Compressive Algorithms

Massimo Fornasier, “Analysis of Partial Differential Equations” Group

main cooperations: I. Daubechies (Univ. Princeton), S. Dahlke (Univ. Marburg)
P. Markowich (RICAM-Univ. Cambridge), R. Ramlau (RICAM),
H. Rauhut (Univ. Vienna), C. Schönlieb (Univ. Cambridge), G. Teschke (Univ. Neubrandenburg)

ÖAW Austrian Academy of Sciences

RICAM JOHANN-RADON-INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Introduction

Compressive algorithms (CA) are a new approach to efficient computing and take advantage of the property of solutions of certain PDEs and variational problems to be characterized by *few* major features which are recovered by adaptive nonlinear iterations. CA are *fast*, tend to use minimal number of degrees of freedom, and are *simple*. CA are very successfully applied in several problems. Their numerical analysis is challenging.

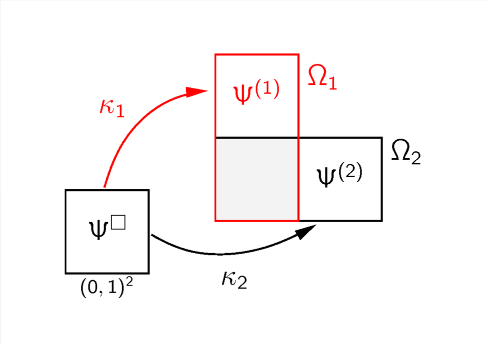
Scope of problems

- *Optimal adaptive frame solvers* (e.g., PDE solutions with limited smoothness): the solution $u^\star$ has a compressed expansion $u^\star = \sum_\lambda u^\star_\lambda \psi_\lambda$ w.r.t. a frame $\Psi = \{\psi_\lambda\}$ (e.g., wavelets, shearlets) constructed, e.g., on overlapping domain decompositions [1-3].
  Let $\Omega \subset \mathbb{R}^d$ Lipschitz domain. H Hilbert space, $H \subset L_2(\Omega) \subset H'$ Gelfand triple. $\mathcal{L} : H \to H'$ linear, elliptic operator. Task: for $f \in H'$, solve adaptively $\mathcal{L}u = f$, when u has limited Sobolev smoothness.

  For the discretization, choose a frame $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ for H, i.e., $\|f\|_{H'} \asymp \|f(\Psi)\|_{\ell_2}$, in particular $F^* : \ell_2 \to H : c \mapsto c^T\Psi$, $\tilde{F} : H \to \ell_2 : u \mapsto \langle u, \tilde{\Psi}\rangle$ are bounded.

  $$\mathcal{L}u^\star = f \Rightarrow \mathbf{L}\mathbf{u}^\star = \mathbf{f}, \quad \begin{cases} \mathbf{L} := \langle \mathcal{L}\Psi, \Psi\rangle, \\ \mathbf{f} := \langle f, \Psi\rangle. \end{cases}$$

  For $H = H^t_0(\Omega)$, reference Riesz basis $\Psi^\square \subset H^t_0(\square)$, $\square := (0,1)^d$; overlapping decomposition $\Omega = \sum_{i=1}^n \Omega_i$; $\kappa_i : \square \to \Omega_i$, C^m-diffeomorphisms, $m \geq t$; appropriate lifting yields frames $\Psi = \bigcup_{i=1}^n \Psi_i$.



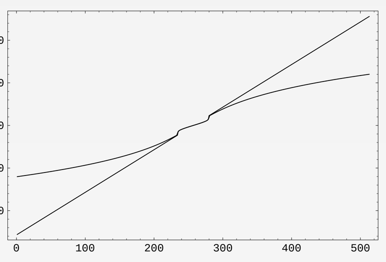
- *Sparse recovery in inverse problems* (e.g., image restoration, tomography): the regularized solution is assumed to have a parsimonious expansion $u^\star = \sum_\lambda u^\star_\lambda \psi_\lambda$ w.r.t. a frame $\Psi = \{\psi_\lambda\}$ [4-9], the problem is regularized by an ℓ_1 or TV constraints:
  $$\mathbf{u}^\star = \arg\min_{\mathbf{u}\in\ell_2(\Lambda)} \|\mathbf{L}\mathbf{u} - \mathbf{f}\|_2^2 + \tau\|\mathbf{u}\|_{\ell_1}, \quad \tau > 0.$$
- *Free-discontinuity problems* (e.g., image segmentation, crack identification via Mumford-Shah functional minimization): the solution $u^\star \in SBV(\Omega)$ is assumed to be smooth except on a *small* discontinuity set $S_{u^\star}$ [8-9]:
  $$u^\star = \arg\min_{u\in SBV(\Omega)} \|u - f\|_2^2 + \alpha\int_{\Omega\setminus S_u} |\nabla u|^p + \beta\mathcal{H}^{d-1}(S_u) \Leftrightarrow$$
  $$(\mathbf{z}^\star, \mathbf{v}^\star) = \arg\min_{\mathbf{z}\in\ell_\infty(\Lambda), \mathbf{v}\in\ell_\infty(\Lambda)_+} \|\mathbf{D}_h^\dagger \mathbf{z} - \mathbf{g}\|_2^2 + \alpha h\sum_\lambda v_\lambda |z_\lambda|^p + \beta\sum_\lambda (1 - v_\lambda)^2$$
  $$\mathbf{z} = \mathbf{D}_h\mathbf{u} \approx \nabla u \qquad \text{(approx. via adaptive finite elements).}$$

General Approach and Tasks

Compressive algorithms are formulated by thresholded gradient iterations [1-3,8-9]:

$$\begin{cases} \mathbf{u}^{(n+1)} = \mathbf{H}_{\gamma_n}(\mathbf{u}^{(n)} + \mathbf{L}_n^*(\mathbf{f} - \mathbf{L}_n\mathbf{u}^{(n)})), \\ \mathbf{L}_n \to \mathbf{L}, \\ \gamma = (\gamma_n)_n \text{ sequence of } \textit{shape} \text{ parameters .} \end{cases}$$

- $\mathbf{H}_\gamma$ acts componentwise as a thresholding function (e.g., soft, hard, firm-thresholding), *penalizes small/promotes big components*, depends on the problem, and on the shape parameter γ;
- the operator $\mathbf{L}$ is *adaptively* approximated by $\mathbf{L}_n$;

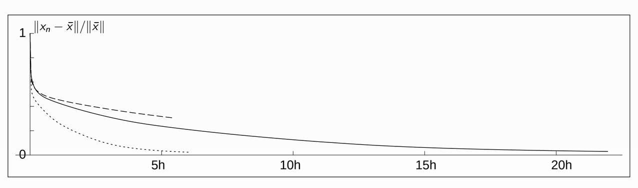


Main Task:
Identify the shape parameters $\gamma = (\gamma_n)_n$ and adaptive approximations $\mathbf{L}_n$ which realize the **best trade-off between rate of convergence and complexity**.

Accelerations

- In inverse problems, a concrete recipe [4] to identify *good* shape parameters $\gamma = (\gamma_n)_n$ is by construction of a suitable (convex) set K and a projection map $\mathbf{P}_K$ such that $\mathbf{P}_K(\mathbf{u}) = \mathbf{H}_\gamma(\mathbf{u})$, $\quad \gamma := \gamma(K, \mathbf{u})$. This leads to the following accelerated projected gradient/steepest descent iteration, which can be several times faster:

$$\begin{cases} \mathbf{u}^{(n+1)} = \mathbf{P}_K(\mathbf{u}^{(n)} + \mathbf{L}_n^*(\mathbf{f} - \mathbf{L}_n\mathbf{u}^{(n)})), \\ \mathbf{L}_n \to \mathbf{L}, \end{cases}$$

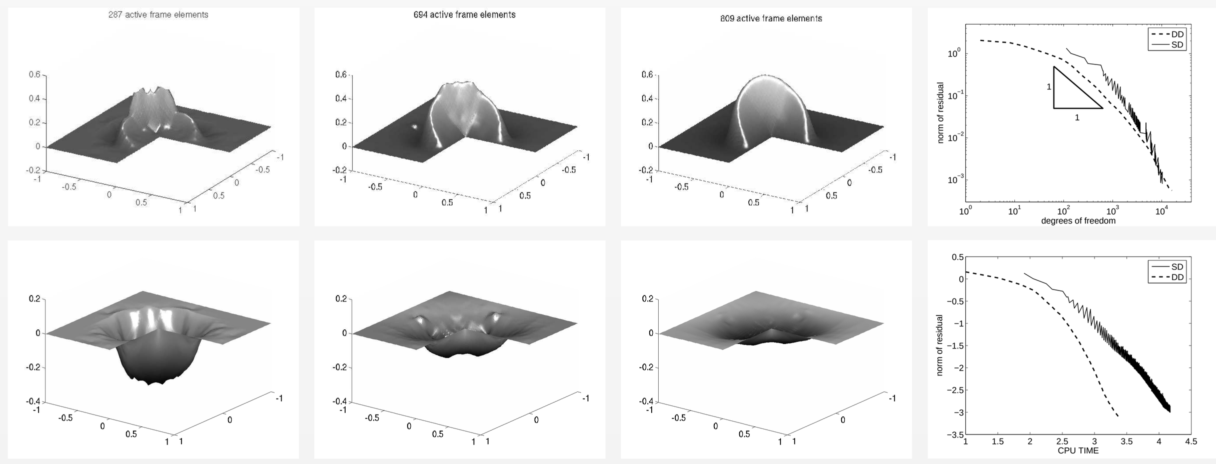


- By splitting the index set $\Lambda = \Lambda_1 \cup \Lambda_2$, domain decomposition/subspace correction strategies [5] can be formulated:

$$\begin{cases} \mathbf{u}_{\Lambda_1}^{(n+1,m+1)} = \mathbf{H}_{\gamma_n}(\mathbf{u}_{\Lambda_1}^{(n+1,m)} + \mathbf{L}_{n,\Lambda_1}^*((\mathbf{f} - \mathbf{L}_{n,\Lambda_2}\mathbf{u}_{\Lambda_2}^{(n,M)}) - \mathbf{L}_{n,\Lambda_1}\mathbf{u}_{\Lambda_1}^{(n+1,m)})) \\ \mathbf{u}_{\Lambda_2}^{(n+1,\ell+1)} = \mathbf{H}_{\gamma_n}(\mathbf{u}_{\Lambda_2}^{(n+1,\ell)} + \mathbf{L}_{n,\Lambda_2}^*((\mathbf{f} - \mathbf{L}_{n,\Lambda_1}\mathbf{u}_{\Lambda_1}^{(n+1,M)}) - \mathbf{L}_{n,\Lambda_2}\mathbf{u}_{\Lambda_2}^{(n+1,\ell)})) \\ \mathbf{u}^{(n+1)} = \mathbf{u}_{\Lambda_1}^{(n+1,M)} + \mathbf{u}_{\Lambda_2}^{(n+1,M)}. \end{cases}$$

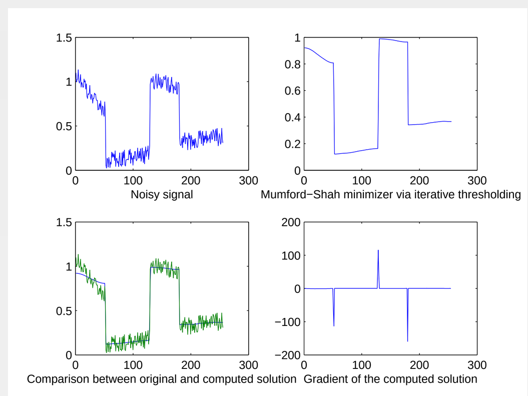
Results in PDEs and Imaging

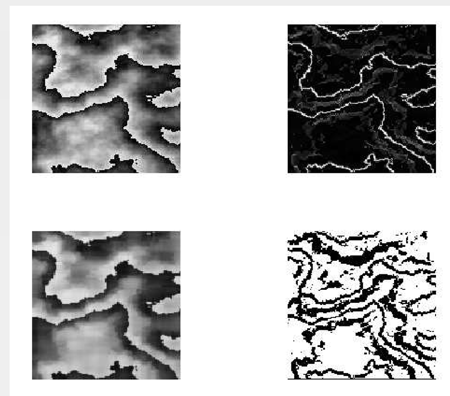
- The Poisson equation on the L-shaped domain is solved with *optimal rate and complexity* by adaptive wavelet frame algorithms [2-3]:



- Image restoration via thresholded iterations to minimize ℓ_1 and total variation constraints [6]:

- Signal/Image denoising/segmentation via thresholded iterations for the Mumford-Shah minimization [8-9]:





Future Work

- Rate of convergence and complexity estimates
- Convergence analysis for nonconvex problems, e.g., the Mumford-Shah functional
- Definition and construction of *optimal splitting* in domain decompositions
- Further accelerations by randomization and probabilistic convergence rate estimates
- Innovative approaches to PDE solution with discontinuous fronts, crack identification in fracture mechanics.

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