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Pattern Recognition 38 (2005) 2074–2087

PATTERN RECOGNITION
THE JOURNAL OF THE PATTERN RECOGNITION SOCIETY
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Fast, robust and efficient 2D pattern recognition for re-assembling fragmented images☆

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Received 22 March 2004; received in revised form 14 March 2005; accepted 14 March 2005

Abstract

We discuss the realization of a fast, robust and accurate pattern matching algorithm for comparison of digital images implemented by discrete Circular Harmonic expansions based on sampling theory. The algorithm and its performance for re-assembling fragmented digital images are described in detail and illustrated by examples and data from the experimentation on an art fresco real problem. Because of the huge database of patterns and the large-scale dimension, the results of the experimentation are relevant to describe the power of discrimination and the efficiency of such method.

Keywords: Art fresco reconstruction; Circular Harmonic expansions; Pattern recognition; Rotation invariance; Fast computation; Sampling theory

1. Introduction

Since 1994, the authors have been involved in the fascinating attempt to recall to life a very important Italian art fresco (A. Mantegna, Cappella Ovetari, Chiesa degli Eremitani in Padova), fragmented in thousands of pieces by an Allied bombing in the Second World War (1944) [1–3]. Recently, a digital cataloging of the fragment images made possible to count their exact number (80735). The distribution of the areas shows that most are relatively small, with an average surface area of 5–6 cm^2, a total area of 77 m^2 versus an original surface of several hundreds square meters. These a priori data demonstrated the lack of continuous fragments for any given fragment and makes extremely improbable that any reconstruction will be successful using methods based on the outline shape of the fragments. There is no information on the possible location of the

☆ The experimental part of this work has been supported by Fondazione Cassa di Risparmio di Padova e Rovigo, in agreement with Soprintendenza al Patrimonio Storico Artistico e Demoetnoantropologico del Veneto, Italy.

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1 The work of M.F. has been developed during his Doctorate in Computational Mathematics at the Department of Pure and Applied Mathematics of the University of Padova, Italy. He also acknowledges the financial support provided through the European Union's Human Potential Programme, under contract HPRN-CT-2002-00285 (HASSIP).

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doi:10.1016/j.patcog.2005.03.014

pieces on the huge original surface and it is unknown also the angle of rotation with respect to the original orientation. Some fairly good quality black and white photographs from between 1900 and 1920 exist, but they suffer from non-linear spectral distortion. A more detailed and complete description of the problem can be found in the contribution [4].

These facts impose that any feasible computer-based solution for a possible recomposition by comparison of the fragments and the fresco digital gray images must be

- *fast*, because of the huge number of fragments and original surface of the fresco;
- *robust*, because of the strong noise presence and intrinsic differences between the images, due to the damage of the bomb and the different photographic techniques;
- *accurate*, because of the fairly small dimensions of the fragments;
- *translation-rotation invariant*, because of the unknown original location and orientation of the fragments.

The request of a fast algorithm excludes the implementation of any comparison *pixel-by-pixel* and suggests that methods based on (compressed) series expansions can be more efficient. Besides other classical expansions, like Laguerre–Gauss [5] or Zernike polynomials [6] (fairly difficult to implement numerically), Circular Harmonic (CH) decompositions have found a relevant role in pattern matching because of their rotation invariance (self-steerable) properties and their effective and successful optical implementations [7–10]. In this paper, we want to present a digital/numerical implementation of compactly supported CHs and an effective 2D pattern recognition algorithm, based on these discrete expansions, which fulfills all the required properties listed above. The algorithm and its performance are described in detail and illustrated by examples and data from the experimentation on the fresco real problem. Because of the huge database of patterns and the large-scale dimension, the results of the experimentation are relevant to describe the power of discrimination and the efficiency of such method. Other problems can be interpreted in such a picture: experiments in character recognition, motion field detection and local rotation registration have also given very good results.

In literature, other kind of expansions have been presented as possible tools for pattern matching: To cite some, 2D (CH) wavelets [11–13] and multiscale self-steerable pyramid decompositions [14,15]. Even if they have given very interesting and promising results on small scale and local registration problems, it is still difficult to implement algorithms where a reasonable and feasible compromise among speed, robustness and location-rotation resolution can be realized on large scales.

The paper is organized as follows: Section 2 illustrates the CH expansions and their properties. In particular, it is shown that the moments constructed by correlation of an image with the CH system is a total information that can be used then for a complete comparison with an other signal. We discuss the discrete implementation of CH expansions by sampling and we will show that by limiting the system to a suitable and computable finite number of elements one can efficiently calculate the moments and preserve with optimal approximation *completeness*, *local orthonormality* and *self-steerability* also in the discrete domain. Section 3 illustrates the pattern recognition algorithm and its complexity is discussed with respect to a reference optimal method. In Section 4, numerical results and robustness of the algorithm in real cases are discussed and compared with the reference optimal method. An appendix collects notations and conventions used in the paper.

2. Discrete compactly supported Circular Harmonics

Compactly supported CHs arise as natural solutions of the Laplace eigenvalue problem on a disk under Dirichlet conditions [16], and they are related to relevant physical problems with rotation invariant symmetries. In fact, since the Laplacian commutes with rotations, CH are also eigenfunctions of any rotation operator. Let us introduce their formal definition as follows. We denote by $L^2(\Omega)$ the Lebesgue space of square-summable functions on $\Omega \subset \mathbb{R}^2$. Assume $\Omega_a \subset \mathbb{R}^2$ is a disk of radius $a > 0$. The system of CH functions on Ω_a is defined in polar coordinates by

$$e_{m,n,a}(r,\theta) = \frac{c_{m,n}}{a} J_m(j_{m,n} r/a) \mathrm{e}^{\mathrm{i} m\theta}, \quad m \in \mathbb{Z},\ n \in \mathbb{N}, \tag{1}$$

$$c_{m,n} = \pi^{-\frac{1}{2}} \left[\frac{\mathrm{d} J_m(s)}{\mathrm{d}s}\bigg|_{s=j_{m,n}} \right]^{-1}, \tag{2}$$

where J_m's are Bessel functions of the first kind of order $m \in \mathbb{Z}$, $(j_{m,n})_{n\in\mathbb{N}}$ is the sequence of their positive zeros [17], and $c_{m,n}$ is a normalization constant. We summarize their properties

(i) CH constitute an orthonormal basis for $L^2(\Omega_a)$ [16], i.e.,

$$\langle e_{m,n,a}, e_{m',n',a} \rangle := \int_{\Omega_a} e_{m,n,a}(x) \overline{e_{m',n',a}(x)}\,\mathrm{d}x = \delta_{(m,n),(m',n')},$$

being $\overline{f(x)}$ the complex conjugate of $f(x)$, and solve the Laplace eigenvalue problem:

$$\begin{aligned} &e_{m,n,a} \in H_0^1(\Omega_a) \cap C^\infty(\overline{\Omega_a}),\\ &\Delta e_{m,n,a} = -\left(\frac{j_{m,n}}{a}\right)^2 e_{m,n,a},\\ &e_{m,n,a}(x) = 0 \quad \forall x \in \partial\Omega_a, \end{aligned} \tag{3}$$

where $H_0^1(\Omega_a)$ is the Sobolev space of functions vanishing on the border $\partial\Omega_a$.

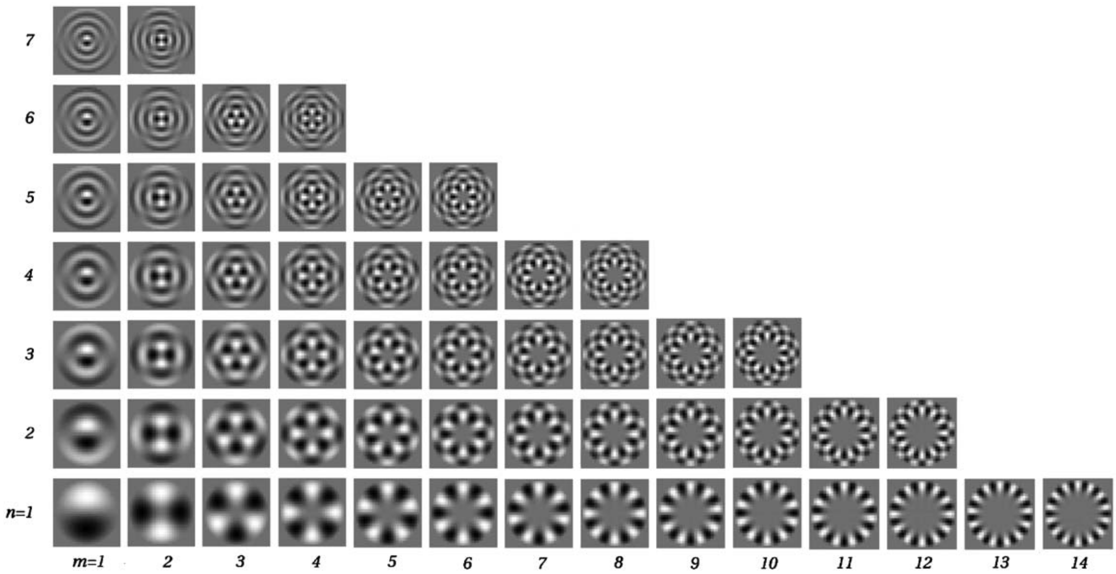

Fig. 1. Real part of some compactly supported CH, ordered by angular (abscissa) and radial (ordinate) frequencies, depending, respectively, on the parameters $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

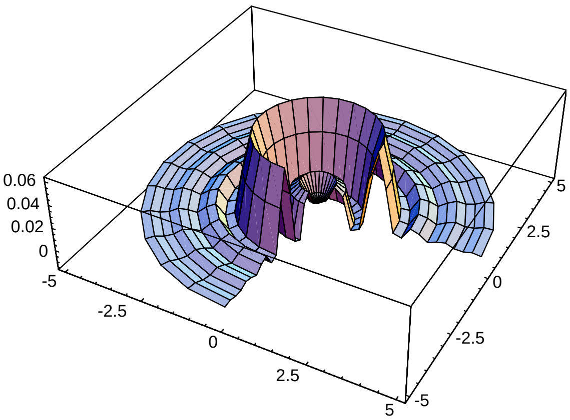

Fig. 2. Absolute value of the Fourier transform of the compactly supported CH function $e_{3,2,10}$. The dominant frequencies at radial distance $j_{3,2}/10$ from the origin, and the fast decay are visualized.

(ii) CH are characterized by special radial and angular frequencies depending on the parameters m and n, respectively, see Fig. 1. The *Fourier transform* of a summable function f on $\mathbb{R}^2$ is

$$\mathscr{F} f(\omega) = \int_{\mathbb{R}^2} f(x) \mathrm{e}^{-2\pi \mathrm{i} \langle \omega, x \rangle_{\mathbb{R}^2}} \, \mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ is the scalar product in $\mathbb{R}^2$. In particular, the Fourier transform $\mathscr{F} e_{m,n,a} \in L^1(\mathbb{R}^2)$ and has quite fast decay ($\mathscr{F} e_{m,n,a}(r, \theta) \sim r^{-5/2}$, see Fig. 2 and Refs. [16,3]). This property is relevant to ensure that CH produce relatively small *aliasing errors* [18] once they are sampled as we will discuss in the following. In particular, the CH basis minimizes aliasing errors among all the possible Fourier–Dini bases which are solutions of the Laplace eigenvalue problem (with Neumann conditions) [16,3], because in general the latter have slower Fourier transform decay.

(iii) Let R_α be the rotation operator of angle α, i.e., in polar coordinates $R_\alpha f(r, \theta) = f(r, \theta + \alpha)$, for all functions f on Ω_a. Then, as already announced, CH are eigenfunctions of any rotation operator (*self-steerability*) [9]

$$R_\alpha e_{m,n,a} = \mathrm{e}^{\mathrm{i} m \alpha} \cdot e_{m,n,a} \qquad (4)$$

for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and this property is used to detect mutual angles in comparisons of digital images, as we will discuss in the following.

2.1. Analysis and synthesis of an image by CH expansions

We describe in this section how moments computed by correlation with respect to CH can be used for a complete analysis of 2D (digital) signals.

For all $f \in L^2(\mathbb{R}^2)$, the moment (or *circular short time Fourier transform* (*CSTFT*) [19,20])

$$\begin{aligned} \mathscr{F}_{\mathrm{CSTFT}}(f)(x, m, n) &= \langle f, e_{m,n,a}(\cdot - x) \rangle \\ &= \int_{\mathbb{R}^2} f(y) \overline{e_{m,n,a}(y - x)} \, \mathrm{d}y, \end{aligned} \qquad (5)$$

localizes the information of f at the space point $x \in \mathbb{R}^2$ and at the frequency $j_{m,n}/a$. As we write $\tau > 0$ we mean

$\tau = (\tau_1, \tau_2) \in \mathbb{R}^2_+$ and we denote $det(\tau) = \tau_1\tau_2$. The multiplication of two vectors $x, y \in \mathbb{R}^2$ is assumed componentwise $x \cdot y := xy := (x_1y_1, x_2y_2)$.

There exists $\tau = \tau(a) > 0$, depending on $a > 0$, such that $\mathbb{R}^2 = \bigcup_{k\in\mathbb{Z}^2} \tau k + \Omega_a$. Therefore, by application of Fornasier [21, Theorem 1] the sequence $(e_{m,n,a}(\cdot - \tau k))_{k,m,n}$ is a global *frame* for $L^2(\mathbb{R}^2)$ (see Ref. [22] for an introduction on frames) generated by the local orthonormal basis $(e_{m,n,a})_{m,n}$ on the disk. Thus, one can invert Eq. (5) by using the *canonical dual frame* $(\widetilde{\mathscr{E}}_{m,n,a,k})_{m,n,k}$ [21–23]

$$f = \sum_{k,m,n} \mathscr{F}_{CSTFT}(f)(\tau k, m, n)\widetilde{\mathscr{E}}_{m,n,a,k}. \tag{6}$$

This means in particular that the sequence $(\mathscr{F}_{CSTFT}(f)(\tau k, m, n))_{k,m,n}$ is a *complete information on the image f* and then can be used for a complete comparison with an other image. We show in the following how to compute $\mathscr{F}_{CSTFT}$ in a *fast and accurate* way and how to discretize formula (5) for use in applications. For functions (on continuous or discrete domains) f, g, h we define $I(h)(x) = \overline{h(-x)}$ the involution operator and with $f * g$ the convolution operator of *f* and *g*. It is not hard to see that $\mathscr{F}_{CSTFT}(f)(\tau k, m, n) = (f * I(e_{m,n,a}))(\tau k)$. Hence, one can consider to implement a fast Fourier transform (FFT)-based discrete convolution at the resolution τ to approximate $\mathscr{F}_{CSTFT}(f)(\tau k, m, n)$. This choice produces the following estimation:

$$\mathscr{F}_{CSTFT}(f)(\tau k, m, n) \approx det(\tau) \sum_{l\in\mathbb{Z}^2} f(\tau l)\overline{e_{m,n,a}(\tau(l-k))}. \tag{7}$$

As we have fixed τ, we will write l^2 instead of $l^2(\tau\mathbb{Z}^2)$ the space of discrete signals with finite energy and we will suppose it endowed with the scalar product $\langle f, g\rangle_{l^2} \equiv det(\tau)\sum_{k\in\mathbb{Z}^2} f(\tau k)\overline{g(\tau k)}$. If f is a discrete signal then we will denote again with $\mathscr{F} f$ or $\hat{f}$ the *discrete Fourier transform* of *f*.

Observe that $\langle f, e_{m,n,a}(\cdot - \tau k)\rangle_{l^2} = det(\tau)\sum_{l\in\mathbb{Z}^2} f(\tau l)\overline{e_{m,n,a}(\tau(l-k))}$. It is known that scalar products are preserved under suitable sampling whenever the signals involved are band-limited. Thus, in such a case a formula like Eq. (7) would be exact and no approximation would appear. For non-band-limited signals *aliasing errors* [18,24] propagate to scalar products and perturb the approximation. Since CH functions are not band-limited, then to minimize the approximation error in Eq. (7) one should consider only those CH functions for which the aliasing errors are small. Fornasier [24, Theorem 3.4] describes how to measure aliasing errors of a function by suitable *function space norms* of the "tails" of its Fourier transform and how to optimize the approximation (7) for all possible choices of *f*: one should in fact select those CH which minimize this norm (as the measure of their aliasing error). In practice, one discretizes the infinite frame $(e_{m,n,a}(\cdot - \tau k))_{m,n,k}$ by sampling it on $\tau\mathbb{Z}^2$ and by selecting only a finite number of

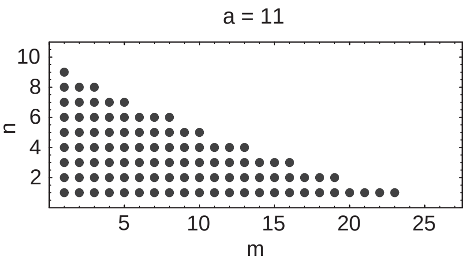

Fig. 3. Example of the set $\Phi_{a,\varepsilon}$ defined as the set of couples (m, n) such that $j_{m,n} \leqslant \frac{2\pi a}{\tau}\varepsilon^{1/2}$ for $\varepsilon = 0.1$ and $\tau = 1$.

elements which guarantee a good approximation. This selection produces a finite sampled sequence (the "$\cdot^s$" stands for "sampled") $(e^s_{m,n,a}(\cdot - \tau k))_{k,m,n}$ that is again a global (discrete) frame for $l^2(\tau\mathbb{Z}^2)$: hence *no truncation error occurs* when one wants to reproduce even a discrete signal from Eq. (7). By an application of Ref. [24, Theorem 3.4] one can show, numerically but also analytically [3], that for any precision $\varepsilon > 0$ and all radius $a > 0$ there exist a resolution $\tau = \tau(a) > 0$ and a *frequency set* $\Phi_{a,\varepsilon}$ of couples (m, n) such that for $(m, n) \in \Phi_{a,\varepsilon}$, the corresponding $e_{m,n,a}$ *minimizes the aliasing error up to* ε. In order to select a suitable method in concrete cases, we observe that the images, divided between channels (RGB or gray levels) can be considered as functions with real values. Since for any real signal h one has $\langle h, e_{-m,n,a}\rangle = (-1)^m\overline{\langle h, e_{m,n,a}\rangle}$ [17,16], the terms with $m < 0$ of the $\Phi_{a,\varepsilon}$ are redundant and maybe deducted from that part of $\Phi_{a,\varepsilon}$ with $m > 0$. Hence, in the following we will not consider $e_{-m,n,a}$ for $m > 0$.

Because of the peculiar shape of $\mathscr{F} e_{m,n,a}$, i.e., with absolute radial and dominant frequencies at $j_{m,n}/a$ and vanishing residual tails (Fig. 2), practically one can select with good approximation $(m, n) \in \Phi_{a,\varepsilon}$ whenever $j_{m,n} \leqslant \frac{2\pi a}{\tau}\varepsilon^{1/2}$. In Figs. 1 and 3 we show some examples of an admissible frequency set $\Phi_{a,\varepsilon}$.

Therefore, for a suitable choice of $\varepsilon > 0$, one can in particular ensure that

- The sequence $(e^s_{m,n,a})_{(m,n)\in\Phi_{a,\varepsilon}}$ is *almost* orthonormal, i.e.,

$$\delta_{(m',n'),(m,n)} = \langle e_{m',n',a}, e_{m,n,a}\rangle \approx \langle e^s_{m',n',a}, e^s_{m,n,a}\rangle_{l^2}. \tag{8}$$

  As a consequence the finite sequence $(e^s_{m,n,a})_{(m,n)\in\Phi_{a,\varepsilon}}$ is linearly independent, $\#\Phi_{a,\varepsilon} = \#\Omega^s_a$, and then $(e^s_{m,n,a})_{(m,n)\in\Phi_{a,\varepsilon}}$ is a local frame for $l^2(\Omega^s_a)$ with $(e^s_{m,n,a}(\cdot - \tau k))_{k,(m,n)\in\Phi_{a,\varepsilon}}$ as respective *discrete* global frame for $l^2(\tau\mathbb{Z}^2)$ [21,23]. The discrete moments (7) are again a complete information for sampled images.
- The discrete compactly supported CH approximatively satisfies Eq. (4) in the weak sense. With this we mean that, for all rotation operator R_α, for all $(m, n) \in \Phi_{a,\varepsilon}$,

and for all band-limited f, one has

$$\begin{aligned}\langle (R_\alpha f)^s, e^s_{m,n,a}\rangle_{l^2} &\approx \langle R_\alpha f, e_{m,n,a}\rangle \\ &= \langle f, R_{-\alpha} e_{m,n,a}\rangle \\ &\approx \mathrm{e}^{\mathrm{i}m\alpha}\langle f^s, e^s_{m,n,a}\rangle_{l^2}. \end{aligned} \tag{9}$$

3. 2D pattern recognition algorithms

We illustrate in this section an efficient pattern matching algorithm based on the CH-moments $\mathscr{F}_{CSTFT}$. We refer the properties of the method directly to the application on the computer-based recomposition of frescoes, where one would like to localize the images of some fragments on an “old” picture of the fresco prior to damage. However, the resulting method can be efficiently applied in other situations where one wants to detect a small particular in a larger image independently of mutual rotation.

Digital images can be represented by matrices and we consider all the following signals sampled at some fixed resolution τ (without loss of generality one can assume $\tau=1$) and we identify them with their sampling matrix. According to the considerations mentioned above, an image supported on a discrete disk of radius $a>0$ can be represented on a finite dimension space, identified by the set $\Phi_{a,\varepsilon}$. This space has an *almost* orthonormal basis (8) consisting of sampled eigenfunctions of rotation operators. The choice of this space also guarantees that the passage from the continuous to the discrete domain is (almost) error free in angle resolution (9).

3.1. Matching coefficient independent of rotation

An *ideal method* of comparison is realized by rotating one image with respect to the other and by detecting the rotation angle which performs the best possible matching, in some measure. Given a CH expansion of an image, one can rotate it just multiplying the moments in a suitable way by eigenvalues of the rotation operator (9):

$$f = \sum_{m,n} f_{m,n}\, e_{m,n,a}, \quad R_\theta f \approx \sum_{m,n} \mathrm{e}^{\mathrm{i}m\theta} f_{m,n} e_{m,n,a}, \tag{10}$$

where $f_{m,n} = \langle f, e_{m,n,a}\rangle_{l^2}$. The approximation symbol “$\approx$” is due to the *aliasing errors* which are in practice supposed negligible, as we have discussed in the previous section. Given two images f and g (defined on a disk) one can define the *matching coefficient* of f and g, depending on the rotation, by

$$\begin{aligned} M(f,g,\theta) &:= \frac{\langle R_\theta f, g\rangle_{l^2}}{\|f\|_2\|g\|_2} \\ &\approx \frac{1}{\|f\|_2\|g\|_2}\sum_{m,n} \mathrm{e}^{\mathrm{i}m\theta} f_{m,n}\overline{g_{m,n}}. \end{aligned} \tag{11}$$

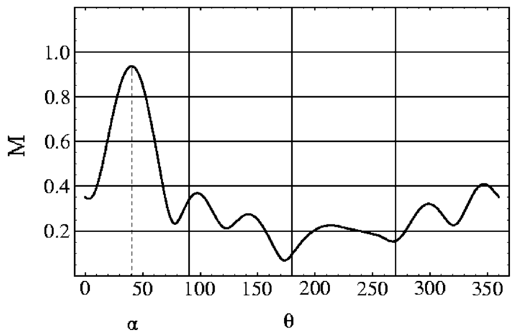

Fig. 4. $|M(f,g,\frac{2\pi}{360}\theta)|$ for $\theta\in[0^\circ,360^\circ)$. The maximum is realized at $\theta=\alpha\approx 40^\circ$, i.e., $|M(f,g,\frac{2\pi}{360}\alpha)| = \max_{\theta\in[0,360^\circ)}|M(f,g,\theta)|$.

This formula is equivalent to measure the angle between $\frac{R_\theta f}{\|f\|_2}$ and $\frac{g}{\|g\|_2}$ as unitary vectors in the finite dimension CH space, indexed by the set $\Phi_{a,\varepsilon}$.

The angle α such that $|M(f,g,\alpha)| = \max_{\theta\in[0,2\pi)}|M(f,g,\theta)|$ is an optimal angle which realizes the best matching (Fig. 4). Let us call this strategy *optimal matching procedure*. In fact, $0\leqslant|M(f,g,\theta)|\leqslant 1$ and it is $|M(f,g,\theta)|=1$ if and only if $f=\lambda g$ except for a mutual rotation of θ, and $\lambda\neq 0$. Hence, the matching coefficient is independent of *contrast* in the sense that if $\lambda>0$ then $M(\lambda f,g,\theta) = M(f,g,\theta)$ [8].

3.2. Computational cost

A direct computation of the maximum of $|M(f,g,\theta)|$ is quite expensive to achieve a good accuracy, because one needs high sampling rate in angle resolution. Hence, it is time consuming and it requires the use of much memory to store all coefficients involved in formula (11) for all the possible positions and angles.

In order to establish a comparison of efficiency with the novel algorithm we are going to illustrate later in this paper, let us estimate here which is the computational cost of the optimal matching procedure. Assume that the sampled CH functions and that the moments $g_{m,n}$ for all the positions are pre-calculated. Denote with $C(\Gamma)$ the number of complex operations to compute the quantity Γ. Because of Eq. (8) if $(m,n)\in\Phi_{a,\varepsilon}$ then one can estimate $m\lesssim 2a$ and $n\lesssim a$ and $\#\Phi_{a,\varepsilon}\sim a^2$. As a consequence, it is not difficult to see that $C(f_{m,n})\sim \pi a^2$ and $C(M(f,g,\theta))\sim 3a^2$. One should multiply this last factor by the number N_{rot} of rotations to be considered, as a relevant constant, and then apply an optimization to detect the best angle α: the complexity of the *Quick Sort algorithm* is $O(N\,\log(N))$ and hence, one has $C(M(f,g,\alpha))\sim 3N_{rot}a^2+\alpha_{QS}N_{rot}\,\log(N_{rot})$. Since the calculation should be executed for all relevant positions of a given $h\times w$ reference image (one should consider all the positions which are in the interior of the reference image excluding a frame of width a), one finally achieves that to compute the best position one should execute ca.

$$\pi a^4 + (h-a)(w-a)(3N_{rot}a^2 + \alpha_{QS}N_{rot}\log(N_{rot})) + \alpha_{QS}(h-a)(w-a)\log((h-a)(w-a))$$

complex operations. Observe that for a given radius $a>0$, to achieve an accuracy comparable with the resolution of the sampling lattice, a suitable choice of N_{rot} is $\sim 2\pi a$.

3.3. The circular sum procedure of computation of $M(f, g, \alpha)$

We want to show here a new procedure of calculation of the matching coefficient, invariant under rotation, by means of a *sequential* (*memory saving*) and *faster* (*FFT based*) comparison of CH-moments where the angle is detected by *implicit* computations. We will show that using this procedure the computational cost can be dramatically reduced in the range of parameters used in concrete applications. We claim also that this novel method of comparison can be as effective and robust as the ideal one, at least for a reasonable class of cases. Before defining the new procedure, we need some further preparation.

If $\widetilde{f}$ is a disk extracted from a fragment (Fig. 5), and f its image under rotation by $-\alpha$ in such a way as to return it to its original orientation, then by Eq. (10)

$$\widetilde{f} \approx \sum_m \sum_n \mathrm{e}^{\mathrm{i}m\alpha} f_{m,n}\, e_{m,n,a} = \sum_m f_m, \tag{12}$$

where $f_m = \sum_n \mathrm{e}^{\mathrm{i}m\alpha} f_{m,n} e_{m,n,a}$ and $(m,n) \in \Phi_{a,\varepsilon}$. If $g^{i,j}$ represents a disk extracted around any position (i,j) of the fresco image G, we could write analogously

$$g^{i,j} = \sum_m g_m^{i,j}, \tag{13}$$

where $g_m^{i,j} = \sum_n g_{m,n}^{i,j} e_{m,n,a}$, $g_{m,n}^{i,j} = \langle g^{i,j}, e_{m,n,a}\rangle_{l^2} \approx \mathscr{F}_{CSTFT}(G)(i,j,m,n)$.

In the following, we assume componentwise operations on the matrices. We consider, now, a matrix F_m with same dimensions $h \times w$ as G with entries consisting of zeros except for the entries representing f_m, translated *modulo* $h \times w$ so that the center of the disk coincides with the left upper corner of G, considered as the origin of the axes. As discussed in Section 2 we can execute a fast comparison of the CHs moments by convolutions and the use of the FFT.

Let us denote $F = \sum_m F_m$ and $F_m^*(k) = F_m(-k)$ modulo $h \times w$, for $k = (i,j) \in \mathbb{Z}^2$. According to Eq. (8), we have

$$\begin{aligned}(F_m^* * G)(i,j) &= \langle f_m, g^{i,j}\rangle_{l^2} \\ &\approx \langle f_m, g_m^{i,j}\rangle_{l^2} \\ &\approx \sum_n \mathrm{e}^{\mathrm{i}m\alpha} f_{m,n}\overline{(g_{m,n}^{i,j})}.\end{aligned} \tag{14}$$

It is well known that a convolution is equivalent to a multiplication in the frequency domain and it is also

$$\mathscr{F}^{-1}(\widehat{F_m^*}\widehat{G})(i,j) = (F_m^* * G)(i,j). \tag{15}$$

Even if the $e_{0,n,a}$'s are the only totally rotation invariant components of the expansion, we will define the matching coefficient computation without counting the contribution of the projection on the $e_{0,n,a}$'s. In fact, these components cannot be useful in order to detect mutual angles. Moreover, this choice makes the comparison independent of *brightness*, since $\langle c, e_{m,n,a}\rangle = 0$ for any constant $c>0$ and $m \neq 0$. Hence, using $m>0$, one defines inductively the procedure for an *implicit approximated calculation of optimal matching and angle*: at some $m>0$, one assumes that a first determination of the optimal angle, say $\alpha_{m-1} \approx \alpha$, is given maybe by means of some calculations on previous coefficients $v_k = \sum_n f_{k,n}\overline{g_{k,n}^{i,j}}$, $k = 1, \ldots, m-1$. Then, one computes a successive approximation/correction of the optimal angle using the next independent complex vector v_m, just rotating back it of α_{m-1}, i.e., multiplying v_m by $\mathrm{e}^{-\mathrm{i}(m-1)\alpha_{m-1}}$, and setting $\alpha \approx \alpha_m = \arg(\mathrm{e}^{-\mathrm{i}(m-1)\alpha_{m-1}}v_m)$. An initial approximation from which to start can be deduced by v_1 whenever f and $g^{i,j}$ can be "close enough" except for a rotation of α. Formally, this can be expressed by the following procedure:

$k \in \mathbb{N}\backslash\{0\}$ and $z, v \in \mathbb{C}\backslash\{0\}$.

We consider the binary operator $\oplus_k$ defined by

$$z \oplus_k v = z + v\left(\frac{z}{|z|}\right)^{1-k}. \tag{16}$$

For $m \in \mathbb{N}\backslash\{0\}$ and $v_1, \ldots, v_m \in \mathbb{C}\backslash\{0\}$ the *circular sum* of order m is the operator $\bigoplus_{k=1}^m$ defined recursively by

$$\begin{aligned}&\textstyle\bigoplus_{k=1}^m v_k = v_1, && m=1,\\ &\textstyle\bigoplus_{k=1}^m v_k = (\bigoplus_{k=1}^{m-1} v_k)\oplus_m v_m, && m>1.\end{aligned} \tag{17}$$

By induction on formula (17) one can easily show the following formal properties of the circular sum:

For $m \in \mathbb{N}\backslash\{0\}$ let $v_1 = \rho_1\mathrm{e}^{\mathrm{i}\alpha}, \ldots, v_m = \rho_m\mathrm{e}^{\mathrm{i}m\alpha} \in \mathbb{C}\backslash\{0\}$, $\rho_k > 0$ for all $k = 1, \ldots, m$ and $\alpha \in [0, 2\pi)$.

(i) (Angle detection property) $\bigoplus_{k=1}^m v_k = \mathrm{e}^{\mathrm{i}\alpha}(\sum_{k=1}^m \rho_k)$.

(ii) (Angle detection in presence of errors) If $v_{m-1} = \rho_{m-1}\mathrm{e}^{\mathrm{i}((m-1)\alpha+\delta_{\alpha_{m-1}})}$ and $v_m = \rho_m\mathrm{e}^{\mathrm{i}(m\alpha+\delta_{\alpha_m})}$ then

$$\bigoplus_{k=1}^m v_k = \mathrm{e}^{\mathrm{i}\alpha}\left(w_{m-1} + \frac{\mathrm{e}^{\mathrm{i}\delta_{\alpha_m}}\rho_m}{sgn(w_{m-1})^{m-1}}\right), \tag{18}$$

where $w_{m-1} = \sum_{k=1}^{m-2}\rho_k + \mathrm{e}^{\mathrm{i}\delta_{\alpha_{m-1}}}\rho_{m-1}$. As we will discuss in Section 4 this last property ensures stability in angle resolution also in presence of strong noise.

We are now ready to define the new procedure for computing the matching coefficient.

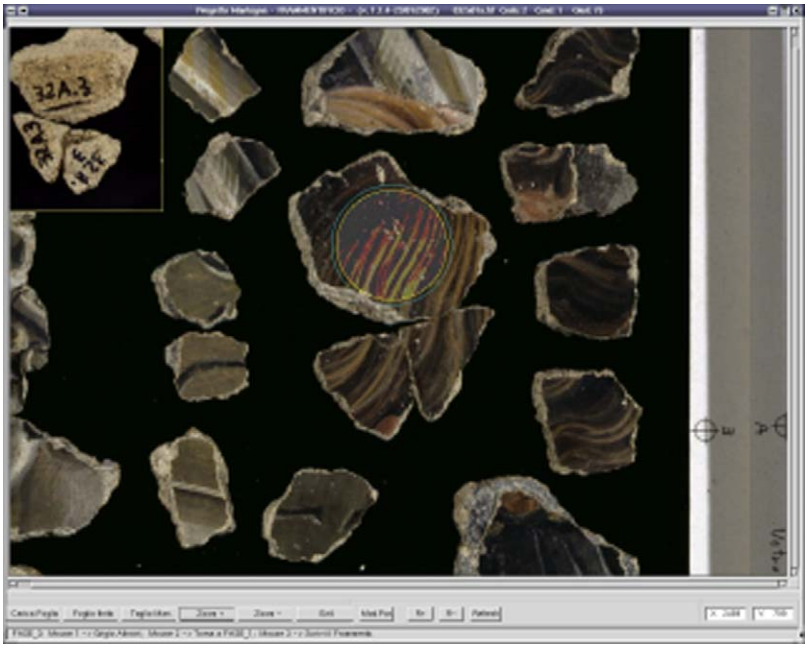

Fig. 5. The figure shows a sheet from the digital catalog of fragment images. In particular a disk portion is extracted from a fragment to be processed for the localization.

With the notation used above, we define the *matching matrix* of f and G by

$$M(F,G) := \frac{1}{\sum_{k=1}^{m} \|f_k\|_2^2} \bigoplus_{k=1}^{m} \frac{\|f_k\|_2}{(\|g_k^{i,j}\|_2)_{i,j}} \mathscr{F}^{-1}(\widehat{F_k^*}\widehat{G}), \tag{19}$$

where $\|f_k\|_2$ is the (discrete) norm of f_k, $(\|g_k^{i,j}\|_2)_{i,j}$ is the matrix of the norms of $g_k^{i,j}$ and $m > 0$ is the maximal integer for which there exists $n \in \mathbb{N}$ such that $(m,n) \in \Phi_{a,\varepsilon}$. The circular sum in Eq. (19) acts componentwise on the matrices.

For each point (i,j) of the image G, the matching matrix returns the matching coefficient given by $0 < |M(F,G)(i,j)| \leqslant 1$, measure of the correspondence with the fragment at (i,j) independently of the relative rotation. In fact, if we suppose there exists a position $(\bar{i},\bar{j})$ in G, such that

$$g^{\bar{i},\bar{j}} = f, \tag{20}$$

from Eq. (14)–(15) one has

$$\mathscr{F}^{-1}(\widehat{F_k^*}\widehat{G})(\bar{i},\bar{j}) \approx \mathrm{e}^{\mathrm{i}k\alpha}\|f_k\|_2^2 \tag{21}$$

for each k. Hence, by the property (i) of the circular sum, we have

$$M(F,G)(\bar{i},\bar{j}) \approx \frac{1}{\sum_{k=1}^{m} \|f_k\|_2^2} \bigoplus_{k=1}^{m} \mathrm{e}^{\mathrm{i}k\alpha}\|f_k\|_2^2 = \mathrm{e}^{\mathrm{i}\alpha}. \tag{22}$$

We deduce that a necessary condition so that a position $(\bar{i},\bar{j})$ is the original one of the fragment, is

$$|M(F,G)(\bar{i},\bar{j})| \approx 1, \quad \arg(M(F,G)(\bar{i},\bar{j})) \approx \alpha. \tag{23}$$

Except for negligible aliasing errors, the condition is sufficient, since we have shown that $(\mathscr{F}_{CSTFT}(G)(i,j,m,n))_{i,j,m,n}$ is a complete information. See Fig. 6 for the result of the application of such procedure on the fragment illustrated in Fig. 5.

Of course one can adaptively construct a subset of $\Phi_{a,\varepsilon}$ depending on the particular fragment, in order to *compress* the information used in the calculation of Eq. (19), reducing much the computational cost. In practice, we have verified that for general fragments most of the energy is concentrated in the lower m-components $m = 1, 2, 3, \ldots$ that can be already enough to have a quite discriminating comparison.

3.4. Computational cost and comparison with the optimal matching procedure

Since $\widehat{G}$, and the norms $\|g_k^{i,j}\|_2^2$ can be pre-calculated, the computational cost of the matching matrix depends essentially on the fragment quantities, the correlation

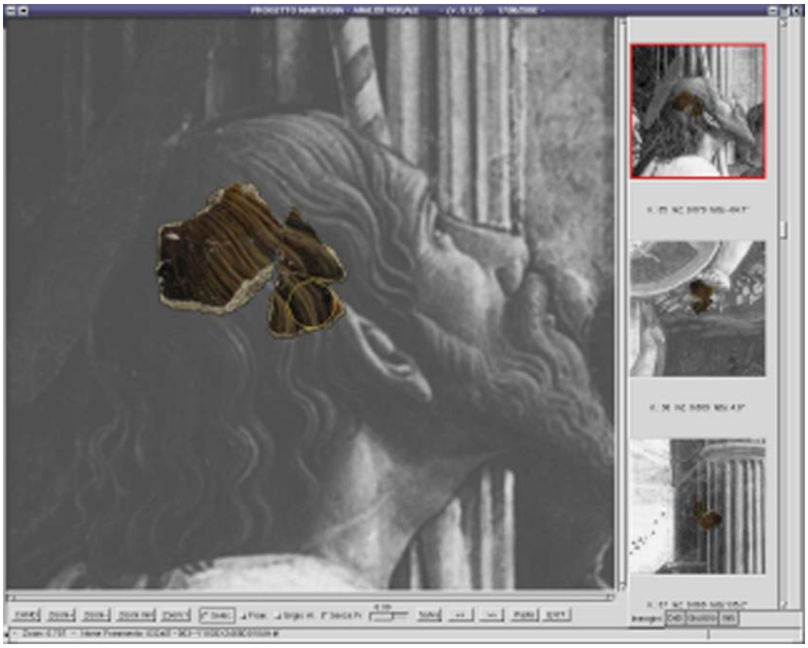

Fig. 6. The fragment illustrated in Fig. 5 has been correctly detected and here it is shown overlapping the corresponding part of the old gray fresco photo dated to 1920.

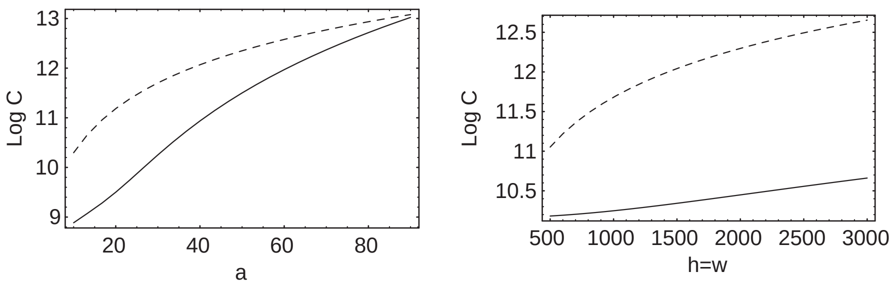

Fig. 7. The computational cost of the optimal matching procedure (dashed curve) and of the circular sum procedure (continuous curve) are shown. With respect to increasing values of radius $6 \leqslant a \leqslant 90$ on the left and fixed size $h = w = 1024$, and increasing values of the dimensions $512 \leqslant h = w \leqslant 3000$ and fixed radius $a = 30$ on the right. With radius $a \approx 20$ and size $h = w = 1024$ the cost can be 50 times smaller.

and the cost of the circular sum. By computations in Section 3.2 one has

- $C(F_k) \sim \pi^2 a^5 + \pi a^3$;
- since F is zero except for $2a$ rows/columns one can compute $\widehat{F_k^*}$ in $C(\widehat{F_k^*}) \sim 2a \cdot \alpha_{FFT}(h \log(h) + w \log(w))$ complex operations, by applying the 1D-FFT first along the rows and then along the columns;
- $C(\widehat{F_k^*}\widehat{G}) \sim hw$;
- $C(\mathscr{F}^{-1}(\widehat{F_k^*}\widehat{G})) \sim \alpha_{FFT}^2 hw \log(hw)$;
- $C(\|f_k\|_2^2) \sim 2a$;
- $C(\bigoplus_{k=1}^m v_k) \sim \frac{7m+m^2}{2}$;

From these estimates one shows that

$$C(M(F,G)) \sim 2a(\pi^2 a^5 + \pi a^3 + a \cdot \alpha_{FFT}(h \log(h) + w \log(w)) + hw + \alpha_{FFT}^2 hw \log(hw) + 2a) + (7a + 2a^2)hw.$$

One should add to this the execution of a Quick Sort on the full matrix to compute the best position. In the range of parameters in concrete applications (usually $6 \leqslant a \leqslant 30$ and $512 \leqslant h, w \leqslant 3000$), one can show that the computational cost is much less than that required by the optimal matching procedure (see Fig. 7).

With one fast calculation one can deduce both the matching coefficient and the rotation for any position of the fresco. This method can be applied in every case where one wants to localize a detail in a big scene independently of relative rotation, brightness, and contrast and it is also effective with very noisy data, proving its robustness (see the following section).

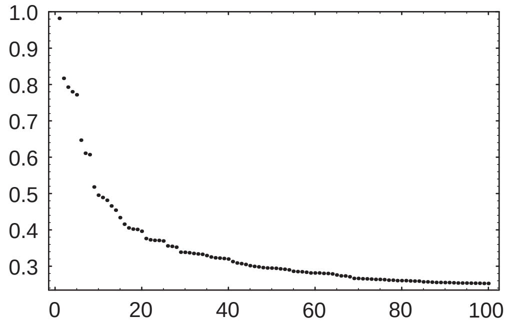

Fig. 9. Matching curve of an auto-fragment for the 100 first best positions. Each position is mapped into its corresponding matching coefficient.

4. Stability and numerical results

4.1. Pattern matching without noise

Extracting from a digital image G an *auto-fragment f*, i.e., a disk portion of a digital image which is rotated with respect to its original orientation, and applying the matching matrix $M(F,G)$, one can rearrange the entries of the matrix by decreasing values of the matching coefficient $|M(F,G)(i,j)|^2$. One would like that the first positions indicate the original position. Moreover, one would like also that the best positions can have matching coefficients clearly much larger (discriminating power) than the others (wrong positions). Let us consider the function, say it *matching curve*, which maps each position into its matching coefficient, rearranged by decreasing values of the matching coefficient.

Figs. 8 and 9 show calculations of best positions applied on auto-fragments. For all tests that have been applied on auto-fragments extracted from 1000 × 1000 pixels random images, the method has been successful in the 100% of the cases, in the sense that the first best position out

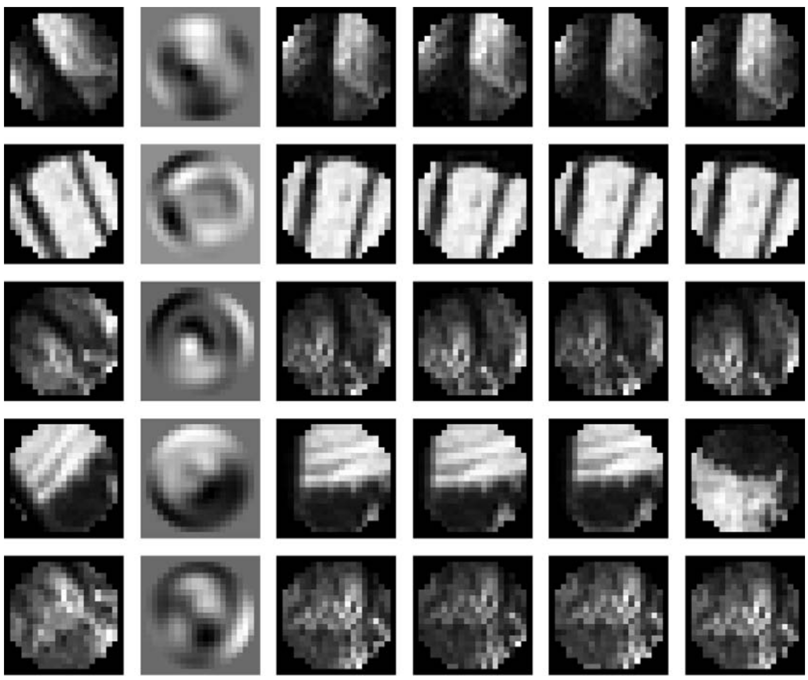

Fig. 8. Four best positions of a series of 5 auto-fragments are shown. The sample auto-fragments extracted from pictures of the fresco and rotated by $-37°$ are listed in the first column. In the second column the projection of each fragment onto the space $\Phi_{a,\varepsilon}$ and $m>0$ is presented. The last four columns show the best positions up to rotation with respect to the matching coefficient the algorithm is able to detect. They look very similar, but in fact they differ by the minimal displacement.

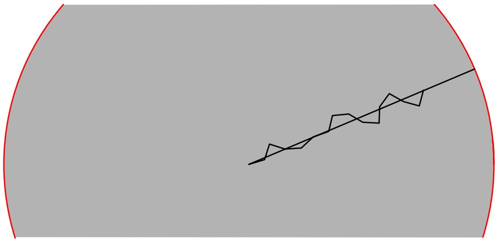

Fig. 10. Angle error compensation by the circular sum auto-correcting property. The straight line indicates the right angle and its length the right matching. A random error is added on each single term of the circular sum. The matching coefficient is reduced from 1 to 0.78.

of 1 000 000 possible competitors has always coincided with the original and the rates of decay in the first 10 positions were of order $\frac{1}{5}$. Moreover, the calculation of the angle by means of $\arg(M(F,G)(i,j))$ is also correct, except for small errors, fast decreasing with increasing resolutions.

4.2. Real-world cases: noise presence

In real-world cases the comparison between images of the same object, but actually taken in different technical and time situations, can be made very difficult by the presence of noise (or just by some strong differences as in the case of the fragments damaged by the bomb) and the correct order in the matching curve will be surely perturbed. We want to show now that the suggested (rotation) invariant (23) is quite robust in real-world cases. The circular sum operator is responsible of the calculation of the matching matrix and one of the essential ingredients of the proposed method. In particular, property (ii) of the circular sum is a specific and suggestive *auto-correcting property*: the errors δ_{α_k} on the angles at some k are compensated by the next term $k+1$ of the circular sum.

One can make an analytic example in a very simplified case:

If $v_k = e^{ik\alpha}$, $k = 1, \dots, m-2, m$ and $v_{m-1} = e^{i((m-1)\alpha+\delta_{\alpha_{m-1}})}$, it is not difficult to show that for m large enough and $\delta_\alpha \approx 0$

$$\bigoplus_{k=1}^{m} v_k \approx e^{i\alpha}((m-2) + e^{i\delta_{\alpha_{m-1}}} + e^{-i\delta_{\alpha_{m-1}}}). \quad (24)$$

This compensating effect is quite desirable in order to stabilize the angle calculation, while the reduction of the matching coefficient cannot be eliminated (see Fig. 10).

Motivated by the fast computation of the matching matrix and by the shown robustness, we have extensively applied the method on the real problem of the localization of fragments of the huge Andrea Mantegna's art fresco ("Stories of St. James and St. Christopher", 12 scenes of 15 m^2 each.). All fragments have been already tested on three of the scenes of 15 m^2 and we have detected the original position for 500–900 fragments on each. Some current results are presented in Figs. 5,6,11,12. In the experimentation, the first 100 best positions returned by the algorithm were considered and discussed by human operators. The calculation of a digitalized fragment of radius $a = 10$ pixels (equivalent to 2–3 cm^2 real dimensions) on one scene of 3000×2400 pixels (equivalent to 15 m^2 real dimensions) takes about 120 s with a C/C++ implementation on a standard PC (AMD K7 Athlon 1 GHz, 500 MB RAM) with the FFTW library to compute discrete Fourier transforms.

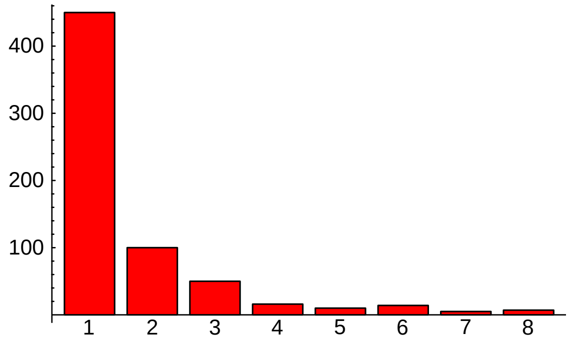

Fig. 11. Distribution of localized fragments with respect to the matching curve rank. The 75% of the localized fragments have been found in the first 3 best positions and the 90% in the first 20. The total number of localized fragments is ca. 800.

4.3. Comparison with the ideal method in presence of noise

In this section, we want to compare the optimal matching procedure (Fig. 4) with the "circular sum procedure" (Fig. 10). Fig. 13 shows the two methods in presence of noise: Montecarlo experiments of matching are applied on a random pattern with respect to itself, affected by white noise of increasing energy. In particular, the percentiles curves (5% − 50% − 95%) related to matching coefficient and angle distributions are depending essentially on the signal/noise (S/N) ratio and on the number of CH-moments used. For a fixed number of CH-moments (typically about 100), the methods look equivalent for ratio $S/N \geqslant 0.8$.

4.4. Increasing the discrimination power

The algorithm returns a sequence of positions ordered by means of the matching coefficient. In fact, the results of the experimentation on the fresco problem show that, for the 90% of the cases, one can reduce the analysis of the possible matching positions from $h \times w = 3000 \times 2400 = 7\,200\,000$ to the first 20 best ones only (Fig. 11). We want to exemplify how the discrimination power can be improved. In fact, one can extract from a fragment more than one disk portion/selection. Given multiple sequences of best positions,

Fig. 12. Some re-composed fragments are shown on the original background.

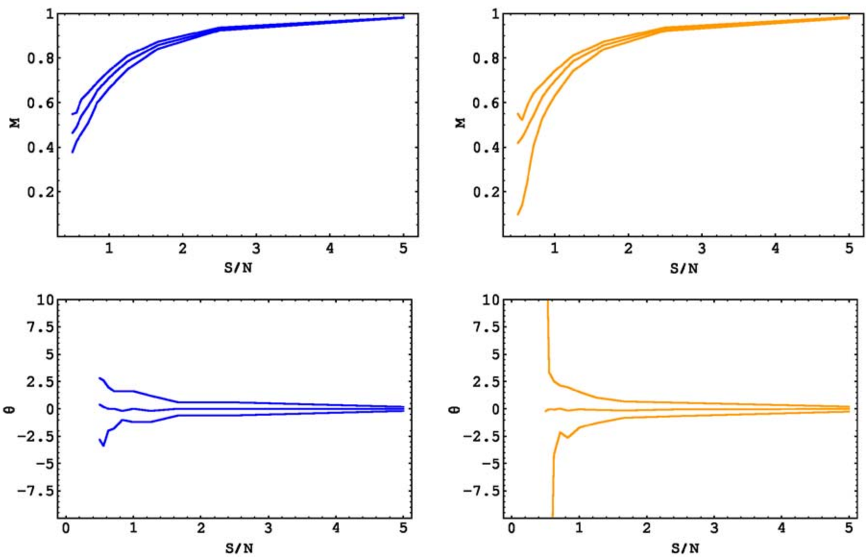

Fig. 13. Percentiles curves (5% – 50% – 95%) of the matching and angle distributions when a random noise of various energy is added to the signal. The percentile values were calculated by the "optimal matching" (left) and by the "circular sum" (right) procedures with respect to increasing S/N ratio. The two methods look equivalent for ratio $S/N \geqslant 0.8$.

maybe computed by the circular sum procedure on different selections, one can keep only those positions which are respecting the original mutual positions on the fragment and that have the same computed angle. Due to the robustness in detecting correct angles, a fast constraint check applied on the first 100 best positions returned by the circular sum procedure of the two selections of the fragment in Fig. 14 reduced the coupled possible positions to 1 only:

Fig. 14. Multiple disk selections extracted from the same fragment.

Fig. 15. The original position on the fresco of the fragment shown in Fig. 14.

$p = (890, 1183)$ and $q = (884, 1189)$ with mutual distance $dist(p, q) \approx 8.48$ (pixels) while the mutual distance of the original selections (Fig. 14) is $dist(sel_1, sel_2) \approx 8.94$; the calculated angles are $\alpha_1{=}3.09$ and $\alpha_2{=}3.11$ (radiant). Therefore, redundant calculations on a fragment ensure a very accurate detection of the original position of the fragment (Fig. 15) with high discrimination power. An other way to improve the discrimination power is to combine our method with fast local registration [25] applied on the best positions returned by the algorithm. The list can consequently be reordered. In fact, fast registration algorithms based on optimization methods works when the images are already "close enough". Because of the robustness and the accuracy in angle detection, the fractional rigid rotation registration can be efficiently realized by using our algorithm. On the other hand, two digital images of the same object cannot in general be sampled exactly on the same grid. Some

Table 1
The five best positions of the matching, with rotation angle and coordinates (X, Y), computed on the same fragment before (left) and after (right) the fractional shift registration

Match.	Angle	Y	X	Match.	Angle	Y	X
0.993	0.232	67	70	0.996	0.228	67	70
0.920	0.193	66	70	0.982	0.212	66	70
0.889	0.244	68	70	0.956	0.231	68	70
0.816	−2.904	**71**	**75**	0.940	0.233	**67**	**69**
0.795	−2932	**72**	**75**	0.934	0.224	**68**	**69**

The positions, after the registration, form a cluster with matching coefficients very close and quite high and with reduced variance for the angles. The first 3 positions, in the same order, are present also before registration (left), while the fourth and fifth positions substitute the corresponding incorrect positions at left (the coordinates of the positions involved in the replacement are indicated with bold numericals).

fractional shifts (or more complex elastic deformations) are usually present. One can simulate this effect by considering an image at resolution τ, shift it of one pixel, for example, in the right direction, and then scaling both, the original and the shifted, to resolution 2τ. The resulting images are representing the same object but in fact they differ of a fractional shift. Table 1 compares the output of our algorithm in a real case with its optimized version by local shift registration.

5. Conclusion

In this paper we propose a fast, robust, and accurate digital image pattern recognition algorithm which is independent of mutual rotation and displacement. The accuracy and the robustness of the procedure are realized by exploiting the independent (orthogonal) information given by the so-called circular short time Fourier transform (CSTFT) at different angular and radial frequencies. The CSTFT is here implemented by discrete scalar product of the digital image with respect to location-shifted and sampled compactly supported Circular Harmonic (CH) functions, selected among those affected by minimal aliasing, up to a prescribed tolerance. The computational efficiency of the algorithm is given by the combined use of the correlation implemented by fast Fourier transforms for the location/position detection, and of a tricky implicit and fast computation of the mutual angle by exploiting self-steerability properties of CH functions. The proved algorithm robustness and efficiency have allowed its use for re-assembling fragments of one of the most important art frescoes of the Italian Renaissance, destroyed by a bombing during the Second World War. Because of the huge database of patterns and the large scale dimension, the results of the experimentation in this concrete problem are relevant to describe the power of discrimination and the efficiency of such method. Up to 90% of the fragments are detected in the first 20 best positions the algorithm returns out of more than 7 millions possible. The implementation of combined redundant computations and registration methods improves further this ratio, realizing in most of the cases an almost completely automatic detection of the fragments.

Acknowledgements

The authors want to thank Claudio Fanin (Istituto Nazionale di Fisica Nucleare, sezione di Padova, Italy) and Giuseppe Galeazzi (Department of Physics “G. Galilei”, University of Padova, Italy) for the fruitful cooperation and their valuable suggestions during the preparation of this work. The authors are indebt with Rocco Cazzato and Germano Costa for the technical support and the management of the Mantegna Laboratory (University of Padova), during the experimental part of the work.

Appendix A. Notations and conventions

For the reader convenience, we list here the more specific and relevant notations we have defined and used in the content in their order of appearance:

We denote by $L^2(\Omega)$ the Lebesgue space of square-summable functions on $\Omega \subset \mathbb{R}^2$; as we write $\tau > 0$ we mean $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2_+$ and $det(\tau) = \tau_1\tau_2$; the multiplication of two vectors $x, y \in \mathbb{R}^2$ is assumed componentwise $x \cdot y := xy := (x_1y_1, x_2y_2)$; by $l^2(\tau\mathbb{Z}^2)$ we denote the space of square-summable sequences indexed on $\tau\mathbb{Z}^2$; as we have fixed τ, we will write l^2 instead of $l^2(\tau\mathbb{Z}^2)$ and we will suppose l^2 endowed with the scalar product $\langle f, g\rangle_{l^2} \equiv det(\tau)\sum_{k\in\mathbb{Z}^2} f(\tau k)\overline{g(\tau k)}$, $\overline{f}$ being the complex conjugation of *f*. For any function (on continuous or discrete domains) h we define $h^*(x) = h(-x)$ and $I(h) = \overline{h^*}$ the involution operator and with $f * g$ the convolution operator of *f* and *g*;

$e_{m,n,a}$ is the CH function on the disk Ω_a centered in 0 with radius $a > 0$; J_m is the Bessel function of the first kind of order $m \in \mathbb{Z}$; $(j_{m,n})_{n\in\mathbb{N}}$ is the sequence of the positive zeros of J_m; $H^1_0(\Omega)$ is the Sobolev space of functions vanishing on the border $\partial\Omega$; R_α is the rotation operator of angle α; $\hat{f}$ or $\mathscr{F}(f)$ indicate the *Fourier transform* of *f*; if *f* is a discrete image then we will denote again by $\mathscr{F} f$ or $\hat{f}$ the *discrete Fourier transform* of *f*; $\mathscr{F}_{CSTFT}$ is the CSTFT; $\Phi_{a,\varepsilon}$ is the *frequency set* of couples (m, n) such that the corresponding $e_{m,n,a}$ *minimizes the aliasing error up to* ε; $f_{m,n} = \langle f, e_{m,n,a}\rangle_{l^2}$ is the CH moment of *f* with respect to the frequency $(m, n) \in \Phi_{a,\varepsilon}$; $M(f, g, \theta)$ is the *matching coefficient* of the images *f* and *g*, depending on the rotation θ; $C(\Gamma)$ is the number of complex operations to compute the quantity Γ; $f_m = \sum_n \langle f, e_{m,n,a}\rangle_{l^2} e_{m,n,a}$; $\bigoplus_{k=1}^m$ *circular sum* procedure of order $m \in \mathbb{N}$; $M(F, G)$ is the matching matrix of the fragment matrix *F* with respect to the image *G*.

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