begin{aligned}\mathcal{J}_p^{surr}(u^*, u^*) = \mathcal{J}_p(u^*) &\leq \mathcal{J}_p(u)\\ &\leq \mathcal{J}_p(u) + \|u - u^*\|^2_{\ell_2(\mathcal{I})} - \|Tu - Tu^*\|^2_{\ell_2(\mathcal{K})} = \mathcal{J}_p^{surr}(u, u^*).\end{aligned}$$

Hence, u^* is also a minimizer of $\mathcal{J}_p^{surr}(\cdot, u^*)$. By using Proposition 4.3 we eventually obtain $u^* = \bar{u} \in \mathrm{Fix}(\mathbb{H})$. We now would like to show that any $u^* \in \mathscr{G}$ is isolated. For this purpose, consider again the partition of the index set $\mathcal{I}$ into $\mathcal{I}_0 = \{i \in \mathcal{I} : |u_i^*| \leq r\}$ and $\mathcal{I}_1 = \{i \in \mathcal{I} : |u_i^*| > r\}$ as in Notation 2.1, and note that $|\mathcal{I}_1| < \infty$, or else $\mathcal{J}_p(u^*)$ would not be finite.

By minimality of u^*, if we fix u_0^*, the vector u_1^* satisfies $u_1^* = \arg\min_{z \in \ell_2^{\mathcal{I}_1}(\mathcal{I})} \mathcal{J}_{p,1}(z)$, where

$$\mathcal{J}_{p,1}(z) := \|T_1 z - (g - T_0 u_0^*)\|^2_{\ell_2(\mathcal{K})} + \gamma \sum_{i \in \mathcal{I}_1} \min\{|z_i|^p, r^p\}. \tag{87}$$

Since all coefficients in u_1^* have absolute value $|(u_1^*)_i| > r$, the vector u_1^* also minimizes the functional

$$\|T_1 z - (g - T_0 u_0^*)\|^2_{\ell_2(\mathcal{K})}, \tag{88}$$

or, else, the vector z^* minimizing (88) would satisfy $\mathcal{J}_{p,1}(z^*) < \mathcal{J}_{p,1}(u_1^*)$, contradicting the minimality of u_1^*. In fact, u_1^* must be the *unique* vector minimizing (88). For, if another vector u' also minimized (88), then the operator T_1 would have a nontrivial null space containing the span of some nonzero vector v, so that all vectors in the affine space $\{u_1^* + tv : t \in \mathbb{R}\}$ would be minimal solutions for (88). In this case, we would have also the freedom of choosing from this affine subspace a vector u' having one coefficient u_i' satisfying $|u_i'| < r$. But such a vector u' satisfies $\mathcal{J}_{p,1}(u') < \mathcal{J}_{p,1}(u_1^*)$, contradicting the minimality of u_1^*.

It follows that the operator T_1 must have trivial null space, and u_1^* is the unique minimal least squares solution to (88), well-known to be explicitly given by

$$u_1^* = T_1^\dagger(g - T_0 u_0^*) = \left(T_1^* T_1\right)^{-1} T_1^*(g - T_0 u_0^*), \tag{89}$$

so that $T_1 u_1^*$ is the unique orthogonal projection of $(g - T_0 u_0^*)$ onto the range of T_1. More explicitly we have that $\mathcal{P}_1 = T_1 T_1^\dagger = T_1(T_1^* T_1)^{-1} T_1^*$ is the orthogonal projection onto the range of T_1, due to the non-triviality of the null space of T_1, and $T_1 u_1^* = \mathcal{P}_1(g - T_0 u_0^*)$. Now, on the other hand, by observing that any optimal variable u_1 for fixed u_0 depends on u_0 via the relationship (89), we easily infer that the vector u_0^* minimizes

$$\mathcal{J}_{r,0}^p(v) = \|\mathcal{P}_1^\perp(T_0 v - g)\|^2_{\ell_2(\mathcal{J})} + \sum_{i \in \mathcal{I}_0} \min\{|v_i|^p, r^p\}, \tag{90}$$

where $\mathcal{P}_1^\perp$ denotes the orthogonal projection operator onto the orthogonal complement of the range of T_1.

Consider the convex functional,

$$\mathcal{F}(v) := \|\mathcal{P}_1^{\perp}(T_0 v - g)\|_{\ell_2(\mathcal{J})}^2 + \|v\|_{\ell_p^{\mathcal{I}_0}(\mathcal{I})}^p, \tag{91}$$

and note that $\mathcal{J}_{r,0}^p(u) \le \mathcal{F}(u)$, while at the same time $\mathcal{J}_{r,0}^p(u_0^*) = \mathcal{F}(u_0^*)$ by virtue of the fact that $|u_i^*| < r$ for $i \in \mathcal{I}_0$. For $p > 1$ it follows that u_0^* is also the unique minimizer of $\mathcal{F}$, due to the strict convexity of $\|\cdot\|_{\ell_p^{\mathcal{I}_0}(\mathcal{I})}^p$. We conclude that for any fixed admissible $\mathcal{I}_0$ and $\mathcal{I}_1$, a global minimizer $u^* = u^*(\mathcal{I}_0, \mathcal{I}_1)$ is uniquely identified. Since global minimizers u^* are also a fixed points of $\mathbb{H}$, they do satisfy a strict separation of the entries supported on the sets $\mathcal{I}_0$ and $\mathcal{I}_1$ respectively, hence such sets are stable under small perturbations of u^*. We conclude that in a neighborhood of u^* there are no other global minimizers, because the sets $\mathcal{I}_0$ and $\mathcal{I}_1$ cannot change there. For $p = 1$ we can guarantee the isolation of a global minimizer when $\mathcal{P}_1^{\perp} T_0$ is injective, implying again the strict convexity of (91).

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