

# Fluid dynamic description of flocking via Povzner–Boltzmann equation

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December 16, 2009

## Abstract

We introduce and discuss the possible dynamics of groups of undistinguished agents, which are interacting according to their relative positions, with the aim of deriving hydrodynamical equations. These models are developed to mimic the collective motion of groups of living individuals such as bird flocks, fish schools, herds of quadrupeds or bacteria colonies. Our starting model for these interactions is the Povzner equation [21], which describes a dilute gas in which binary collisions of elastic spheres depend of their relative positions. According to the Cucker and Smale model [9], we will consider binary interactions between agents that are dissipative collisions in which the coefficient of restitution depends on their relative distance. Under the assumption of weak dissipation, it is shown that the Povzner equation is modified through a correction in the form of a nonlinear friction type operator. Using this correction we obtain formally from the Povzner equation in a direct way a fluid dynamic description of a system of weakly interacting agents interacting in a dissipative way, with a coefficient of restitution which depends on their relative distance.

**Key words.** Swarming, Povzner equation, dissipative collisions.

**AMS(MOS) subject classification.** 76P05, 82C40.

## 1 Introduction

The aim of this paper is to discuss some questions connected with the modeling of the evolution of groups of agents, which are interacting according to their relative positions.

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In the pertinent literature, these models are developed to mimic the collective motion of groups of living individuals such as bird flocks, fish schools, herds of quadrupeds or bacteria colonies [5, 11, 13, 19, 20, 24]. Once initialized with a certain velocity distribution, these groups are developing particular profiles with time, like flocking of birds, in which all birds fly with the same velocity close each other. Among other approaches, these groups can be described at a mesoscopic level within the concepts of classical statistical mechanics, by means of methods borrowed from the kinetic theory of rarefied gases [14, 6]. In discrete models of flocking [9, 10] it appears clearly that the formation of flocking structures is heavily dependent of the loss of kinetic energy. This property remains true at a continuous level, where the loss of kinetic energy is used to describe the large-time behavior of the kinetic equation [14, 6]. At a kinetic level, this establishes a link between groups of agents and granular gases. In granular gases, the dissipation of kinetic energy causes in fact a series of non-trivial effects, as formation of clusters and other spatial structures [12, 17], non-Maxwellian velocity distributions, anomalous diffusion, and others.

The recent history of the kinetic description of granular gases, however, shows clearly that, while extremely powerful to give the precise evolution of the particle density from a theoretical point of view, the numerical approach to real problems owing to Boltzmann-type equations has a prohibitive cost. In most of the applications, in fact, rapid granular flows are described at the macroscopic level by means of equations for fluid dynamics, modified to account for dissipation due to collisions among particles. This was the approach of Haff, which, in his pioneering paper [15], gave a macroscopic description of the behavior of a granular material treating the individual grains as the molecules of a granular fluid, without resorting to the mesoscopic picture (the Boltzmann or Enskog kinetic equations). On the other hand, in agreement with the well established derivation of conservative fluid dynamics from the Boltzmann equation [2, 3], kinetic theory can be used as the basis for a deeper understanding of macroscopic equations even for dissipative flows [4, 23].

The hydrodynamic description of a granular gas is well-understood in presence of weak dissipation [23]. Weak dissipative granular gases with variable restitution coefficient, can be described by introducing a correction to the classical Boltzmann collision operator. This correction is represented by a nonlinear friction type operator, with a kernel which depends of the variable restitution coefficient. This representation allows to obtain a formal derivation of hydrodynamic equations for weakly dissipative granular gases, by closing the conservation equations with respect to the classical Maxwellian function.

We will adopt the same strategy here. The main difference between granular gas molecules and groups of agents is that in the former case, molecules collide as rigid-spheres when they are at the distance of a diameter, while in the latter case, agents interact at any distance (long-range interactions). In all the other aspects, collisions are very similar. In a granular gas binary collision, mass and momentum are preserved, while the energy is dissipated (the dissipation depending of the relative velocity). In a binary interaction of birds, mass and momentum are preserved, while the energy is dissipated (the dissipation depending of the relative positions). More precise hypotheses will be made in the next section, where previous models will be checked in some detail.

The connection between solutions of the Euler equations for compressible fluids, and the solutions of an equation describing the dynamics of a system of particles undergoing elastic collisions at a stochastic distance has been enhanced some years ago by Lachowicz and Pulvirenti [16]. There, it was discovered that the underlying kinetic equation able to represent the one-particle dynamics as the number of particles tends to infinity is the Povzner equation [21], which was originally introduced by Povzner for purely mathematical reasons, considering a smearing process for the pair collisions. In Povzner equation, in fact, the unit vector which is responsible of the post-collision velocities depends of the relative positions of particles.

We will briefly present the result of [16] in Section 2, where also Povzner equation will be described in some details. Corrections to the Povzner collision operator which take into account dissipation will be the main contribution in the direction of the flocking modeling. The passage to fluid dynamics will be described in Section 3, where the corrections to the classical Euler equation derived from the dissipative correction to the Povzner equation will be dealt with. A description of the possible steady states of these Euler equations, and their connection with the flocking phenomenon will conclude our analysis. Some numerical experiments in the one-dimensional case will be presented in Section 4.

## 2 The flocking dynamics of a bird population

### 2.1 The dynamics of a stochastic particle system

In [16] Lachowicz and Pulvirenti established an interesting connection between solutions of the Euler equations for compressible fluids, and the solutions of an equation describing the dynamics of a system of particles undergoing elastic collisions at a stochastic distance. More precisely, consider density, velocity and temperature fields  $\rho(x, t)$ ,  $u(x, t)$  and  $T(x, t)$  which constitute a (smooth) solution of the Euler equations (up to some time  $t_0$  before the appearance of the first singularity), and construct a local Maxwellian function  $\mathcal{M}$  whose mean density, velocity and temperature are given by  $\rho$ ,  $u$  and  $T$ , respectively [8]

$$\mathcal{M}(x, v, t) = \frac{\rho(x, t)}{(2\pi T(x, t))^{3/2}} \exp\left(-\frac{(v - u(x, t))^2}{2T(x, t)}\right). \quad (1)$$

Consider also a system of  $N$  particles located at the points  $x_1, x_2, \dots, x_N$  on a domain of  $\mathbb{R}^3$ , which move freely unless a pair of them undergo an elastic collision, expressed by the formula

$$v'_i = v_i - \frac{1}{2}((v_i - v_j) \cdot n_{ij})n_{ij}, \quad v'_j = v_j + \frac{1}{2}((v_i - v_j) \cdot n_{ij})n_{ij}. \quad (2)$$

where the unit vector  $n_{ij}$  is given by

$$n_{ij} = \frac{x_i - x_j}{|x_i - x_j|}. \quad (3)$$

As usual,  $v'_i$  and  $v'_j$  denote the outgoing velocities, where the ingoing velocities are given by  $v_i$  and  $v_j$ , provided that  $(v_i - v_j) \cdot n_{ij} < 0$ . Each binary collision takes place according to a stochastic law. The collision times for each pair  $i$  and  $j$  of particles are independent Poisson processes with intensity given by  $\varphi(x_i, x_j, v_i, v_j)|v_i - v_j|$ , and  $\varphi$  is given by

$$\varphi(x_i, x_j, v_i, v_j) = \frac{3}{N\delta^3} \frac{1}{\epsilon} \chi(|x_i - x_j| \leq \delta) \chi(|v_i - v_j| \leq \theta). \quad (4)$$

In (4)  $\chi(I)$  is the characteristic function of the subset  $I$ .

The evolution of the system of particles is described by the  $N$ -particle distribution function  $f^N(x_1, v_1, \dots, x_N, v_N, t)$  which gives the probability density for finding the  $N$  particles in the points  $x_1, \dots, x_N$  with velocities  $v_1, \dots, v_N$  at the time  $t \geq 0$ . Let the  $s$ -particle distribution functions be defined by the marginals

$$f^{N,s}(x_1, v_1, \dots, x_s, v_s) = \int f^N(x_1, v_1, \dots, x_N, v_N) dx_{s+1} dv_{s+1} \cdots dx_N dv_N.$$

Then, under some additional hypotheses on the regularity of the solutions to the Euler system in the time interval  $[0, t_0]$ , it is proven in [16] that, for all  $\sigma > 0$  there exist  $\epsilon_0(\sigma)$ ,  $\delta_0(\sigma, \epsilon)$ ,  $\theta_0(\sigma, \epsilon, \delta)$  and  $N_0(\sigma, \epsilon, \delta, \theta)$  such that if  $\epsilon \leq \epsilon_0$ ,  $\delta \leq \delta_0$ ,  $\theta \geq \theta_0$ ,  $N \geq N_0$

$$\sup_{t \in [0, t_0]} \|\mathcal{M} - f^{N,1}\| < \sigma,$$

where  $f^{N,1}$  is the 1-particle marginal corresponding to the  $N$ -particle distribution function  $f^N(x_1, v_1, \dots, x_N, v_N, t)$  with initial conditions

$$f^{N,s}(x_1, v_1, \dots, x_s, v_s, t = 0) = \prod_{j=1}^s \mathcal{M}(0; x_j, v_j).$$

The analysis of [16] shows that, as the number of particles tends to infinity, the 1-particle marginal  $f^{N,1} = f$  satisfies the (elastic) Povzner's equation [21]. This kinetic equation was introduced by Povzner in 1962, through a modification of Boltzmann collision operator consisting in a smearing process for the pair collisions. The modified Povzner's collision operator looks as follows

$$Q_P(f, f)(x, v) = \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw B(x - y, v - w) (f(x, v_*) f(y, w_*) - f(x, v) f(y, w)). \quad (5)$$

In (5)  $B$  is the collision kernel, while  $(v_*, w_*)$  are the pre-collision velocities of the so-called inverse collision, which generate the pair  $(v, w)$ . The relationship between the pair  $(v, w)$  and the post-collision velocities  $(v^*, w^*)$  is expressed by

$$v^* = (I - A(x - y))v + A(x - y)w, \quad (6a)$$

$$w^* = A(x - y)v + (I - A(x - y))w. \quad (6b)$$

where  $A$  is a  $3 \times 3$  matrix and  $I$  the identity matrix. In Povzner's equation the matrix  $A$  is such that momentum and energy are preserved in a collision. The conservation of energy  $(v^*)^2 + (w^*)^2 = v^2 + w^2$ , yields straightforwardly

$$2(A - I)A(v - w) = 0, \quad \text{for all } v, w \in \mathbb{R}^3,$$

or

$$A^2 = A.$$

In what follows we choose the matrix  $A$  to be 1-rank, and expressed with respect to the unit vector  $n = n(x - y) = \frac{x-y}{|x-y|}$

$$A(x - y) = n \cdot n^\top.$$

Consequently

$$v^* = v - (v - w) \cdot n(x - y)n(x - y), \quad w^* = w + (v - w) \cdot n(x - y)n(x - y). \quad (7)$$

Note that the main difference between the classical Boltzmann equation and the Povzner's one is that the positions of the particles enter into the interaction rule. From a mathematical point of view, Povzner's equation was introduced to overcome the main difficulty in solving the classical Boltzmann equation, where binary collisions between particles happen at the same point  $x = y$ . Maybe for this reason, Povzner's equation has been usually ignored by the physicists, while only few mathematical papers deal with it [1, 7, 16].

Going back to the result of [16], it was proven that, in presence of a stochastic law for binary interactions, if the particles are initially identically and independently distributed according to a distribution density  $F = F(x, v)$ , then at later times they are identically and independently distributed according to a solution of a kinetic equation with collision integral given by the Povzner's operator (5), with initial datum  $f_o = F$ , and rate function

$$B(x - y, v - w) = \frac{1}{2} \chi(|x - y| < \delta) \chi(|v - w| < \theta) |(v - w) \cdot n|,$$

where  $\delta$  and  $\theta$  are positive constants, and  $\chi(E)$  denotes the characteristic function of the set  $E \subseteq \mathbb{R}^3$ . Note that, at least formally, the Boltzmann equation for the hard sphere model is obtained from the Povzner's equation for  $\theta = +\infty$  by making  $\delta$  tend to zero

$$Q_P(f, f)(x, v) \rightarrow Q_B(f, f)(x, v),$$

where

$$Q_B(f, f)(x, v) = \frac{1}{2} \int_{\mathbb{R}^3} dw \int_{S^2} dn |(v - w) \cdot n| (f(x, v_*) f(x, w_*) - f(x, v) f(x, w)). \quad (8)$$

## 2.2 A dissipative correction

In [9, 10], Cucker and Smale proposed a model to describe a population of  $N$  agents, which, while interacting according to their relative positions, are developing particular profiles with time, like flocking of birds, in which all birds fly with the same velocity close each other. In [9], the main hypothesis which justifies the large-time behavior of the population is that every bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other birds. That is, given a population of  $k$  birds, at time  $t \in \mathbb{N}$ , and for  $i$ -th bird,

$$v_i(t+1) - v_i(t) = \sum_{j=1}^k a_{ij} (v_j(t) - v_i(t)). \quad (9)$$

where the weights  $a_{ij}$  quantify the way the birds influence each other. In [9] it is assumed that this influence is a function of the distance between birds, namely

$$a_{ij} = \frac{K}{(\sigma^2 + |x_i - x_j|^2)^\beta} \quad (10)$$

for some fixed  $K$ ,  $\sigma > 0$  and  $\beta \geq 0$ .

Condition (9) can be fruitfully rephrased in a different way [6]. Suppose that we have a population composed by two birds, say  $i$  and  $j$ . Then, if their velocities are modified according to (9)

$$v_i(t+1) = (1 - a_{ij})v_i(t) + a_{ij}v_j(t), \quad (11a)$$

$$v_j(t+1) = a_{ij}v_i(t) + (1 - a_{ij})v_j(t). \quad (11b)$$

Thus, while the momentum is preserved after the interaction, so that

$$v_i(t+1) + v_j(t+1) = v_i(t) + v_j(t),$$

the energy increases or decreases according to the value of  $a_{ij}$

$$v_i^2(t+1) + v_j^2(t+1) = v_i^2(t) + v_j^2(t) - 2a_{ij}(1 - a_{ij})(v_i - v_j)^2. \quad (12)$$

If  $a_{ij} < 1$ , the energy is dissipated. Note that in this case the relative velocity is decreasing, since

$$|v_i(t+1) - v_j(t+1)| = |1 - 2a_{ij}||v_i(t) - v_j(t)| < |v_i(t) - v_j(t)|, \quad (13)$$

and the velocities of the two birds tend towards the mean velocity  $(v_i + v_j)/2$ .

In the general case of a population of  $k$  birds, the binary law (11) is taken into account together with the assumption that the  $i$ -th bird modifies its velocity giving the same weight to all the other velocities. In consequence of this,

$$v_i(t+1) = \frac{1}{k} \sum_{j=1}^k \{(1 - a_{ij})v_i(t) + a_{ij}v_j(t)\}, \quad (14)$$

that is a different way to write formula (9).

The stochastic dynamics of particles introduced in [16] is perfectly adaptable to the context of a population of living individuals. Substituting particles with birds, and changing consequently the interaction intensity  $\varphi$  in (4), allows to obtain a reasonable model to describe the time-space evolution of a population of birds, and, at the same time it establishes an interesting connection with the fluid dynamic picture (Euler equations), in presence of a large population.

To adapt the particle system by Lachowicz and Pulvirenti to the present context, we need however to introduce various modifications. The first one is related to the fact that, since the flocking phenomena are heavily dependent from dissipation, the elastic interaction considered in [16] has to be suitably modified to account for dissipation.

Second, the interaction rules (11) considered according to the Cucker–Smale model, while dissipative and able to reproduce the flocking phenomenon, do not represent a dissipative correction of an elastic collision of type (2). In fact, letting the dissipation to zero in (11), do not produce an elastic collision, but a collision in which birds simply do not exchange their velocities. This is related to the fact that the interaction rule given by (11) is such that the relative position influences the interactions only through the modulus of the distance, while the angles formed by the relative position and the respective velocities does not play any role.

To overcome this unpleasant effect, a different more sophisticated interaction can be considered, such that agents tend to dissipate their relative velocity along the relative direction. This agrees with the realistic assumption that birds which are approaching tend to diminish their relative velocity along their relative positions, and the same happens in the opposite situation where they are going away.

We assume that the microscopic dynamics of two agents  $(x, v)$  and  $(y, w)$  is governed by the interaction coefficient  $0 < e(|x - y|) < 1$  which relates the components of the agents velocities along before and after an interaction. The post interaction velocities  $(v^*, w^*)$  are such that

$$(v^* - w^*) \cdot \frac{x - y}{|x - y|} = -e(|x - y|) (v - w) \cdot \frac{x - y}{|x - y|}. \quad (15)$$

Thanks to (15), and assuming the conservation of momentum, one finds the change of velocity for the interacting agents as

$$v^* = v - \frac{1}{2}(1 + e)((v - w) \cdot n)n, \quad w^* = w + \frac{1}{2}(1 + e)((v - w) \cdot n)n, \quad (16)$$

where  $n = n(x - y) = (x - y)/|x - y|$ . A Povzner–type (conservative) interaction is obtained for  $e = 1$ . For dissipative interactions  $e$  decreases with increasing degree of dissipation.

The choice

$$e(|x - y|) = 1 - \gamma a(|x - y|), \quad (17)$$

where  $a(|x - y|)$  is given like in (10) by

$$a(|x - y|) = \frac{K}{(\sigma^2 + |x - y|^2)^\beta}, \quad (18)$$

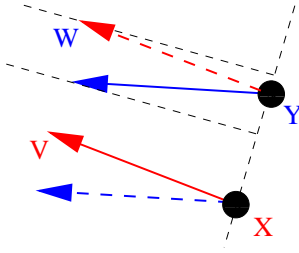


Figure 1: Example of interaction rule as in the Cucker-Smale-Povzner model where the particle with position  $x$  and velocity  $v$  averages its velocity with a particle in position  $y$  and velocity  $w$  according to (16).

is consistent with the Cucker-Smale approach. Note that the constant  $\gamma$  has to be chosen so that  $e(|x - y|) < 1$ .

### 2.3 Boltzmann-like model for dissipative interactions

According to the rule (16), the one-particle distribution function  $f(x, v, t)$  considered by Lachowicz and Pulvirenti [16], in presence of a large population ( $N \rightarrow \infty$ ) satisfies a Povzner-type equation for dissipative interactions, which reads [4]

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \bar{Q}_P(f, f)(x, v, t), \quad (19)$$

where  $\bar{Q}_P$  is now a dissipative collision operator, which describes the change in the density function due to creation and annihilation of agents velocities in binary interactions

$$\bar{Q}_P(f, f)(x, v) = \sigma^2 \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw B(|x - y|) (\Gamma(|x - y|) f(x, v_*) f(y, w_*) - f(x, v) f(y, w)). \quad (20)$$

The constant  $\sigma^2$  accounts for the interaction frequency. Like in Povzner's equation, in (20) the velocities  $(v_*, w_*)$  are the pre-interaction velocities. The factor  $\Gamma$  in the gain term appears respectively from the Jacobian of the transformation  $dv^* dw^*$  into  $dv dw$ . For a restitution coefficient which depends only on positions, like in (17),  $\Gamma(|x - y|) = e(|x - y|)^{-2}$ . Finally,  $B(\tau)$  represents the collision rate function, which gives the probability that a collision between agents happen at a distance  $\tau$ . We recall that  $B(\tau) = \chi(\tau < \delta)/\delta^3$  leads to a model similar to the one studied by Lachowicz and Pulvirenti [16]. We also stress that, differently from (5), we do not assume that the collision rate function depends on the term  $|(v - w) \cdot n|$ , which is not considered here.

To avoid the presence of the function  $\Gamma$ , and to study approximations to the Povzner operator (20) it is extremely convenient to write the operator (20) in weak form. More precisely, let us define with  $\langle \cdot, \cdot \rangle$  the inner product in  $L_1(\mathbb{R}^3)$ . For all smooth functions  $\varphi(v)$ , it holds

$$\begin{aligned} \langle \varphi, \bar{Q}_P(f, f)(x, v) \rangle &= \sigma^2 \int_{\mathbb{R}^3} \varphi(v) \bar{Q}_P(f, f)(x, v) dv \\ &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(|x - y|) (\varphi(v^*) - \varphi(v)) f(x, v) f(y, w) dv dw dy \end{aligned}$$

Let  $(v', w')$  be the post-interaction velocities in a Povzner elastic interaction with  $(v, w)$  as incoming velocities. Denoting by  $q$  the relative velocity,  $q = v - w$ ,

$$v' = v - (q \cdot n)n, \quad w' = w + (q \cdot n)n. \quad (21)$$

As before, the unit vector is  $n = n(x - y) = \frac{x-y}{|x-y|}$ . Using (16) and (21) one obtains

$$v^* = v' + \frac{1}{2}(1 - e)(q \cdot n)n, \quad w^* = w' - \frac{1}{2}(1 - e)(q \cdot n)n. \quad (22)$$

If we assume that the coefficient of restitution has the form (17),

$$v^* - v' = \gamma a(|x - y|)(q \cdot n)n. \quad (23)$$

Let us consider a Taylor expansion of  $\varphi(v^*)$  around  $v'$ . Thanks to (23) we get

$$\begin{aligned} \varphi(v^*) &= \varphi(v') + \gamma a(|x - y|) \nabla \varphi(v') \cdot (q \cdot n)n + \\ &\frac{1}{2} \gamma^2 a(|x - y|)^2 \sum_{i,j} \frac{\partial^2 \varphi(v')}{\partial v'_i \partial v'_j} (q \cdot n)^2 n_i n_j + \dots \end{aligned} \quad (24)$$

If the interactions are nearly elastic, so that  $\gamma \ll 1$ , we can cut the expansion (24) after the first-order term. Inserting (24) into (21) gives

$$\begin{aligned} \langle \varphi, \bar{Q}_P(f, f) \rangle &\approx \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(|x - y|) (\varphi(v') - \varphi(v)) \\ &\quad + \gamma \nabla \varphi(v') \cdot a(|x - y|)(q \cdot n)n f(x, v) f(y, w) dv dw dy \\ &= \langle \varphi, \mathcal{Q}_P(f, f) \rangle + \gamma \langle \varphi, \mathcal{I}(f, f) \rangle. \end{aligned} \quad (25)$$

It is a simple matter to recognize that in (25)  $\mathcal{Q}_P(f, f)$  is a Povzner's collision operator of the type (5), since the post-interaction velocity  $v'$  into (25) is obtained from the pre-interaction velocities  $(v, w)$  through the elastic interaction (21).

Let us now study in more detail the second contribution to the inner product (25). First of all, let us set

$$b(|x - y|) = B(|x - y|)a(|x - y|). \quad (26)$$

Using the properties of the transformation (21), we obtain

$$\begin{aligned} \langle \varphi, \mathcal{I}(f, f) \rangle &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \varphi(v') \cdot n b(|x - y|)(q \cdot n) f(x, v, t) f(y, w, t) dv dw dy \\ &= -\sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \varphi(v) \cdot n b(|x - y|)(q \cdot n) f(x, v', t) f(y, w', t) dv dw dy \\ &= \sigma^2 \int_{\mathbb{R}^3} dv \varphi(v) \operatorname{div}_v \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} n b(|x - y|)(q \cdot n) f(x, v', t) f(y, w', t) dw dy \right). \end{aligned} \quad (27)$$

In fact, the transformation  $dv dw$  into  $dv' dw'$  given by (21) is such that  $q' \cdot n = -q \cdot n$ , while its Jacobian is equal to unity.

The last equality follows from the divergence theorem. The second contribution to the inner product (25) defines the dissipative interaction operator

$$\mathcal{I}(f, f)(x, v, t) = \sigma^2 \operatorname{div}_v \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} n(q \cdot n) b(|x - y|) f(x, v', t) f(y, w', t) dw' dy \right). \quad (28)$$

Finally, for nearly elastic interactions, with a restitution coefficient satisfying (17), the dissipative Povzner equation can be modeled at the leading order as

$$\left( \frac{\partial f}{\partial t} + v \cdot \nabla_x f \right) (x, v, t) = \mathcal{Q}_P(f, f)(x, v, t) + \gamma \mathcal{I}(f, f)(x, v, t), \quad (29)$$

where  $\mathcal{Q}_P$  is inspired by the classical elastic Povzner collision operator, and  $\mathcal{I}$  is a dissipative nonlinear friction operator which is based on elastic interactions between agents.

**Remark 2.1** The model kinetic equation obtained here as a first-order correction to an elastic Povzner-type equation is valid, at least formally, for weakly dissipative interactions, i.e., when the value of  $\gamma$  is sufficiently small. Other moderately dissipative regimes would require the inclusion of higher order terms in the expansion (24). In particular, the second-order term in this expansion gives a diffusive correction to the Povzner equation. The analysis done in this section is close to the analogous one done for the Boltzmann equation for granular dissipative gases [23]. This last problem has been studied systematically both from a numerical and theoretical point of view (cfr. [23] and the references therein). Numerical computations [18] showed that for a one-dimensional (in the velocity space) dissipative Boltzmann equation the results relative to the full equation are in good agreement with those relative to the equation with a first-order correction, even in regimes of moderate inelasticity. The same analysis showed that higher-order corrections (of order bigger than two) introduce problems in the numerical spectral approximation, without essential improvements in the accuracy of the solution. This suggests that, also in presence of agents interactions the first-order correction gives a reasonable good approximation to the dissipative Povzner equation.

**Remark 2.2** The dissipative correction (28) to the elastic Povzner equation appears very similar to the corresponding one obtained in [14] as the mean field limit of the Cucker–Smale model. Also, the Povzner-like model introduced in [6] is apparently close to the one considered in this paper. These models, which are based on the Cucker–Smale interaction (11), however are not consistent with the elastic description of the system introduced in [16]. In fact, the exact continuum description of the Cucker–Smale interaction (11)

$$v^* = (1 - \gamma a)v + \gamma aw, \quad w^* = av + (1 - \gamma a)w,$$

leads to  $v^* = v$  and  $w^* = w$  as  $\gamma \rightarrow 0$ . Consequently, the limit  $\gamma \rightarrow 0$  gives a dynamics in which all particles in the system move freely without interactions.

### 3 Passage to fluid dynamics

#### 3.1 Exact computations

The main purpose of this short section is to study in some detail the main properties of the nonlinear friction operator  $\mathcal{I}$ . Choosing  $\varphi = 1$  into (27) shows that

$$\begin{aligned} & \langle 1, \mathcal{I}(f, f)(x, t) \rangle \\ &= \sigma^2 \int_{\mathbb{R}^d} \operatorname{div}_v \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} n(q \cdot n) b(|x - y|) f(x, v', t) f(y, w', t) dw' dy \right) dv = 0, \end{aligned} \quad (30)$$

so that the mass is a collision invariant. Choosing now  $\varphi = v_i$  into (27), and using the first equality, owing to the fact that  $\nabla \varphi(v') \cdot n = n_i$  we obtain

$$\begin{aligned} \langle v, \mathcal{I}(f, f)(x, t) \rangle &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} n b(|x - y|) (q \cdot n) f(x, v, t) f(y, w, t) dv dw dy \\ &= \sigma^2 \int_{\mathbb{R}^3} n b(|x - y|) (u(x, t) - u(y, t)) \cdot n \rho(x, t) \rho(y, t) dy. \end{aligned} \quad (31)$$

In (31),  $\rho(x, t)$  and  $u(x, t)$  denote as usual the local mass density and bulk velocity, that is

$$\rho(x, t) = \int_{\mathbb{R}^3} f(x, v, t) dv, \quad u(x, t) = \frac{1}{\rho(x, t)} \int_{\mathbb{R}^3} v f(x, v, t) dv. \quad (32)$$

Note that the mean velocity is not a local collision invariant, while by symmetry

$$\int_{\mathbb{R}^3} \langle v, \mathcal{I}(f, f)(x, t) \rangle dx = 0. \quad (33)$$

Finally, taking  $\varphi = v^2/2$  into (27), and using the identity  $v' \cdot n = w \cdot n$  we obtain

$$\begin{aligned} & \langle v^2/2, \mathcal{I}(f, f)(x, t) \rangle \\ &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w \cdot n b(|x - y|) ((v - w) \cdot n) f(x, v, t) f(y, w, t) dv dw dy \\ &= \sigma^2 \int_{\mathbb{R}^3} b(|x - y|) \rho(x, t) \rho(y, t) n \cdot u(x, t) n \cdot u(y, t) dy \\ &- \sigma^2 \rho(x, t) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(|x - y|) |w \cdot n|^2 f(y, w, t) dw dy. \end{aligned} \quad (34)$$

We remark that, while equality (31) is expressed in terms of macroscopic quantities, the same is not true for equality (34), where one can not in general express the last integral in terms of macroscopic quantities. This is possible at least when the density  $f(x, v, t)$  is isotropic in the velocity variable,  $f(x, v, t) = f(x, |v|, t)$ . In this case, in fact

$$\int_{\mathbb{R}^3} |v \cdot n|^2 f(x, v, t) dv = \frac{1}{3} \int_{\mathbb{R}^3} |v|^2 f(x, v, t) dv,$$

and

$$\int_{\mathbb{R}^3} |v|^2 f(x, v, t) dv = \rho(x, t) |u(x, t)|^2 + 2\rho(x, t) e(x, t). \quad (35)$$

In (35),  $e(x, t)$  is the internal energy per unit mass, given by

$$e(x, t) = \frac{3}{2}T(x, t) = \frac{1}{2\rho(x, t)} \int_{\mathbb{R}^3} |v - u(x, t)|^2 f(x, v, t) dv. \quad (36)$$

A second useful representation can be obtained, owing to the second line equality in (27),

$$\begin{aligned} & \langle v^2/2, \mathcal{I}(f, f)(x, t) \rangle \\ &= -\sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot n b(|x - y|) ((v - w) \cdot n) f(x, v', t) f(y, w', t) dv dw dy. \end{aligned} \quad (37)$$

In this case, since the change of variables  $v \leftrightarrow w$ , and, at the same time  $x \leftrightarrow y$  gives  $v' \leftrightarrow w'$  and  $n \leftrightarrow -n$

$$\begin{aligned} & \int_{\mathbb{R}^3} \langle v^2/2, \mathcal{I}(f, f)(x, t) \rangle dx \\ &= -\frac{\sigma^2}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(|x - y|) ((v - w) \cdot n)^2 f(x, v', t) f(y, w', t) dv dw dx dy < 0. \end{aligned} \quad (38)$$

Hence, the nonlinear friction operator is globally dissipative.

To close this section, we remark that, since the Maxwellian function  $\mathcal{M} = \mathcal{M}(x, v, t)$  defined in (1) is isotropic in the velocity variable, hence

$$\begin{aligned} & \int_{\mathbb{R}^3} \langle v^2/2, \mathcal{I}(\mathcal{M}, \mathcal{M})(x, t) \rangle dx \\ &= \sigma^2 \int_{\mathbb{R}^3} b(|x - y|) \rho(x, t) \rho(y, t) \left[ n \cdot u(x, t) n \cdot u(y, t) - \left( \frac{1}{3} |u(y, t)|^2 + T(y, t) \right) \right] dy. \end{aligned} \quad (39)$$

## 3.2 Hydrodynamic limit and the Euler equations.

The goal of this section is to derive, in some particular regimes, a fluid dynamic description of the population of agents. Motivated by the analysis of [16], and in analogy with the basic concepts of kinetic theory of gases [8], a fluid dynamic description can be obtained from the Povzner-type equation in presence of two scalings. First, the collision rate  $B(\tau)$  in the collision integral (5) has to take into account mainly interactions which happen at a small distance. Second, the mean free path has to be taken small. In addition, the fluid dynamic description has to retain memory of the dissipative interaction dynamics. Among others, a choice of interactions functions which satisfies all these requirements is the following

$$B(|x - y|) = B_\delta(|x - y|) = \frac{1}{\delta^3} \left( 1 + \frac{|x - y|^2}{\delta^2} \right)^{-2}, \quad (40)$$

and

$$a(|x - y|) = a_\delta(|x - y|) = \frac{K \delta^3 \left(1 + \frac{|x-y|^2}{\delta^2}\right)^2}{(\sigma^2 + |x - y|^2)^{\beta+1}}. \quad (41)$$

The previous choices are heavily justified from the modelling point of view. The interaction kernel  $B_\delta$  in (40) implies that interactions are more frequent if the distance between agents is small, and this property improves as soon as  $\delta \rightarrow 0$ .

This is clearly related to the fact that in a dense group of agents a single agent modifies its velocity according to the nearest agents (in reality, the dimension of the agents allow to see only the nearest ones). Second, the interaction rate  $a_\delta$ , which characterizes the amplitude of the modification of the velocity due to the presence of the other agents, is constructed to retain memory of the presence of the underlying agents even in a situation where the agents themselves are dense.

Note that, under these assumptions

$$b(|x - y|) = B_\delta(|x - y|)a_\delta(|x - y|) = \frac{K}{(\sigma^2 + |x - y|^2)^{\beta+1}}. \quad (42)$$

is independent of  $\delta$ , while

$$B_\delta(|x - y|)a_\delta^2(|x - y|) = \frac{K \delta^3 \left(1 + \frac{|x-y|^2}{\delta^2}\right)^2}{(\sigma^2 + |x - y|^2)^{2\beta+2}} \leq \frac{C}{\delta}. \quad (43)$$

This implies that, provided  $\gamma \cong \delta$ ,  $\beta > 0$ , the higher order term in the expansion (24), which behaves like  $\gamma^2 B_\delta a_\delta^2$  remains uniformly small. Next, we substitute  $\sigma^2$  with  $\sigma^2/\epsilon$ , where  $\epsilon \ll 1$ . In this case we can formally derive the fluid dynamics equations in the regime of small dissipative interactions.

Let us make use of the splitting method, very popular in the numerical approach to the Boltzmann equation. To solve the complete dissipative Boltzmann–Povzner equation (19), at each time step we consider sequentially the elastic Boltzmann–Povzner equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathcal{Q}_P(f, f)(x, v, t), \quad (44)$$

and the dissipative correction

$$\frac{\partial f}{\partial t} = \frac{\gamma}{\epsilon} \mathcal{J}(f, f)(x, v, t). \quad (45)$$

Thanks to the result of Lachowicz and Pulvirenti [16] we know that the solution to (44) is well approximated by a Maxwellian function  $\mathcal{M}$  whose moments satisfy the Euler system. Substituting this Maxwellian function in equation (45), the right-hand side can be evaluated exactly using (31) and (38). Hence, when  $\gamma$  and  $\epsilon$  are of the same order, so that  $\gamma/\epsilon \rightarrow c$ , we obtain that, for large values of  $N$  and small values of  $\epsilon$  the stochastic system is well approximated by the following Euler system for density  $\rho(x, t)$ , bulk velocity  $u(x, t)$  and temperature  $T(x, t)$

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0, \\
\frac{\partial}{\partial t}(\rho u_i) + \operatorname{div}(\rho u u_i + \rho T e_i) &= \lambda \mathcal{A}_i(\rho, u)(x, t), \\
\frac{\partial}{\partial t} \left( \rho \left( \frac{2}{3} T + \frac{1}{2} u^2 \right) \right) + \operatorname{div} \left( \rho u \left( \frac{1}{2} u^2 + \frac{5}{2} T \right) \right) &= \lambda \mathcal{B}(\rho, u, T)(x, t). \tag{46}
\end{aligned}$$

In (46)  $e_i$  is the component of the unit vector  $e$  in the  $i$ -th direction,

$$\mathcal{A}(\rho, u)(x, t) = \sigma^2 \int_{\mathbb{R}^3} n b(|x - y|) (u(x, t) - u(y, t)) \cdot n \rho(x, t) \rho(y, t) dy, \tag{47}$$

and

$$\begin{aligned}
&\mathcal{B}(\rho, u, T)(x, t) = \\
&= \sigma^2 \int_{\mathbb{R}^3} b(|x - y|) \rho(x, t) \rho(y, t) \left[ n \cdot u(x, t) n \cdot u(y, t) - \left( \frac{1}{3} |u(y, t)|^2 + T(y, t) \right) \right] dy. \tag{48}
\end{aligned}$$

**Remark 3.1** In [14], starting from the kinetic model obtained as the mean field limit of the Cucker and Smale model, Ha and Tadmor tried to introduce a set of hydrodynamical equations, which are given in the form of a system of Euler-type equations, with correction terms which are very similar to the  $\mathcal{A}$  and  $\mathcal{B}$  terms considered here. Their equations, however, constitute a system for the thirteen moments, one obtains evaluating the evolution of mass, momentum and energy, so that the system is not closed.

**Remark 3.2** The procedure we used to recover the Euler system (46) is based on several truncations and asymptotics of the true Povzner-type equation (19). While the final result is largely formal, nevertheless the main steps of this procedure can be justified. In particular, the truncation of Taylor's formula (24) after the first-order term does not affect Euler equations in all cases in which the coefficient  $\gamma^2 B_\delta a_\delta^2 = o(\gamma)$  uniformly with respect to the position variable. In this case, in fact,  $\gamma^2 B_\delta a_\delta^2 / \epsilon = o(\gamma) / \epsilon \rightarrow 0$  if  $\gamma / \epsilon \rightarrow \lambda$ , and we lose the contribution of this term in the Euler system. Note that the choice of the interaction term  $a(|x - y|)$  of the type considered by Cucker and Smale (decay of order  $\beta$ ), implies a weaker interaction term at the level of Euler equations. In these equations the interaction term  $b(|x - y|)$  decays at the order  $\beta + 1$ .

**Remark 3.3** A different choice of the interaction parameters leads to different correction terms in the Euler system. In particular, leaving the interaction function  $a(|x - y|)$  unchanged, i.e., without any dependence on  $\delta$  to compensate the vanishing of  $B_\delta$  for  $\delta \rightarrow 0$ , implies that the correction terms  $\mathcal{A}$  and  $\mathcal{B}$  collapse into the analogous ones obtained for a granular gas [23]. In this case

$$\mathcal{A}_\delta(\rho, u)(x, t) = \sigma^2 \int_{\mathbb{R}^3} n B_\delta(|x - y|) a(|x - y|) (u(x, t) - u(y, t)) \cdot n \rho(x, t) \rho(y, t) dy \rightarrow 0 \tag{49}$$

as  $\delta$  tends to 0, while

$$\mathcal{B}_\delta(\rho, u, T)(x, t) \rightarrow -\frac{4}{3}\sigma^2\rho^2(x, t)T(x, t). \quad (50)$$

### 3.3 Long-time solutions

The presence of the corrections to the classical Euler equations, allows to classify some possible steady solutions of system (46). Let  $\Omega$  be a bounded subset of  $\mathbb{R}^3$ . Then, it is immediate to conclude that

$$\rho(x) = 0, \quad \text{if } x \in \mathbb{R}^3 \setminus \Omega,$$

and

$$u(x) = \bar{u}, \quad \rho(x) = \bar{\rho}(x) \quad \text{if } x \in \Omega$$

imply  $\mathcal{A}(\rho, u)(x) = 0$ . Moreover, under the same hypotheses on  $\rho$  and  $u$ ,

$$\mathcal{B}(\rho, u, T) = 0 \quad \text{if } x \in \mathbb{R}^3 \setminus \Omega,$$

and

$$\mathcal{B}(\rho, u, T) = -\sigma^2\bar{\rho}(x) \int_{\mathbb{R}^3} b(|x-y|)\bar{\rho}(y)T(y) dy \quad \text{if } x \in \Omega.$$

Therefore,  $\rho(x) = \bar{\rho}(x)$ ,  $u(x) = \bar{u}$ ,  $T(x) = 0$  in  $\Omega$  implies  $\mathcal{B}(\rho, u, T) = 0$ . Then, the condition

$$\operatorname{div}(\bar{\rho}(x)\bar{u}) = 0 \quad \text{if } x \in \Omega \quad (51)$$

is enough to guarantee that  $\rho(x) = \bar{\rho}(x)$ ,  $u(x) = \bar{u}$ ,  $T(x) = 0$  in  $\Omega$  and  $\rho(x) = 0 \in \mathbb{R}^3 \setminus \Omega$  is a steady solution of the Euler equations. Note that condition (51) only implies that  $\bar{\rho}(x)$  is constant along the direction of the constant mean velocity  $\bar{u}$ . Last, let us remark that

$$\begin{aligned} & \int_{\mathbb{R}^3} \mathcal{B}(\rho, u, T)(x, t) dx \\ &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(|x-y|)\rho(x, t)\rho(y, t) \left[ n \cdot u(x, t) n \cdot u(y, t) - \left( \frac{1}{3}|u(y, t)|^2 + T(y, t) \right) \right] dy dx \\ &= \sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(|x-y|)\rho(x, t)\rho(y, t) \left[ n \cdot u(x, t) n \cdot u(y, t) - \left( \frac{1}{3}|u(x, t)|^2 + T(x, t) \right) \right] dy dx \\ &= -\sigma^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(|x-y|)\rho(x, t)\rho(y, t) \left[ T(x, t) + T(y, t) + (n \cdot u(x, t) - n \cdot u(y, t))^2 \right] dy dx. \end{aligned} \quad (52)$$

Therefore, integrating with respect to the  $x$ -variable the third equation in (46) we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left( \rho \left( \frac{2}{3}T + \frac{1}{2}u^2 \right) \right) (x, t) dx = \int_{\mathbb{R}^3} \mathcal{B}(\rho, u, T)(x, t) dx < 0. \quad (53)$$

Hence the functional

$$\mathcal{F}(t) = \int_{\mathbb{R}^3} \left( \rho \left( \frac{2}{3}T + \frac{1}{2}u^2 \right) \right) (x, t) dx \quad (54)$$

is a Lyapunov (energy) functional for system (46). Moreover, the entropy production

$$\mathcal{E}(t) = - \int_{\mathbb{R}^3} \mathcal{B}(\rho, u, T)(x, t) dx > 0, \quad (55)$$

Considering now that  $\mathcal{E}(t) = 0$  if and only if either both  $T(x) = 0$  and  $u(x) = \bar{u}$ , or  $\rho(x) = 0$ , starting with a configuration which is compactly supported, the solution to system (46) would be forced both to remain compactly supported in space and to converge towards one of the steady states characterized above.

## 4 Numerical experiments

In this section we would like to illustrate the properties of the system (46) in a simple one dimensional situation. Actually, we were expecting that the contributions of the right-hand side operators  $\mathcal{A}$  and  $\mathcal{B}$  could produce a regularization and stabilization effect, especially for the expected dissipative properties of  $\mathcal{B}$ , but while developing the numerical experiments we realized that the presence of these additional terms in the Euler system imposed us a much more conservative approach, with the need of very small time steps, resulting in a rather high computational cost. This tendency to unstable behavior currently prevents us to consider 2D simulations which need *ad hoc* new methods for addressing the system (46), and are a matter of current investigation.

We considered the problem on a bounded interval  $\Omega = [0, 1]$ , by imposing additionally periodic boundary conditions. We used a ENO scheme for the approximation of fluxes in the space discretization and a third-order Runge-Kutta method for the time discretization, see [22] for an introduction. Interestingly, the results seem to be less oscillatory when we use lower order approximation in the ENO scheme, so eventually we used just the first order approximation. We tried an approach both with and without transformation to the characteristic variables, and the option without transformation gives more stable results. The integral operators  $\mathcal{A}$  and  $\mathcal{B}$  on the right-hand side are approximated by a simple first order quadrature formula, in order to limit the computational costs. We considered a 100 grid point discretization in space and a rather conservative time step  $\Delta t = 10^{-6}$  in order to ensure the stability of the scheme for longer time before blow-up; note that for the Euler system with homogenous right-hand side (i.e., setting  $\mathcal{A} = 0$  and  $\mathcal{B} = 0$ ), we can use  $\Delta t = 10^{-3}$  with the same scheme and therefore the source of possible instabilities are necessarily related to the particular form of the right-hand side. For our experiment reported in Figure 2 and Figure 3 we considered initial data  $\rho_0(x) = \rho(x, 0)$  being a Gaussian centered at 0.5, the initial velocity  $u_0(x) = u(x, 0) = -\sin(2\pi x)$ , and the initial temperature  $T_0(x) = T(x, 0) = 1$ . Note that in this case  $\bar{u} = 0$ , hence, if a steady state is reached, the velocity should be zero as well as the temperature, as conjectured in Section 3.3. The parameter  $\beta$ , measuring the strength of the interaction  $a(|x - y|)$ , is chosen  $\beta = 0.1$ .

The evolution starts with the initial density concentrated around  $x = 0.5$  which first splits apart, then it merges at  $x = 0$  and  $x = 1$  (periodically), then again in  $x = 0.5$ , and so on. The height of the peak at  $x = 0.5$  is growing at (almost) every

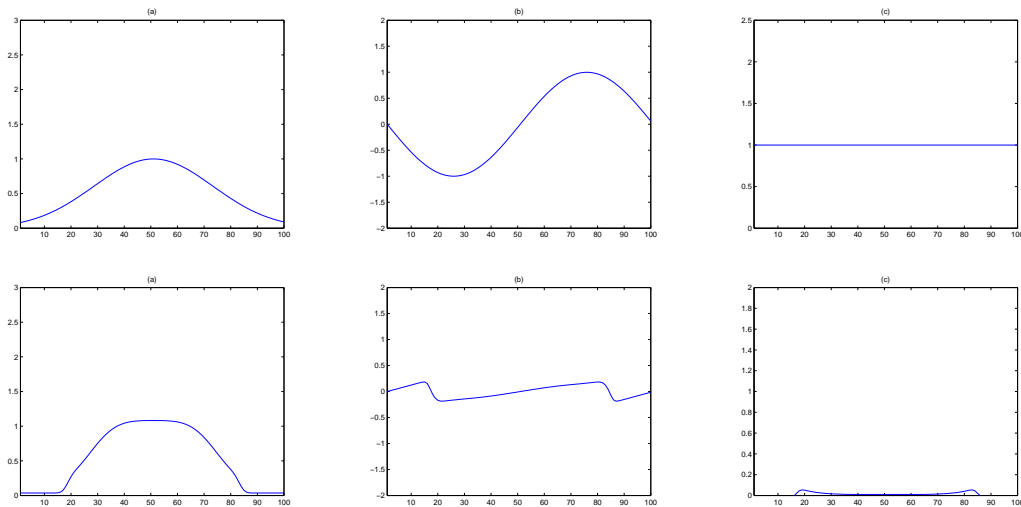


Figure 2: We illustrate respectively in (a) the density  $\rho(x, t)$ , in (b) the velocity  $u(x, t)$ , and in (c) the temperature  $T(x, t)$  at time  $t = 0$  (top) and after  $t = 3100$  time-units of the numerical simulation (bottom).

new merging. Periodically the peak becomes so concentrated that the gradient blows up, and eventually the numerics breaks down. Therefore we believe that the last part of the numerical simulation is not completely stable and reliable. Nevertheless, this numerical test confirms the dissipative nature of the system (46) since the total energy of the system decreases in time, in particular, both velocity and temperature are tending to 0, see Figure 2 and Figure 3, as expected in Section 3.3.

## 5 Conclusions

In this paper we derived Euler-type equations for a population of agents mutually interacting at a kinetic level in such a way that their mutual velocities are dissipated along the joining line position. The underlying kinetic model is the Povzner equation [21], which describes long-range elastic collisions between molecules lying in different positions in space, suitably corrected to take into account the dissipation. Owing to the pioneering result of Lachowicz and Pulvirenti [16], one can resort to the standard fractional step method to obtain the modified system of Euler equations, in which new correction terms are present into the equations for velocity and energy. These corrections are highly nonlocal, and their strength depends of the interaction function introduced by Cucker and Smale [9]. From a formal point of view, it is easily seen that the evolution of macroscopic quantities is driven by the dissipation of energy, which forces the evolution of the system towards a configuration in which the mean velocity is constant, and the internal energy vanishes. It would be certainly interesting to render

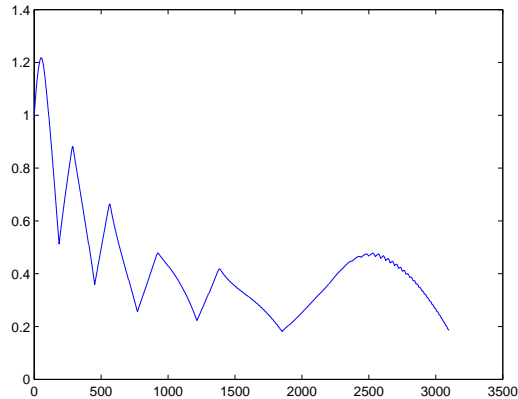


Figure 3: We illustrate the time decay of the maximal velocity, i.e.,  $U(t) := \max_{x \in \Omega} |u(x, t)|$ .

this behavior rigorous. Numerical simulations show indeed that this is the case.

**Acknowledgment:** Massimo Fornasier and Jan Haskovec acknowledge the financial support provided by the FWF project Y 432-N15 START-Preis “Sparse Approximation and Optimization in High Dimensions”. Giuseppe Toscani acknowledges the financial support from the Italian MIUR project “Kinetic and hydrodynamic equations of complex collisional systems”.

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