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Adaptive Frame Methods for Nonlinear Variational Problems

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Abstract In this paper we develop adaptive numerical solvers for certain nonlinear variational problems. The discretization of the variational problems is done by a suitable frame decomposition of the solution, i.e., a complete, stable, and redundant expansion. The discretization yields an equivalent nonlinear problem on the space of frame coefficients. The discrete problem is then adaptively solved using approximated nested fixed point and Richardson type iterations. We investigate the convergence, stability, and optimal complexity of the scheme. A theoretical advantage, for example, with respect to adaptive finite element schemes is that convergence and complexity results for the latter are usually hard to prove. The use of frames is further motivated by their redundancy, which, at least numerically, has been shown to improve the conditioning of the discretization matrices. Also frames are usually easier to construct than Riesz bases. We present a construction of divergence-free

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wavelet frames suitable for applications in fluid dynamics and magnetohydrodynamics.

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1 Introduction

Adaptive numerical methods have yielded very promising results [6,7,10,22,43] when applied to a large class of operator equations, in particular, PDE and integral equations. In classical schemes the adaptivity is realized at the level of the discretization and in the finite element space. The finite element space is refined and enriched locally at each iteration step depending on some (a posteriori) error estimators [22,32]. A novel paradigm for adaptive schemes has been recently proposed in [6,7], where the discretization via wavelet decompositions is fixed at the beginning. The adaptivity is indeed realized at the level of the solver of the equivalent bi-infinite system of linear equations. The basic idea is to transform the original PDE into a discrete (infinite) linear problem on $\ell_2(\mathcal{N})$, the space of wavelet coefficients. The discrete problem is then solved with the help of approximate iterative schemes. The advantage of the latter approach is the fact that its convergence and stability can be proved and its asymptotic optimal complexity can be estimated. On the contrary, it has been a very hard technical problem to obtain such nice theoretical estimates for classical finite element methods, although some important theoretical results have recently appeared in [4,36,37] for linear elliptic equations. A version of the paradigm in [6,7] has been recently proposed also for nonlinear problems in [8]. It is again based on (wavelet) bases discretizations.

One drawback of the wavelet approach is the construction of the wavelet system itself especially on domains with complicated geometry or manifolds [15–17], see also [42, p. 104, sec. More General Domains]. The wavelet bases constructed so far exhibit relatively high condition numbers or limited smoothness. The patching used to construct global smooth wavelets by domain decomposition techniques appears complicated and, in most cases, makes the conditioning even worse. The global smoothness of the basis, when implementing adaptive schemes in [6,7], is a necessary condition for getting compressibility (i.e., finitely banded approximations) of (infinite) discretization matrices, especially for high order operators. This bottleneck has led to generalizations of the wavelet approach. These generalizations are based on *frame discretizations*, i.e., stable, redundant, non-orthogonal expansions [11,35], which are much more flexible and simpler to construct even on domains of complicated geometry. Frame construction is usually implemented by Overlapping Domain Decompositions (ODD) so no patching at the interfaces is needed to obtain global smoothness. Moreover, the use of frames, due to their intrinsic redundancy, improves the conditioning of the corresponding discretization matrices. Certainly, ODD generates regions of the domain where the side effect of the redundancy is that functions are no longer uniquely representable by the global frame system. At first sight, it may seem that redundancy contradicts the minimality requirement on the amount of information being used to approximate the solution. Especially in fluid mechanics, accurate simulations already require processing of huge amounts of data. How can one attempt such computations if the information is also made redundant? A figurative answer to this question is the so-called "dictionary example": The larger and richer is my dictionary the *shorter* are the phrases I compose. The use of proper terminology avoids long circumlocutions for describing an object. Of course, the key point is the capability to choose the right terminology. Back to mathematical terms, the combination of adaptivity (i.e., the capability to choose the right terminology) and redundancy (i.e., the richness or non-uniqueness of representations) indeed gives rise to compressed and accurate approximations [20,25,29,38,39]. Numerical experiments in [13,44] show that frames improve conditioning without increasing the effective dimension of the problem, and that the frame approach has optimal complexity. The results included in [33,34] illustrate that frames can be naturally used for domain decomposition methods, where the overlapping patches induce Schwarz alternating iterations. Thus, together with adaptive schemes, we expect that the frame approach will produce a significant breakthrough in wavelet methods.

This encourages us to present and analyze a new adaptive scheme based on (wavelet) frame discretizations for an abstract nonlinear variational problem. Such nonlinear problems correspond, for example, to well-posed quadratic saddle-point problems arising in fluid dynamics and magnetohydrodynamics, see [5,24,30]. In particular, in this paper we transform the abstract nonlinear variational problem into an equivalent nonlinear discrete problem on $\ell_2(\mathcal{N})$ by using suitable frame expansions. We show how the discrete problem can be solved *adaptively* by means of nested fixed point and approximated Richardson iterations. We discuss convergence, stability, and, under certain additional assumptions, computational cost (quasi-optimal complexity) of the proposed adaptive procedure. Indeed the emphasis in this paper is on the theoretical investigation of the applicability of frame discretizations for solving nonlinear variational problems. Our results confirm that the redundancy of a frame (which corresponds to an improved flexibility in practice) is not necessarily in opposition to optimality. To emphasize the flexibility of frames, we present a construction of divergence-free wavelet frames on domains which are not necessarily affine images of $(0,1)^n$. To our knowledge, constructions of such wavelet bases or frames have not appeared in the literature so far.

The paper is organized as follows: In Section 2, we present the abstract nonlinear variational problem. We reformulate it as an equivalent fixed point iteration, where, at each iteration step, a linear (non–symmetric) elliptic operator equation is to be solved. Next, we present a way for discretizing the fixed point iteration. We translate the original nonlinear problem into a problem on $\ell_2(\mathcal{N})$, the space of suitable *frame* coefficients. The concept of frames, i.e., stable, redundant, and complete expansions, is recalled in Subsection 2.1. In Subsection 2.2 we show how a linear (non-symmetric) elliptic operator equation is discretized by means of frame expansions and how Algorithm 1 is used to approximate its solution adaptively, up to any prescribed accuracy.

Using Algorithm 1 as the main building block, we formulate in Section 3 a fully discrete and finite version of the fixed point iteration. We show that this discrete version of the fixed point iteration converges to one of the possible sets of frame coefficients of the true solution of the original problem. In Section 4 we discuss under which conditions on the building block procedures in Algorithm 1 the suggested scheme performs quasi-optimally with respect to suitable *sparseness classes* of frame coefficients. In particular, we show how the flexibility and redundancy of frames lead to technical difficulties, which do not arise in case of Riesz bases, when showing complexity estimates.
Frames of divergence–free functions are crucial for applications in fluid dynamics and magnetohydrodynamics due to the incompressibility of the fluid flow and the conservation of charge. Due to intrinsic technical difficulties, no divergence–free wavelet *bases* have been constructed on general polygonal domains, see [42, p. 104, sec. More General Domains]. Therefore, in Section 5, we present a new construction of suitable divergence-free wavelet *frames* on general polygonal domains. This allows for applications of the general adaptive scheme we propose.

Throughout this paper '$a \sim b$' means that both quantities are uniformly bounded by some constant multiple of each other. Likewise, '$a \lesssim b$' means that there exists a positive constant C such that $a \leq Cb$. We determine the constants explicitly only if their value is crucial for further analysis. The symbol $\|\cdot\|$, when applied to bounded operators, denotes the operator norm from its domain space to its image space, these are not always explicitly specified for notational simplicity.

2 From Variational Formulation to Frame Discretization

Our analysis and numerical schemes can be used for the numerical treatment of several classical problems arising, e.g., in fluid dynamics and magnetohydrodynamics. The problems of fluid dynamics to which our adaptive scheme is applicable are modelled using stationary, incompressible Navier–Stokes equations, see [24]. The ones arising in magnetohydrodynamics are modelled using the fluid dynamics and electromagnetic field equations, see [5,30]. The latter equations come into play, if, e.g., the fluid occupying a 3–dimensional bounded region $\Omega \subset \mathbb{R}^3$ is electrically conducting and there is a given externally generated magnetic field.
The above mentioned physical problems fit the following abstract setting: There exist separable Hilbert spaces $\mathbf{V}$ and $\mathcal{H}$ such that $\mathbf{V} \subset \mathcal{H} \subset \mathbf{V}'$ with bounded and dense inclusions. The triple $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$ is called a Gelfand triple. The duality between $\mathbf{V}'$ and $\mathbf{V}$ is identified on $\mathcal{H}$ using the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ of $\mathcal{H}$. There exists an operator $A : \mathbf{V} \to \mathbf{V}'$ such that $\langle Av, w \rangle_{\mathbf{V}' \times \mathbf{V}} := a(v, w)$ defines an elliptic bilinear form, i.e., there exist positive constants α, β such that $\alpha \|v\|_{\mathbf{V}}^2 \leq a(v, v) \leq \beta \|v\|_{\mathbf{V}}^2$ for all $v \in \mathbf{V}$. The ellipticity of a implies that $\|Av\|_{\mathbf{V}'} \sim \|v\|_{\mathbf{V}}$ and that A is a boundedly invertible operator with $\|A^{-1}\| \leq \alpha^{-1}$. We also assume that there exists a trilinear form a_1 inducing a bounded bilinear operator $A_1 : \mathbf{V} \times \mathbf{V} \to \mathbf{V}'$, defined by $\langle A_1(v, w), z \rangle_{\mathbf{V}' \times \mathbf{V}} := a_1(v, w, z)$ for $v, w, z \in \mathbf{V}$.

We study the following abstract problem:

Find $u \in \mathbf{V}$ *such that*

$$Au + A_1(u,u) = \ell, \tag{1}$$

where $\ell \in \mathbf{V}'$ *is a functional on* $\mathbf{V}$*.*

The local solvability of (1) is ensured by the following well-known result.

Theorem 1 *Let* $H : \mathbf{V} \to \mathbf{V}$ *be given by* $H(v) := A^{-1}(\ell - A_1(v,v))$*. The following hold true*

(a) If $0 < r < \frac{\alpha}{2\|A_1\|}$*, then* $H_{|B_r}$ *is a contraction with Lipschitz constant* $L := \frac{2\|A_1\|r}{\alpha}$*, where* B_r *is the closed ball of radius* r *centered in* 0*;*

(b) If $0 < r < \frac{\alpha}{\|A_1\|}$ *and* $\|\ell\|_{\mathbf{V}'} \le r(\alpha - \|A_1\|r)$*, then* $H_{|B_r} : B_r \to B_r$*;*

(c) If $\|\ell\|_{\mathbf{V}'} \le \frac{\alpha^2}{4\|A_1\|}$*, then the equation* (1) *has a unique solution* $u \in \mathbf{V}$ *with* $\|u\| < \frac{\alpha}{2\|A_1\|}$ *and the solution is computable by the fixed point iteration*

$$\begin{aligned} u_{n+1} &= Hu_n, \quad u_0 = 0, \quad n \in \mathbb{N}_0, \\ u &= \lim_{n\to\infty} u_n. \end{aligned} \tag{2}$$

Note that (2) is equivalent to

$$Au_{n+1} = \ell - A_1(u_n, u_n), \quad n \in \mathbb{N}_0, \quad u_0 = 0. \tag{3}$$

Under our assumptions on A, the equations in (3) are elliptic operator equations. In the rest of this section we derive a discrete problem equivalent to the abstract nonlinear problem in (1) and prove in Theorem 4 that the solution of the discrete problem exists and is unique under certain assumptions on the parameters of the original problem and frames chosen for discretization. The discrete problem is obtained using suitable *stable*, *redundant*, and *nonorthogonal* expansions, so–called Gelfand frames for the Gelfand triple $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$. We also show that the derived discrete fixed point iteration can be realized efficiently using the key routine **SOLVE** to approximate the solution of the elliptic problems in (3) adaptively.

Remark 1 To give concrete examples of spaces $\mathbf{V}$, denote by $H^s(\Omega)$ the Sobolev space of functions with the norm $||v||_{H^s(\Omega)} = \Big(\sum_{|\alpha|\le s} ||D^\alpha v||^2_{L_2(\Omega)}\Big)^{\frac{1}{2}}$. For vector–valued functions $\mathbf{v} = (v_1, v_2, v_3)$ we write $\mathbf{H}^s(\Omega)$ and $\mathbf{L}_2(\Omega)$. Let also $\mathbf{H}^1_0(\Omega)$ denote

$$\mathbf{H}^1_0(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\},$$

$\dot{H}^1(\Omega)$ and $\dot{L}_2(\Omega)$ the subspaces of $H^1(\Omega)$ and $L_2(\Omega)$ consisting of functions with mean zero, respectively. In the case of fluid dynamics the space $\mathbf{V}$, see [24], can be given by

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \ : \ \int_\Omega (\nabla \cdot \mathbf{v}) q = 0 \quad \text{for all} \quad q \in \dot{L}_2(\Omega)\}.$$

When modelling the stationary flow of a viscous, incompressible, electrically conducting fluid in [30], the solution space $\mathbf{V}$ for the velocity $\mathbf{v}$ and the electric current $\boldsymbol{K}$ is

$$\mathbf{V} := \{(\mathbf{v}, \boldsymbol{K}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}_2(\Omega) \ : \ \int_\Omega (\nabla \cdot \mathbf{v}) q + \int_\Omega \boldsymbol{K} \cdot (\nabla \psi) = 0 \quad \text{for all} \quad (q, \psi) \in \dot{L}_2(\Omega) \times \dot{H}^1(\Omega)\}.$$

2.1 Gelfand Frames

The sequence space $\ell_2(\mathcal{N})$ on the countable index set $\mathcal{N} \subset \mathbb{R}^d$ is induced by the norm

$$\|\mathbf{c}\|_{\ell_2(\mathcal{N})} := \left(\sum_{n \in \mathcal{N}} |c_n|^2 \right)^{1/2}, \quad \mathbf{c} = \{c_n\}_{n \in \mathcal{N}} \in \ell_2(\mathcal{N}).$$

The space $\ell_0(\mathcal{N}) \subset \ell_2(\mathcal{N})$ is the subspace of sequences with compact support. Denote by $\langle \cdot, \cdot \rangle_\mathcal{H}$ and $\| \cdot \|_\mathcal{H}$ the inner product and the norm on the separable Hilbert space $\mathcal{H}$, respectively. A sequence $\mathcal{F} := \{f_n\}_{n \in \mathcal{N}}$ in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if

$$\|f\|_\mathcal{H}^2 \sim \sum_{n \in \mathcal{N}} \left| \langle f, f_n \rangle_\mathcal{H} \right|^2, \quad \text{for all } f \in \mathcal{H}. \tag{4}$$

Due to (4) the corresponding operators of analysis and synthesis given by

$$F : \mathcal{H} \to \ell_2(\mathcal{N}), \quad f \mapsto \left(\langle f, f_n \rangle_\mathcal{H} \right)_{n \in \mathcal{N}}, \tag{5}$$

$$F^* : \ell_2(\mathcal{N}) \to \mathcal{H}, \quad \mathbf{c} \mapsto \sum_{n \in \mathcal{N}} c_n f_n, \tag{6}$$

are bounded. In particular, we have the following orthogonal decomposition of $\ell_2(\mathcal{N})$

$$\ell_2(\mathcal{N}) = \operatorname{ran}(F) \oplus \ker(F^*), \tag{7}$$

and

$$\mathbf{Q} := F(F^*F)^{-1}F^* : \quad \ell_2(\mathcal{N}) \to \operatorname{ran}(F), \tag{8}$$

is the orthogonal projection onto $\operatorname{ran}(F)$. The frame $\mathcal{F}$ is a Riesz basis for $\mathcal{H}$ if and only if $\ker(F^*) = \{0\}$. In general, $\{0\}$ is a proper subspace of $\ker(F^*)$. In other words, due to the redundancy of the frame there may exist sequences $\mathbf{c} = \{c_n\}_{n \in \mathcal{N}} \neq \mathbf{d} = \{d_n\}_{n \in \mathcal{N}}$ in $\ell_2(\mathcal{N})$ such that $\sum_{n \in \mathcal{N}} c_n f_n = \sum_{n \in \mathcal{N}} d_n f_n$. In particular, the redundancy may lead to the situation when a small perturbation $\mathbf{d}$ of the coefficient sequence $\mathbf{c}$ has no effect on the synthesis operator. This possible reduction effect on errors, noise, and numerical round-offs is the motivation for using frames for the applications, where tolerance to errors is required. The intrinsic stability of frames is also expected to play a role in the conditioning of the discretizations of operator equations and leads to additional robustness that the discretization inherits from the frame. It has been confirmed by numerical experiments in [44] that by increasing the redundancy of the frame we improve the conditioning of discretization matrices.

The composition $S := F^*F$ is a boundedly invertible (positive and self–adjoint) operator called the *frame operator* and $\tilde{\mathcal{F}} := \{S^{-1}f_n\}_{n\in\mathcal{N}}$ is again a frame for $\mathcal{H}$, called the *canonical dual frame*, with corresponding analysis and synthesis operators

$$\tilde{F} := F(F^*F)^{-1}, \quad \tilde{F}^* := (F^*F)^{-1}F^*. \tag{9}$$

The importance of the canonical dual frame is its use in the reconstruction of any $f \in \mathcal{H}$, i.e.

$$f = SS^{-1}f = \sum_{n\in\mathcal{N}} \langle f, S^{-1}f_n\rangle_{\mathcal{H}} f_n = S^{-1}Sf = \sum_{n\in\mathcal{N}} \langle f, f_n\rangle_{\mathcal{H}} S^{-1}f_n. \tag{10}$$

Since a frame is typically overcomplete, there exist several non–canonical duals $\{\tilde{f}_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ such that

$$f = \sum_{n\in\mathcal{N}} \langle f, \tilde{f}_n\rangle_{\mathcal{H}} f_n. \tag{11}$$

A more general definition of frames is required for Banach spaces. Let $\mathcal{B}$ be a Banach space continuously and densely embedded in $\mathcal{H}$

$$\mathcal{B} \subseteq \mathcal{H} \simeq \mathcal{H}' \subseteq \mathcal{B}'. \tag{12}$$

If the right inclusion is dense, then $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ is called a *Gelfand triple*. The symbol $\simeq$ stands for the canonical Riesz identification of $\mathcal{H}$ with its dual $\mathcal{H}'$.

Definition 1 A frame $\mathcal{F}$ (here $\tilde{\mathcal{F}}$ is some dual frame, e.g., the canonical dual frame) for $\mathcal{H}$ is called a *Gelfand frame* for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, if $\mathcal{F} \subset \mathcal{B}$, $\tilde{\mathcal{F}} \subset \mathcal{B}'$ and there exists a Gelfand triple $\big(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\big)$ of sequence spaces such that

$$F^* : \mathcal{B}_d \to \mathcal{B},\ F^*\mathbf{c} = \sum_{n\in\mathcal{N}} c_n f_n \quad \text{and} \quad \tilde{F} : \mathcal{B} \to \mathcal{B}_d,\ \tilde{F}f = \big(\langle f, \tilde{f}_n\rangle_{\mathcal{B}\times\mathcal{B}'}\big)_{n\in\mathcal{N}} \tag{13}$$

are bounded operators.

Remark 2 1. If $\mathcal{F}$ is a Gelfand frame for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ with respect to the Gelfand triple of sequences $\big(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\big)$, then by duality the operators

$$\tilde{F}^* : \mathcal{B}_d' \to \mathcal{B}',\ \tilde{F}^*\mathbf{c} = \sum_{n\in\mathcal{N}} c_n \tilde{f}_n \quad \text{and} \quad F : \mathcal{B}' \to \mathcal{B}_d',\ Ff = \big(\langle f, f_n\rangle_{\mathcal{B}'\times\mathcal{B}}\big)_{n\in\mathcal{N}} \tag{14}$$

are bounded, see, e.g., [27] for details.

2. If $\mathcal{B} = \mathcal{H}$ then Definition 1 becomes the definition of frames for Hilbert spaces.

3. Even if $\mathbf{V} \subset \mathcal{H} = \mathbf{L}^2(\Omega) \subset \mathbf{V}'$ is a Hilbert space, using the notation "$\mathcal{B}$" we emphasize that the frame $\mathcal{F}$ is *not* a Hilbert space frame for $\mathbf{V}$. It is

a frame for $\mathcal{H}$, which also characterizes $\mathbf{V}$ (as a subspace of $\mathcal{H}$) with frame coefficients $\langle f, \tilde{f}_n \rangle_{\mathcal{H}}$ belonging to $\mathbf{V}_d \subset \ell_2(\mathcal{N})$ and computed using $\mathcal{H}$-inner product.

4. Definition 1 generalizes the following case to pure frames: Consider a wavelet system $\Psi := \{\psi_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j}$ on Ω ($\mathcal{J}_j$ is a suitable set of indexes depending on the scale j, see [42] for details), $\mathcal{B} = H^s(\Omega)$, $\mathcal{H} = L_2(\Omega)$ and

$$\mathcal{B}_d = \ell_{2,2^{s\cdot}} := \{\mathbf{d} := \{d_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j} : \left(\sum_{j\geq -1}\sum_{k\in\mathcal{J}_j} 2^{2sj}|d_{j,k}|^2\right)^{1/2} < \infty\}.$$

It is well known that if Ψ is a Riesz basis for $L_2(\Omega)$ and its elements, together with those of its biorthogonal dual basis $\tilde{\Psi} := \{\tilde{\psi}_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j}$, are compactly supported, smooth enough, and with a sufficient number of vanishing moments, then $H^s(\Omega)$ is fully characterized by Ψ in the sense that $f \in H^s(\Omega)$ if and only if

$$f = \sum_{j\geq -1}\sum_{k\in\mathcal{J}_j} \langle f, \tilde{\psi}_{j,k}\rangle_{L_2(\Omega)}\psi_{j,k} \tag{15}$$

and

$$\|f\|_{H^s(\Omega)} \sim \left(\sum_{j\geq -1}\sum_{k\in\mathcal{J}_j} 2^{2sj}|\langle f, \tilde{\psi}_{j,k}\rangle_{L_2(\Omega)}|^2\right)^{1/2}. \tag{16}$$

See [11] for the same characterization by using pure wavelet frames, constructed by Overlapping Domain Decomposition. Note that there exists a natural unitary isomorphism from $\ell_{2,2^{s\cdot}}$ into ℓ_2 given by

$$D_{H^s(\Omega)} : \ell_{2,2^{js\cdot}} \to \ell_2, \quad \mathbf{d} := \{d_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j} \mapsto D_{H^s(\Omega)}\mathbf{d} := \{2^{js}d_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j}. \tag{17}$$

2.2 Adaptive Numerical Frame Schemes for Elliptic Operator Equations

To implement the fixed point iteration described in (2), we first study the solvability (for fixed $u^{(n)}$) of the linear operator equations in (3). Generally, such equations are of the form

$$Au = f, \tag{18}$$

where A, as before, is a boundedly invertible operator from Hilbert space $\mathbf{V}$ into its dual $\mathbf{V}'$, $\|Au\|_{\mathbf{V}'} \sim \|u\|_{\mathbf{V}}$, $u \in \mathbf{V}$, and $f \in \mathbf{V}'$. We also have that $a(v,w) := \langle Av, w\rangle_{\mathbf{V}'\times\mathbf{V}}$, defines a bilinear form on $\mathbf{V}$, where $\langle\cdot,\cdot\rangle_{\mathbf{V}\times\mathbf{V}'}$ defines the dual pairing of $\mathbf{V}$ and $\mathbf{V}'$. The form a is *elliptic*, i.e., there exist positive constants α, β such that $\alpha\|v\|^2_{\mathbf{V}} \leq a(v,v) \leq \beta\|v\|^2_{\mathbf{V}}$ for all $v \in \mathbf{V}$, and a is *non-symmetric*. The non-symmetry assumption on a is motivated by applications in [5,30].

Throughout the paper, let $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ be a Gelfand frame for the Gelfand triple $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$ with $(\mathbf{V}_d, \ell_2(\mathcal{N}), \mathbf{V}'_d)$ being the corresponding Gelfand triple of sequence spaces. Moreover, keeping in mind REMARK 4. in Subsection 3.1, let $D_{\mathbf{V}} : \mathbf{V}_d \to \ell_2(\mathcal{N})$ be a unitary isomorphism, so that its $\ell_2(\mathcal{N})$–adjoint $D^*_{\mathbf{V}} : \ell_2(\mathcal{N}) \to \mathbf{V}'_d$ is also an isomorphism.

We show how the Gelfand frame setting is used for the adaptive numerical treatment of elliptic operator equations (18). Following, e.g. [6,11,35], we use frame expansions to convert problem (18) into an operator equation on $\ell_2(\mathcal{N})$. The redundancy of the frame leads to a singular discretization matrix. Nevertheless, in Theorem 2 we show that this can be handled in practice and that the solution of (18) can be computed by a version of the damped Richardson iteration applied to the associated normal equations. The resulting scheme is not directly implementable since we have to deal with infinite matrices and vectors. Therefore, similarly to [6,**?**,**?**], we also show how the scheme can be transformed into an implementable scheme using "finite" versions of the building blocks procedures we introduce in Subsection 2.2.2. The result is a convergent adaptive frame algorithm.

2.2.1 A series representation

We start by generalizing Lemma 4.1 and Theorem 4.2 given in [11] to the case of non–symmetric a.

Lemma 1 *The operator*

$$\mathbf{A} := (D^*_{\mathbf{V}})^{-1} F A F^* D^{-1}_{\mathbf{V}} \tag{19}$$

is a bounded operator from $\ell_2(\mathcal{N})$ to $\ell_2(\mathcal{N})$. Moreover, $\mathbf{A}$ is boundedly invertible on its range $\operatorname{ran}(\mathbf{A}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F)$.

Proof Since $\mathbf{A}$ is a composition of bounded operators $D^{-1}_{\mathbf{V}} : \ell_2(\mathcal{N}) \to \mathbf{V}_d$, $F^* : \mathbf{V}_d \to \mathbf{V}$, $A : \mathbf{V} \to \mathbf{V}'$, $F : \mathbf{V}' \to \mathbf{V}'_d$ and $(D^*_{\mathbf{V}})^{-1} : \mathbf{V}'_d \to \ell_2(\mathcal{N})$, $\mathbf{A}$ is a bounded operator from $\ell_2(\mathcal{N})$ to $\ell_2(\mathcal{N})$. Moreover, from the decomposition (19) we get

$$\ker(\mathbf{A}) = \ker(F^* D^{-1}_{\mathbf{V}}), \quad \operatorname{ran}(\mathbf{A}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F). \tag{20}$$

Define $\mathbf{L} := (D^*_{\mathbf{V}})^{-1} F F^* D^{-1}_{\mathbf{V}}$. Note that $\ker(\mathbf{L}) = \ker(F^* D^{-1}_{\mathbf{V}})$ and $\operatorname{ran}(\mathbf{L}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F)$. The fact that $\ell_2(\mathcal{N}) = \ker(\mathbf{L}^*) \oplus \operatorname{ran}(\mathbf{L})$ implies, due to the self-adjointness of $\mathbf{L}$, that

$$\ell_2(\mathcal{N}) = \ker(F^* D^{-1}_{\mathbf{V}}) \oplus \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F). \tag{21}$$

Therefore, $\mathbf{A}_{|\operatorname{ran}(\mathbf{A})} : \operatorname{ran}(\mathbf{A}) \to \operatorname{ran}(\mathbf{A})$ is boundedly invertible.

Let $\mathbf{P} : \ell_2(\mathcal{N}) \to \operatorname{ran}(\mathbf{A})$ be the orthogonal projection of $\ell_2(\mathcal{N})$ onto $\operatorname{ran}(\mathbf{A})$.

Theorem 2 *Let $\mathbf{f} := (D^*_{\mathbf{V}})^{-1}Ff$ and $\mathbf{A}$ as in* (19). *The solution $u \in \mathbf{V}$ of* (18) *is computed by*

$$u = F^* D^{-1}_{\mathbf{V}} \mathbf{u} \tag{22}$$

where

$$\mathbf{u} = \left(\alpha^* \sum_{n=0}^{\infty} (\mathrm{id} - \alpha^* \mathbf{A}^* \mathbf{A})^n_{|\operatorname{ran}(\mathbf{A})} \right) \mathbf{A}^* \mathbf{f}, \tag{23}$$

with $0 < \alpha^* < 2/\lambda_{\max}$*,* $\lambda_{max} = \|\mathbf{A}^*\mathbf{A}\|_2$ *with* $\|\cdot\|_2$ *being the usual spectral norm.*

Proof We have $u = \sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathcal{H}} f_n$ in $\mathcal{H}$. Since $\mathcal{F}$ is a Gelfand frame, $F^*\tilde{F} : \mathbf{V} \to \mathbf{V}$ is bounded and implies $u = F^*\tilde{F}u = \sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathbf{V}\times\mathbf{V}'} f_n$ in $\mathbf{V}$. Moreover, (18) is equivalent to the following system of equations

$$\sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathbf{V}\times\mathbf{V}'} \langle Af_n, f_m\rangle_{\mathbf{V}'\times\mathbf{V}} = \langle f, f_m\rangle_{\mathbf{V}'\times\mathbf{V}}, \quad m \in \mathcal{N}. \tag{24}$$

Let $\mathbf{u} := \mathbf{P} D_{\mathbf{V}} \tilde{F} u$. Then (24) can be rewritten as

$$\mathbf{A}\mathbf{u} = \mathbf{f}. \tag{25}$$

Multiplying both sides of (25) by $\mathbf{A}^*$ we get the normal equation

$$(\mathbf{A}^*\mathbf{A})\,\mathbf{u} = \mathbf{A}^*\mathbf{f}. \tag{26}$$

Since $\mathbf{A}^*\mathbf{A}$ is self–adjoint and positive–definite on $\operatorname{ran}(\mathbf{A})$, we obtain (23).

2.2.2 Numerical realization

By Theorem 2, the computation of $\mathbf{u}$ in (25) amounts to the damped Richardson iteration

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} - \alpha^* \mathbf{A}^* (\mathbf{A}\mathbf{u}^{(i)} - \mathbf{f}), \quad i \in \mathbb{N}_0, \quad \mathbf{u}^{(0)} = \mathbf{0}. \tag{27}$$

The iteration in (27) cannot be practically realized for infinite vectors $\mathbf{u}^{(i)}$, $i \in \mathbb{N}_0$. To avoid this we make use of the procedures (see [6–8, 35] for details on their analysis and implementation) :

- $\mathbf{RHS}[\varepsilon, \mathbf{f}] \to \mathbf{f}_\varepsilon$: determines for $\mathbf{f} \in \ell_2(\mathcal{N})$ a vector $\mathbf{f}_\varepsilon \in \ell_0(\mathcal{N})$ such that
$$\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon; \tag{28}$$
- $\mathbf{APPLY}[\varepsilon, \mathbf{A}, \mathbf{v}] \to \mathbf{w}_\varepsilon$: determines for a bounded linear operator $\mathbf{A}$ on $\ell_2(\mathcal{N})$ and for $\mathbf{v} \in \ell_0(\mathcal{N})$ a vector $\mathbf{w}_\varepsilon \in \ell_0(\mathcal{N})$ such that
$$\|\mathbf{A}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon; \tag{29}$$
- $\mathbf{COARSE}[\varepsilon, \mathbf{v}] \to \mathbf{v}_\varepsilon$: determines for $\mathbf{v} \in \ell_0(\mathcal{N})$ a vector $\mathbf{v}_\varepsilon \in \ell_0(\mathcal{N})$ such that
$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon. \tag{30}$$

We discuss the properties of the routines **RHS**, **APPLY** and **COARSE** in Section 4, where we study the complexity and the computational cost required to approximate the solution of the original problem up to some prescribed tolerance. Let $\rho := \rho(\alpha^*) := \|\,(\mathrm{id} - \alpha^*\mathbf{A}^*\mathbf{A})_{|\operatorname{ran}(\mathbf{A})}\,\|_2 = \max\{\alpha^*\lambda_{\max} - 1, 1 - \alpha^*\lambda_{\min}\} < 1$, where $\lambda_{\min} := \|(\mathbf{A}^*\mathbf{A}_{|\operatorname{ran}(\mathbf{A})})^{-1}\|_2$. We define the inexact version of the damped Richardson iteration (27):

Algorithm 1

SOLVE$[\epsilon, \mathbf{A}, \mathbf{f}] \rightarrow \mathbf{v}_\epsilon$:
Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.
$j := 0,\ \mathbf{v}^{(0)} := 0,\ \epsilon_0 := \|\mathbf{A}^{-1}_{|\operatorname{ran}(\mathbf{A})}\| \|\mathbf{f}\|_{\ell_2(\mathcal{N})}$
While $\epsilon_j > \epsilon$ do
  $j := j + 1$
  $\epsilon_j := 3\rho^K \epsilon_{j-1}/\theta$
  $\mathbf{g}^{(j)} := \mathbf{RHS}[\frac{\theta\epsilon_j}{12\alpha^* K\|\mathbf{A}^*\|}, \mathbf{f}]$
  $\mathbf{f}^{(j)} := \mathbf{APPLY}[\frac{\theta\epsilon_j}{12\alpha^* K}, \mathbf{A}^*, \mathbf{g}^{(j)}]$
  $\mathbf{v}^{(j,0)} := \mathbf{v}^{(j-1)}$
  For $k = 1, ..., K$ do
    $\mathbf{w}^{(j,k-1)} := \mathbf{APPLY}[\frac{\theta\epsilon_j}{12\alpha^* K\|\mathbf{A}^*\|}, \mathbf{A}, \mathbf{v}^{(j,k-1)}]$
    $\mathbf{v}^{(j,k)} := \mathbf{v}^{(j,k-1)} - \alpha^* \left(\mathbf{APPLY}[\frac{\theta\epsilon_j}{12\alpha^* K}, \mathbf{A}^*, \mathbf{w}^{(j,k-1)}] - \mathbf{f}^{(j)}\right)$
  od
  $\mathbf{v}^{(j)} := \mathbf{COARSE}[(1-\theta)\epsilon_j, \mathbf{v}^{(j,K)}]$
od
$\mathbf{v}_\epsilon := \mathbf{v}^{(j)}$.

For details on the numerical implementation of procedure **SOLVE** to, e.g., the Poisson equation on the L–shaped domain see [11,13,35,44].

Note that the parameter θ plays an important role in complexity estimates for **COARSE** given in Section 4. The proof of the convergence of Algorithm 1 is equal to that of [35, Proposition 2.1] and of [11, Theorem 4.2] except for the fact that here we make use of the damped Richardson iteration in (27) on normal equations, due to the non–symmetry of a. Thus, the following result holds.

Theorem 3 *Under the assumptions of Theorem 2, let* $\mathbf{u} \in \ell_2(\mathcal{N})$ *be a solution of* (25). *Then* **SOLVE**$[\epsilon, \mathbf{A}, \mathbf{f}]$ *produces finitely supported vectors* $\mathbf{v}^{(j,K)}, \mathbf{v}^{(j)}, \mathbf{v}_\epsilon$ *such that*

$$\left\|\mathbf{P}(\mathbf{u} - \mathbf{v}^{(j)})\right\|_{\ell_2(\mathcal{N})} \leq \epsilon_j, \quad j \in \mathbb{N}_0. \tag{31}$$

In particular,

$$\|u - F^* D_{\mathbf{V}}^{-1} \mathbf{v}_\epsilon\|_{\mathbf{V}} \leq \|F^*\| \|D_{\mathbf{V}}^{-1}\| \epsilon. \tag{32}$$

Moreover, it holds that

$$\left\|\mathbf{P}\mathbf{u} - (\mathrm{id} - \mathbf{P})\mathbf{v}^{(j-1)} - \mathbf{v}^{(j,K)}\right\|_{\ell_2(\mathcal{N})} \leq \frac{2\theta\epsilon_j}{3}, \quad j \geq 1. \tag{33}$$

The implementation of the damped Richardson iteration (27) on normal equations might exhibit a slow convergence, if the relaxation parameter α^* is small. To improve the efficiency of the proposed scheme, the generalizations of Algorithm 1 towards, e.g., gradient iterations as suggested in [19,44], appear in [13]. In the latter paper the reader can also find numerical implementations and tests which confirm the optimality of Algorithm 1, compare Section 4.

3 Implementation of the Fixed Point Iteration

We discretize problem (1) and use Algorithm 1 to implement the fixed point iteration in (2). Let

i) $\mathbf{A} := (D^*_{\mathbf{V}})^{-1}FAF^*D^{-1}_{\mathbf{V}} : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$, be bounded and boundedly invertible on its range $\mathrm{ran}(\mathbf{A}) = \mathrm{ran}((D^*_{\mathbf{V}})^{-1}F)$;
ii) $\mathbf{l} := (D^*_{\mathbf{V}})^{-1}F\ell \in \mathrm{ran}(\mathbf{A}) \subset \ell_2(\mathcal{N})$;
iii) $\mathbf{A1}(\cdot) := (D^*_{\mathbf{V}})^{-1}FA_1(F^*D^{-1}_{\mathbf{V}}\cdot, F^*D^{-1}_{\mathbf{V}}\cdot) : \ell_2(\mathcal{N}) \to \mathrm{ran}(\mathbf{A}) \subset \ell_2(\mathcal{N})$;
iv) $\mathbf{H}(\cdot) := (\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}(\mathbf{l} - \mathbf{A1}(\cdot)) : \ell_2(\mathcal{N}) \to \mathrm{ran}(\mathbf{A})$.

Remark 3 Note that $\mathrm{ran}(\mathbf{A}) = \mathrm{ran}((D^*_{\mathbf{V}})^{-1}F)$ implies that the operators in i)–iv) map $\ell_2(\mathcal{N})$ into $\mathrm{ran}(\mathbf{A})$. By definition of $\boldsymbol{P}$ we also have $\mathbf{H}(\boldsymbol{P}\mathbf{v}) = \boldsymbol{P}\mathbf{H}(\mathbf{v}) = \mathbf{H}(\mathbf{v})$ for any $\mathbf{v} \in \ell_2(\mathcal{N})$.

Thus, the variational problem in (1) is equivalent to the following discrete problem.

Problem 3: *Find* $\mathbf{u} \in \mathrm{ran}(\mathbf{A}) \subset \ell_2(\mathcal{N})$ *such that*

$$\mathbf{Au} + \mathbf{A1}(\mathbf{u}) = \mathbf{l}, \tag{34}$$

or, equivalently, such that $\mathbf{u}$ *is a fixed point of* $\mathbf{H}$ *in* $\mathrm{ran}(\mathbf{A})$*, i.e*

$$\mathbf{u} = \mathbf{H}(\mathbf{u}), \quad \mathbf{u} \in \mathrm{ran}(\mathbf{A}). \tag{35}$$

Next, define the closed ball of radius $r > 0$ in $\mathrm{ran}(\mathbf{A})$ by

$$\boldsymbol{B}_r := \{\mathbf{u} \in \mathrm{ran}(\mathbf{A}) : \|\mathbf{u}\|_{\ell_2(\mathcal{N})} \le r\}.$$

Theorem 4 *Under the assumptions and notations specified above, the following statements hold*

a) If $0 < r < \left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$*, then* $\mathbf{H}_{|\boldsymbol{B}_r}$ *is a contraction with Lipschitz constant* $L := r\left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right) < 1$*;*
b) if $0 < r < \left(\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$ *and* $\|\mathbf{l}\| \le r\left(\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^{-1} - \|A_1\|\|F\|^3 r\right)$ *then* $\mathbf{H}(\boldsymbol{B}_r) \subseteq \boldsymbol{B}_r$ *;*
c) if $\|\mathbf{l}\| < \left(4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|\right)^{-1}$ *then* (35) *has a unique solution* $\mathbf{u} \in \boldsymbol{B}_{r^*}$*, for some suitable* r^* *such that* $0 < r^* < \left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|\|A_1\|\right)^{-1}$.

Proof Due to $\|D_{\mathbf{V}}\| = \|D^{-1}_{\mathbf{V}}\| = \|D^*_{\mathbf{V}}\| \equiv 1$ and $\|F\| = \|F^*\|$, for $\mathbf{u}, \mathbf{v} \in \boldsymbol{B}_r$

$$\begin{aligned}\|\mathbf{Hu} - \mathbf{Hv}\| &\le \|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|(D^*_{\mathbf{V}})^{-1}F\left(A_1(F^*D^{-1}_{\mathbf{V}}\mathbf{u}, F^*D^{-1}_{\mathbf{V}}\mathbf{u})\right.\\ &\quad - \left.A_1(F^*D^{-1}_{\mathbf{V}}\mathbf{v}, F^*D^{-1}_{\mathbf{V}}\mathbf{v})\right)\|_{\ell_2(\mathcal{N})}\\ &\le 2r\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\|\mathbf{u} - \mathbf{v}\|_{\ell_2(\mathcal{N})}.\end{aligned}$$

Thus, $\mathbf{H}$ is a contraction due to $L < 1$. For b) observe that by iv) and the estimate above

$$\|\mathbf{Hu}\|_{\ell_2(\mathcal{N})} \le \|\mathbf{A}^{-1}_{|\,\mathrm{ran}\,\mathbf{A}}\|\left(\|\mathbf{l}\| + \|A_1\|\|F\|^3 r^2\right) \le r.$$

c) We show that if $\|\mathbf{l}\| < \left(4\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|\right)^{-1}$, then there exists r^* with $0 < r^* < \left(2\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$ such that $\|\mathbf{l}\| < r^*\left(\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|^{-1} - \|A_1\|\|F\|^3 r^*\right)$. Using a) and b) we get that $\mathbf{H}|_{\boldsymbol{B}_{r^*}}$ is a contractive mapping of $\boldsymbol{B}_{r^*}$ into itself and, thus, has a unique fixed point. To see that such r^* exists, consider $h(r) = r\left(\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|^{-1} - \|A_1\|\|F\|^3 r\right)$, a quadratic mapping, and note that h assumes values from 0 to $\left(4\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|\right)^{-1}$ as r varies from 0 to $\left(2\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$.

Corollary 1 *If* $\|\mathbf{l}\| < (4\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$*, then the solution* $\mathbf{u}$ *in* $\boldsymbol{B}_{r^*}$ *of* (35) *is given by the following discrete fixed point iteration*

$$\mathbf{u}_{n+1} = \mathbf{H}\mathbf{u}_n, \quad n \geq 0, \quad \mathbf{u}_0 = \mathbf{0} \in \operatorname{ran}(\mathbf{A}), \tag{36}$$

$$\mathbf{u} = \lim_{n\to\infty} \mathbf{u}_n. \tag{37}$$

The discrete fixed point iteration cannot be implemented for infinite vectors. We propose the following new approximating adaptive scheme **FIXPT**, where the procedure **SOLVE** introduced in Subsection 2.2.2 replaces the exact computation of $\mathbf{u}_{n+1}$ in (36).

Algorithm 2

FIXPT$[\varepsilon, \mathbf{A}, \mathbf{A1}, \mathbf{l}] \to \mathbf{u}_\varepsilon$:

$i := 0,\ \mathbf{v}_0 := 0,\ 0 < \varepsilon_0 < r^*;\ \varepsilon_0 \neq L;$

While $\left(\varepsilon_i > \frac{\varepsilon_0 - L}{\varepsilon_0\left(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^i\right)}(\varepsilon - L^i r^*)\right)$ do

  $\varepsilon_{i+1} := \varepsilon_0^{i+1}$

  $\mathbf{v}_{i+1} = \mathbf{SOLVE}[\varepsilon_{i+1}, \mathbf{A}, \mathbf{l} - \mathbf{A1}(\mathbf{v}_i)]$

  $i := i + 1$

od

$\mathbf{u}_\varepsilon := \mathbf{v}_i$.

Remark 4 The Algorithm 2 is a perturbation of the exact fixed point iteration creating sequences $\{\mathbf{v}_i\}_{i\in\mathbb{N}_0}$ of finite vectors. In general such vectors do not belong to $\operatorname{ran}(\mathbf{A})$ and there is not much hope that $\lim_{n\to\infty}\mathbf{v}_n = \mathbf{u}$. We can try to see whether $\lim_{n\to\infty}\mathbf{P}\mathbf{v}_n = \mathbf{u}$. It can happen that, because of an accumulation of perturbation errors, there exists $n \in \mathbb{N}$ large enough such that $\mathbf{P}\mathbf{v}_n \notin \boldsymbol{B}_{r^*}$!

To prove the convergence of Algorithm 2 to the solution of problem (35) we start with the following auxiliary Lemma.

Lemma 2 *If* $\|\mathbf{l}\| < (4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$ *and if the quantities*

$$\begin{aligned}\mathcal{E}_n := \sum_{k=2}^{n+1}\varepsilon_k + (1+L)\sum_{h=0}^{n-3}\sum_{k=3}^{n-h}\varepsilon_k L^{n-h-k} + (\varepsilon_1 \\ +\ L(\varepsilon_0 + \|(\mathbf{A}_{|\,\mathrm{ran}(\mathbf{A})})^{-1}\|\|\mathbf{l}\|))\sum_{k=0}^{n-1}L^k \\ +\ \varepsilon_0 + \|\mathbf{A}^{-1}_{|\,\mathrm{ran}(\mathbf{A})}\|\|\mathbf{l}\| \le r^*,\ \textit{for all } n\in\mathbb{N}_0,\end{aligned}$$

with L and r^* as in Theorem 4 a) and c), respectively, then the sequence $\{\mathbf{Pv}_i\}_{i\in\mathbb{N}}$ resulting from the application of Algorithm 2 all lies in $\boldsymbol{B}_{r^*}$.

Proof The proof is by induction on n. For $n=1$ we get that $\mathbf{Pv}_1 \in \boldsymbol{B}_{r^*}$

$$\|\mathbf{Pv}_1\|_{\ell_2(\mathcal{N})} = \|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_{\mathbf{0}},\mathbf{A},\mathbf{l}] - \mathbf{H}(0)\right) + \mathbf{H}(0)\|_{\ell_2(\mathcal{N})} \le \mathcal{E}_0 \le r^*.$$

Since $\mathbf{H}(\mathbf{v}) = \mathbf{H}(\mathbf{Pv}) = \boldsymbol{P}\mathbf{H}(\mathbf{v})$, by (31) it holds $\|\mathbf{P}\ \mathbf{SOLVE}[\varepsilon_{n+1},\mathbf{A},\mathbf{l}-\mathbf{A}1(\mathbf{v}_n)] - \mathbf{P}\ \mathbf{H}(\mathbf{v}_n)\| \le \varepsilon_{n+1}$. By the hypothesis on $\|\mathbf{l}\|$, ensuring that $\mathbf{H}$ is a contraction with the Lipschitz constant L, we obtain

$$\begin{aligned}\|\mathbf{P}(\mathbf{v}_2-\mathbf{v}_1)\|_{\ell_2(\mathcal{N})} &= \|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_2,\mathbf{A},\mathbf{l}-\mathbf{A}1(\mathbf{v}_1)]\right) \\ &-\ \mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_1,\mathbf{A},\mathbf{l}-\mathbf{A}1(\mathbf{v}_0)]\right)\|_{\ell_2(\mathcal{N})} \\ &= \|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_2,\mathbf{A},\mathbf{l}-\mathbf{A}1(\mathbf{v}_1)] - \mathbf{H}(\mathbf{v}_1)\right) + \mathbf{H}(\mathbf{v}_1) - \mathbf{H}(\mathbf{v}_0) \\ &+ \mathbf{H}(\mathbf{v}_0) - \mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_1,\mathbf{A},\mathbf{l}-\mathbf{A}1(\mathbf{v}_0)]\right)\|_{\ell_2(\mathcal{N})} \\ &\le \varepsilon_2+\varepsilon_1+L\|\mathbf{Pv}_1\|_{\ell_2(\mathcal{N})}.\end{aligned}$$

The induction assumption is that $\mathbf{Pv}_1,...,\mathbf{Pv}_n \in \boldsymbol{B}_{r^*}$ with $\|\mathbf{Pv}_n\|_{\ell_2(\mathcal{N})} \le \mathcal{E}_{n-1}$. We have

$$\begin{aligned}\|\mathbf{P}(\mathbf{v}_{n+1}-\mathbf{v}_n)\|_{\ell_2(\mathcal{N})} &\le \varepsilon_{n+1}+\varepsilon_n + L\|\mathbf{P}(\mathbf{v}_n-\mathbf{v}_{n-1})\|_{\ell_2(\mathcal{N})} \\ &= \varepsilon_{n+1} + (1+L)\sum_{k=3}^{n}\varepsilon_k L^{n-k} + L^{n-1}(\varepsilon_1 + L\|\mathbf{Pv}_1\|_{\ell_2(\mathcal{N})}).\end{aligned}$$

Using the above estimate and the triangular inequality we get $\mathbf{Pv}_{n+1} \in \boldsymbol{B}_{r^*}$.

Remark 5 Note that the assumption on $\mathcal{E}_n$ in the above Theorem is not restrictive. It holds that

- $\displaystyle\sum_{k=2}^{n+1}\varepsilon_k = \sum_{k=0}^{n+1}\varepsilon_0^k - 1 - \varepsilon_0 = \frac{1-\varepsilon_0^{n+2}}{1-\varepsilon_0} - 1 - \varepsilon_0 \le \frac{\varepsilon_0^2}{1-\varepsilon_0} \to 0$ for $\varepsilon_0 \to 0$;
- We bound the map

$$n \mapsto \sum_{h=0}^{n-3}\sum_{k=3}^{n-h}\varepsilon_k L^{n-h-k} \le C(\varepsilon_0, L),$$

with $C(\varepsilon_0,L)\to 0$ for $\varepsilon_0 \to 0$, uniformly with respect to n;

– $\left(\varepsilon_1 + L(\varepsilon_0 + \|\mathbf{A}^{-1}_{|\operatorname{ran}(\mathbf{A})}\|\|\mathbf{l}\|\right)\sum_{k=0}^{\infty} L^k + (\varepsilon_0 + \|\mathbf{A}^{-1}_{|\operatorname{ran}(\mathbf{A})}\|\|\mathbf{l}\|)$ is small enough whenever ε_0 and $\|\mathbf{l}\|$ are small. Note that $\|\mathbf{l}\| \lesssim \|F\|\|\ell\|$, where $\|\ell\|$ depends usually on the size of the norms of the forcing terms and boundary data. Note that, for the nonlinear problems we consider, the existence and uniqueness of the solutions are ensured under certain smallness assumptions on the data. We cannot expect numerical implementations to be less restrictive. Such restrictions do represent the worst case analysis.

We are now ready to prove the main convergence result.

Theorem 5 *If $\|\mathbf{l}\| < (4\|(\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$ and $\mathcal{E}_n \le r^*$ for all $n \in \mathbb{N}$, then*

$$\mathbf{H}(\mathbf{u}) = \mathbf{u} = \lim_{n\to\infty} \mathbf{P}\mathbf{v}_n = \mathbf{P}\left(\mathbf{FIXPT}[0, \mathbf{A}, \mathbf{A}1, \mathbf{l}]\right), \tag{38}$$

where $\{\mathbf{v}_i\}_{i\in\mathbb{N}}$ are obtained by Algorithm 2. After n iterations of Algorithm 2 we get

$$\|\mathbf{P}\mathbf{v}_{n+1} - \mathbf{u}\|_{\ell_2(\mathcal{N})} \le \varepsilon_0 \frac{\varepsilon_0^n(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n)}{\varepsilon_0 - L} + L^n\|\mathbf{u}\| \le \varepsilon_0 \frac{\left(\varepsilon_0^n(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n\right)}{\varepsilon_0 - L} + L^n r^*. \tag{39}$$

Moreover, for $\varepsilon > 0$ we have

$$\|\mathbf{u} - \mathbf{P}\left(\mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}1, \mathbf{l}]\right)\| \le \varepsilon, \tag{40}$$

and the number N of iterations to achieve the accuracy $\varepsilon > 0$ is estimated by

$$N \sim -\log(\varepsilon). \tag{41}$$

Proof It holds that

$$\mathbf{P}\mathbf{v}_{n+1} = (\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_{n+1}, \mathbf{A}, \mathbf{l} - \mathbf{A}1(\mathbf{v}_n)]\right) - \mathbf{H}(\mathbf{P}\mathbf{v}_n)) + (\mathbf{H}(\mathbf{P}\mathbf{v}_n) - \mathbf{H}(\mathbf{u})) + \mathbf{u}$$

and, thus,

$$\begin{aligned}
\|\mathbf{P}\mathbf{v}_{n+1} - \mathbf{u}\| &\le \|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_{n+1}, \mathbf{A}, \mathbf{l} - \mathbf{A}1(\mathbf{v}_n)]\right) \\
&\quad - \mathbf{H}(\mathbf{P}\mathbf{v}_n)\|_{\ell_2(\mathcal{N})} + \|\mathbf{H}(\mathbf{P}\mathbf{v}_n) - \mathbf{H}(\mathbf{u})\|_{\ell_2(\mathcal{N})} \\
&\le \varepsilon_{n+1} + L\|\mathbf{P}\mathbf{v}_n - \mathbf{u}\|_{\ell_2(\mathcal{N})} \\
&\le \varepsilon_{n+1} + L(\varepsilon_n + L\|\mathbf{P}\mathbf{v}_{n-1} - \mathbf{u}\|_{\ell_2(\mathcal{N})}) \\
&\le \varepsilon_0 \sum_{k=0}^{n} \varepsilon_0^{n-k} L^k + L^n\|\mathbf{u}\|_{\ell_2(\mathcal{N})} \\
&= \varepsilon_0 \frac{\varepsilon_0^n\left(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n\right)}{\varepsilon_0 - L} + L^n\|\mathbf{u}\|_{\ell_2(\mathcal{N})} \to 0, \quad n \to \infty.
\end{aligned}$$

Therefore, $\mathbf{H}(\mathbf{u}) = \mathbf{u} = \lim_{n\to\infty} \mathbf{P}\mathbf{v}_n = \mathbf{P}\left(\mathbf{FIXPT}[0, \mathbf{A}, \mathbf{A}1, \mathbf{l}]\right)$. Moreover, since $\|\mathbf{u}\|_{\ell_2(\mathcal{N})} \le r^*$, the second inequality in (39) is valid and we get

$$\varepsilon_0 \frac{\varepsilon_0^n(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n)}{\varepsilon_0 - L} + L^n r^* \le \varepsilon \tag{42}$$

if and only if

$$\varepsilon_n \leq \frac{\varepsilon_0 - L}{\varepsilon_0 \left(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n\right)}(\varepsilon - L^n r^*), \tag{43}$$

which is the criterion to exit the loop in Algorithm 2. This implies (40). From (42) we get that the necessary number of iterations to achieve the accuracy ε is given by (41).

Similarly to the proof of formula (32) (see [11, Theorem 4.2]) we finally show the following.

Corollary 2 *If* $\|\mathbf{l}\| < (4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$ *and* $\mathcal{E}_n \leq r^*$ *for all* $n \in \mathbb{N}$*, then*

$$u = \sum_{n \in \mathcal{N}} \left(D_{\mathbf{V}}^{-1}\mathbf{FIXPT}[0, \mathbf{A}, \mathbf{A}1, \mathbf{l}]\right)_n f_n, \tag{44}$$

is a solution in $\mathbf{V}$ *of the fixed point problem* $H(u) = u$ *and we have that for* $\varepsilon > 0$

$$\|u - \sum_{n \in \mathcal{N}} \left(D_{\mathbf{V}}^{-1}\mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}1, \mathbf{l}]\right)_n f_n\|_{\mathbf{V}} \lesssim \varepsilon. \tag{45}$$

Remark 6 The fact that at each iteration we compute the approximation up to a perturbation/tolerance ε_i also means that the scheme is stable. In other words, not only can ε_i be interpreted as the numerical approximation/accuracy we achieve at each step, but also as the error tolerance we can afford, without spoiling convergence. Moreover, the scheme is *fully adaptive* in the sense that the iterations are enforced (by the use of suitable implementation of **COARSE**, see below) to work only with minimal number of relevant quantities (frame coefficients), in order to keep the prescribed accuracy–complexity balance.

4 Quasi–Optimal Complexity of the Algorithm

In this section we present the complexity analysis of Algorithm 2. From formula (41), we already know that to achieve a prescribed accuracy $\varepsilon > 0$ we need to execute $N \sim -\log(\varepsilon)$ iterations. Therefore, having an estimation of the cost of each iteration, the asymptotic analysis of the complexity of the suggested adaptive scheme can be done. This is one of the very interesting theoretical advantages of the adaptive (wavelet) frame approach, together with the fact that one can prove both convergence and stability of the adaptive scheme as shown in the previous section.

Of course, the main ingredient of iterations in Algorithm 2 is the procedure **SOLVE**. As announced in Subsection 2.2.2, we discuss here an implementation of such a procedure and study its complexity. To this end, we should illustrate how the building block procedures **RHS**, **APPLY**, and **COARSE** can be implemented and estimate their computational cost. Therefore, in this section we focus on the main *properties and requirements* of these building blocks so that Algorithms 1 and 2 have certain complexity.

We refer the interested reader to [7,8,11,35] for the descriptions of procedures **RHS**, **APPLY**, and **COARSE** that fulfill the requirements stated below and needed for the complexity estimates. The complexity estimates for even more general algorithms than Algorithm 2 for linear and nonlinear variational problems, but under the more restrictive assumption that the discretizing frame $\mathcal{F}$ is a Riesz basis, were given in [7,8]. In order to describe the complexity in the more general case of pure frame discretizations a bit more (technical) effort and preparation is needed.

An algorithm for computing a finite approximation $\mathbf{u}_\varepsilon$ of $\mathbf{u}$ up to ε, for $\mathbf{u}$ given implicitly as a solution of some equation, is called *optimal* if $\#\,\mathrm{supp}(\mathbf{u}_\varepsilon)$ (the number of elements of the support of $\mathbf{u}_\varepsilon$) is not asymptotically larger (for $\varepsilon \to 0$) than the same quantity obtained by a direct computation of any other approximation of $\mathbf{u}$ using the same tolerance ε, for $\mathbf{u}$ being given explicitly. In addition to this, the optimality is fully realized if the complexity to compute $\mathbf{u}_\varepsilon$ does not exceed $\#\,\mathrm{supp}(\mathbf{u}_\varepsilon)$ asymptotically (for $\varepsilon \to 0$). In other words, we do want the number of algebraic operations to be comparable to the size of what is being computed. For discretizations by means of Riesz bases optimal algorithms can be realized, see [6–8], for example. Analogous algorithms for frames may exhibit "arbitrarily small reductions" with respect to the expected optimality (see the following Theorem 5.1 for the precise statement). Since the techniques for estimating complexity we use are similar to those in [35, Theorem 3.12], we also encounter similar difficulties to achieve the optimality for **FIXPT** when dealing with pure frames. We call this situation *quasi–optimal*. We start by defining a so–called *sparseness class* $\mathcal{A}^s$ of vectors, $s \in \mathbb{R}_+$. $\mathcal{A}^s$ will turn out to be such that, if $\mathbf{u} \in \mathcal{A}^s$, then the size of the support of $\mathbf{u}_\varepsilon = \mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}1, \mathbf{l}]$ and the computational cost for obtaining $\mathbf{u}_\varepsilon$ can be estimated a priori.

An optimal sparseness class is modelled as follows: For given $s > 0$ we define the space

$$\mathcal{A}^s_{weak} := \{\mathbf{c} \in \ell_2(\mathcal{N}) :\ \|\mathbf{c}\|_{\mathcal{A}^s_{weak}} := \sup_{n\in\mathbb{N}} n^{1/2+s}|\gamma_n(\mathbf{c})| < \infty\}, \tag{46}$$

where $\gamma_n(\mathbf{c})$ is the $n-$th largest coefficient in modulus of $\mathbf{c}$. It turns out that $\|\cdot\|_{\mathcal{A}^s_{weak}}$ is a quasi–norm and we refer to [6,21] for further details on the quasi–Banach spaces $\mathcal{A}^s_{weak}$. Such spaces can be usually found in literature under the notation ℓ^w_τ (weak–ℓ_τ) where $\tau = (1/2+s)^{-1} \in (0,2)$, and they are nothing but particular instances of Lorentz sequence spaces. Let us only mention that

$$\|\mathbf{c}\|_{\mathcal{A}^s_{weak}} \sim \sup_{N\in\mathbb{N}} N^s\|\mathbf{c}-\mathbf{c}_N\|_{\ell_2(\mathcal{N})}, \tag{47}$$

where $\mathbf{c}_N$ is the best $N-$term approximation of $\mathbf{c}$, i.e., the subsequence of $\mathbf{c}$ consisting of the N largest coefficients in modulus of $\mathbf{c}$. In particular, (47) implies that for all $\varepsilon > 0$ there exists $N_\varepsilon > 0$ large enough such that for all $\mathbf{c}$ the best $N_\varepsilon-$term approximation $\mathbf{c}_{N_\varepsilon}$ has the following properties

(i) $\|\mathbf{c}-\mathbf{c}_{N_\varepsilon}\|_{\ell_2(\mathcal{N})} \le \varepsilon$;
(ii) $\#\,\mathrm{supp}(\mathbf{c}_{N_\varepsilon}) \lesssim \varepsilon^{-1/s}\|\mathbf{c}\|^{1/s}_{\mathcal{A}^s_{weak}}$;

(iii) $\|\mathbf{c}_{N_\varepsilon}\|_{\mathcal{A}^s_{weak}} \le \|\mathbf{c}\|_{\mathcal{A}^s_{weak}}$.

Furthermore, from (47) we get the following useful technical estimate

$$\|\mathbf{c}\|_{\mathcal{A}^{\tilde{s}}_{weak}} \lesssim (\#\operatorname{supp}(\mathbf{c}))^{\tilde{s}-s} \|\mathbf{c}\|_{\mathcal{A}^s_{weak}} \tag{48}$$

whenever $0 < s < \tilde{s}$ and $\mathbf{c}$ has finite support. This optimal class of vectors perfectly fits with complexity estimates for adaptive schemes for elliptic linear equations [6,7,11,35]. For more general nonlinear problems a "weaker" version of the sparseness class has been introduced and denoted by $\mathcal{A}^s_{tree}$ in [8].

It is not known whether there exist frames for which the solutions of generic nonlinear equations, particularly for Navier-Stokes and magnetohydrodynamics equations, can have frame coefficients belonging to the sparseness classes described above. Therefore, we discuss next the requirements, fulfilled by $\mathcal{A}^s_{weak}$ and $\mathcal{A}^s_{tree}$, that a generic sparseness class $\mathcal{A}^s$ should have to ensure quasi–optimality of our scheme. In case the solution belongs to any sparseness class with such properties, the algorithm will behave as ensured theoretically. The introduction of $\mathcal{A}^s$ is also motivated by the need to simplify the presentation of our complexity result. The numerical tests in [1,42] for turbulent flows motivate our assumption that the (wavelet) frame coefficients of the corresponding solutions do belong to some of these generic sparseness classes (i.e., only few significant wavelet coefficients can be expected to be relevant in the representation of the solution).

For $s > 0$ and for a nondecreasing function $T : \mathbb{N} \to \mathbb{N}$ such that $N \lesssim T(N)$ we call any space $\mathcal{A}^s$ a *T–sparseness class* if $\mathcal{A}^{\tilde{s}}_{weak} \subset \mathcal{A}^s \subseteq \mathcal{A}^s_{weak}$ and $\mathcal{A}^{\tilde{s}} \subset \mathcal{A}^s$ for all $\tilde{s} > s$ and if for all $\mathbf{u} \in \mathcal{A}^s$ and for all $\varepsilon > 0$ there exists a finite vector $\mathbf{u}_\varepsilon$ with the properties

a) $\|\mathbf{u} - \mathbf{u}_\varepsilon\| \le \varepsilon$;
b) $T(\#\operatorname{supp}(\mathbf{u}_\varepsilon)) \lesssim \varepsilon^{-1/s} \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s}$;
c) $\|\mathbf{u}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}$.

In particular we assume that there exists a constant $C_1(s)$ such that $\|\mathbf{u} + \mathbf{v}\|_{\mathcal{A}^s} \le C_1(s)(\|\mathbf{u}\|_{\mathcal{A}^s} + \|\mathbf{v}\|_{\mathcal{A}^s})$. Of course, $\mathcal{A}^s_{weak}$ is itself a sparseness class with $T = I$. Moreover, there exist other T–sparseness classes different from $\mathcal{A}^s_{weak}$, for example, the class $\mathcal{A}^s_{tree}$ defined in [8, Formula (6.7)], that also turns out to be relevant in our context.

Note that, for $\tilde{\tilde{s}} > \tilde{s} > s > 0$, by the inclusions $\mathcal{A}^{\tilde{\tilde{s}}}_{weak} \subset \mathcal{A}^{\tilde{s}} \subset \mathcal{A}^s \subseteq \mathcal{A}^s_{weak}$ and (48), we have

$$\|\mathbf{c}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\mathbf{c}\|_{\mathcal{A}^{\tilde{\tilde{s}}}_{weak}} \lesssim (\#\operatorname{supp}(\mathbf{c}))^{\tilde{\tilde{s}}-s} \|\mathbf{c}\|_{\mathcal{A}^s_{weak}} \lesssim (T(\#\operatorname{supp}(\mathbf{c})))^{\tilde{\tilde{s}}-s} \|\mathbf{c}\|_{\mathcal{A}^s}, \tag{49}$$

for all finite vectors $\mathbf{c}$.

Now we are ready to formulate our main conceptual requirements. For a fixed $s > 0$

(A1) Let $\theta < 1/3$. We assume that for any $\varepsilon > 0$, $\mathbf{v} \in \mathcal{A}^s$ and any finitely supported $\mathbf{w}$ such that

$$\|\mathbf{v} - \mathbf{w}\| \le \theta\varepsilon,$$

for $\mathbf{w}^* = \mathbf{COARSE}[(1-\theta)\varepsilon, \mathbf{w}]$ it holds that

$$T(\#\operatorname{supp}(\mathbf{w}^*)) \lesssim \varepsilon^{-1/s}\|\mathbf{v}\|_{\mathcal{A}^s}^{1/s},$$

and

$$\|\mathbf{w}^*\|_{\mathcal{A}^s} \le C_2(s)\|\mathbf{v}\|_{\mathcal{A}^s},$$

for some constant $C_2(s) > 0$. Moreover, we assume that the number of algebraic operation needed to compute $\mathbf{w}^* := \mathbf{COARSE}[\varepsilon, \mathbf{w}]$ for any finite vector $\mathbf{w}$ can be estimated by $\lesssim T(\#\operatorname{supp}(\mathbf{w})) + o(\varepsilon^{-1/s}\|\mathbf{w}\|_{\mathcal{A}^s}^{1/s})$. See [35, Proposition 3.2] and [8, Proposition 6.3] for examples of procedures **COARSE** with such properties. As it will be clear in the proof of Theorem 6 the use of the procedure **COARSE** is fundamental to ensure that the supports of the iterates generated by the algorithm can be controlled.

(A2) The vector $\mathbf{w}_\varepsilon := \mathbf{APPLY}[\varepsilon, \mathbf{A}, \mathbf{v}]$ is such that $\|\mathbf{w}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\mathbf{v}\|_{\mathcal{A}^s}$, $T(\#\operatorname{supp}(\mathbf{w}_\varepsilon)) \lesssim \varepsilon^{-1/s}\|\mathbf{v}\|_{\mathcal{A}^s}^{1/s}$, and it is computed with a number of algebraic operations estimable by $\lesssim \varepsilon^{-1/s}\|\mathbf{v}\|_{\mathcal{A}^s}^{1/s} + T(\#\operatorname{supp}(\mathbf{v}))$. See [35, Proposition 3.8] and [8, Corollary 7.5] for examples of procedures **APPLY** with such properties.

(A3) A crucial procedure of the iterative approximate fixed point scheme is the realization of $\mathbf{g}_i^{(j)} := \mathbf{RHS}[\frac{\theta\varepsilon_j}{12\alpha K}, \boldsymbol{\ell} - \mathbf{A}_1(\mathbf{v}_i)]$ for each step i and j of the outer and inner loops, respectively. In particular, it requires computing efficiently a finite approximation to $\mathbf{A}_1(\mathbf{v}_i)$, where $\mathbf{A}_1$ is some nonlinear operator and $\mathbf{v}_i$ is a given finite vector. In the work [2,8,9,18] an effective way for solving this problem for multiscale and wavelet expansions has been proposed. Note that the efficient evaluation of nonlinear functionals (as the ones we consider for, e.g., Navier-Stokes equations) on coefficients as in [2,8,9,18] is valid as soon as
a) the reference bases are multiscale, i.e., the size of the supports of the basis functions decays dyadically for increasing levels;
b) their corresponding coefficients can be structured in suitable trees;
c) such bases characterize Besov spaces by certain norm equivalences.
Under these assumptions, the procedure is realized into two steps: i) optimal computation of a near-best tree approximation [3] of the coefficients of the nonlinear map; ii) numerical integration of the coefficients on average at unit cost. While for ii) only results for Riesz bases are currently available, see e.g. [18], the results in [9], which allow for an efficient realization of step i), can be straightforwardly extended to frames. In Section 5 we present a construction of suitable wavelet frames that enjoy the above mentioned multiscale properties (a-c) and, therefore, allow for the use of the cited results. As in [9, Assumption E, Section 7.2] we assume that the numerical integration can be realized on average at unit cost.

On the basis of these observations we assume the existence of a procedure **RHS** such that $\|\mathbf{g}_i^{(j)} - (\mathbf{l} - \mathbf{A}_1(\mathbf{v}_i))\|_{\ell_2} \le \frac{\theta\epsilon_j}{12\alpha K}$, $T(\#\operatorname{supp}(\mathbf{g}_i^{(j)})) \lesssim \epsilon_j^{-1/s}\|\mathbf{v}_i\|_{\mathcal{A}^s}^{1/s}$, $\|\mathbf{g}_i^{(j)}\|_{\mathcal{A}^s} \lesssim 1 + \|\mathbf{v}_i\|_{\mathcal{A}^s}$, and the number of algebraic operations needed to compute $\mathbf{g}_i^{(j)}$ is bounded by $\lesssim \epsilon_j^{-1/s}\|\mathbf{v}_i\|_{\mathcal{A}^s}^{1/s} + T(\#\operatorname{supp}(\mathbf{v}_i))$. The constants C silently appearing in the previous estimates may depend on $\|\mathbf{v}_i\|_2$ (as in [9, Theorem 3.4]), i.e., $C = C(\|\mathbf{v}_i\|_2)$ and are increasing as functions of their argument. As we will show in (58), $\|\mathbf{v}_i\|_2 \lesssim \|\mathbf{v}_i\|_{\mathcal{A}^{\tilde{s}}} \lesssim \varepsilon_i^{1-\frac{\tilde{\tilde{s}}}{s}} \|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{\tilde{\tilde{s}}}{s}}$ and $\tilde{\tilde{s}}$ is arbitrarily close to s, we can always assume that $C \lesssim 1 + C(\|\mathbf{u}\|_{\mathcal{A}^s})$ are uniformly bounded. A careful inspection of the proofs in [9] allow for C to depend on $\|\mathbf{P}\mathbf{v}_i\|_2$ in certain cases. Since by Lemma 2 $\mathbf{P}\mathbf{v}_i$ all lies in $\boldsymbol{B}_{r^*}$, we can assume in these cases the constants uniformly bounded by $C(\|\mathbf{u}\|_2) \leq C(r^*)$.

In the following the subscript index i refers to the iterations in the outer loop of the fixed point iteration and the superscript j refers to the inner loop iterations in **SOLVE**. Moreover, ε_i and ϵ_j refer to the outer and inner loop tolerances, respectively. All estimations below hold asymptotically for $\varepsilon \to 0$ (ε as in Algorithm 2).

Theorem 6 *For $0 < s < \tilde{s} < \tilde{\tilde{s}}$ let $\mathcal{A}^{\tilde{s}}$ be a T-sparseness class and $\mathbf{u} \in \mathcal{A}^s$, the solution of* (34) *as in Corollary 1. Assume that*

(i) (A1)-(A3) hold for all $s \in (0, \tilde{s}]$;
(ii) $\mathbf{P}$ is bounded on $\mathcal{A}^t$ for all $t \in (0, \tilde{s}]$;
(iii) $K > 0$ and $0 < \theta < 1/3$ in Algorithm 1 are chosen so that

$$C_1(s)C_2(s)\|\operatorname{id} - \mathbf{P}\|(3\rho^K/\theta)^{\tilde{\tilde{s}}/s-1} < 1. \tag{50}$$

Here the norm $\|\operatorname{id} - \mathbf{P}\|$ is the norm of $\operatorname{id} - \mathbf{P}$ as an operator on $\mathcal{A}^s$;
(iv) the constants L, ε_0 and r^* satisfy

$$L < \varepsilon_0 < r^* \quad \text{and} \quad \rho^{-K}\frac{L}{\varepsilon_0 - L} + \delta_0 \leq \frac{1}{2}, \quad \text{for some } \delta_0 > 0. \tag{51}$$

Then, under the assumptions of Theorem 5, for any $\varepsilon > 0$ and $\delta > 0$ such that $\tilde{\tilde{s}}/\tilde{s} = 1 + \delta$, the finite vector $\mathbf{u}_\varepsilon := \mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}1, \mathbf{l}]$ satisfies

a) $\|\mathbf{u} - \mathbf{P}\mathbf{u}_\varepsilon\|_{\ell_2(\mathcal{N})} \leq \varepsilon$;
b) $\#\operatorname{supp}(\mathbf{u}_\varepsilon) \lesssim \varepsilon^{-(1+\delta)/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s}$;
c) the number of algebraic operations needed to compute $\mathbf{u}_\varepsilon$ is $\lesssim \varepsilon^{-(1+\delta)/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s}$.

Proof For the proof of part a) see Theorem 5. Next, we show part b). Assume that $\{\mathbf{v}_i\}_{i\in\mathbb{N}_0}$ is the sequence of vectors generated in Algorithm 2. We want to show that $T(\#\operatorname{supp}(\mathbf{v}_i)) \lesssim \varepsilon_i^{-(1+\delta)/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s}$ for i large enough. Since $\mathbf{P}$ is bounded on $\mathcal{A}^t$ for all $t \in (0, \tilde{s}]$, it is also bounded on $\mathcal{A}^s$. Thus $\mathbf{P}\mathbf{u} \in \mathcal{A}^s$. Then for $\epsilon_j > 0$ there exists a finite vector $(\mathbf{P}\mathbf{u})_{\epsilon_j}$ such that $\|\mathbf{P}\mathbf{u} - (\mathbf{P}\mathbf{u})_{\epsilon_j}\|_{\ell_2(\mathcal{N})} \leq \frac{\theta}{6}\epsilon_j$ and $\#\operatorname{supp}((\mathbf{P}\mathbf{u})_{\epsilon_j}) \lesssim T(\#\operatorname{supp}((\mathbf{P}\mathbf{u})_{\epsilon_j})) \lesssim \epsilon_j^{-1/s}\|\mathbf{P}\mathbf{u}\|_{\mathcal{A}^s}^{1/s} \lesssim \epsilon_j^{-1/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{1/s}$. Therefore, by (49) we have

$$\begin{aligned} &\epsilon_j^{\tilde{\tilde{s}}/s-1}\|(\mathbf{P}\mathbf{u})_{\epsilon_j}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s-1}\|(\mathbf{P}\mathbf{u})_{\epsilon_j}\|_{\mathcal{A}^s} \\ &\lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s-1}\|\mathbf{P}\mathbf{u}\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s-1}\|\mathbf{u}\|_{\mathcal{A}^s} = \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s} \end{aligned} \tag{52}$$

for any $\tilde{\tilde{s}} > \tilde{s} > s > 0$. From (33) we get that

$$\left\|\mathbf{P}\mathbf{v}_i^{ex} - (\mathrm{id} - \mathbf{P})\mathbf{v}_i^{(j-1)} - \mathbf{v}_i^{(j,K)}\right\|_{\ell_2(\mathcal{N})} \leq \frac{2\theta\epsilon_j}{3} \tag{53}$$

with $\mathbf{v}_i^{ex} := \mathbf{H}(\mathbf{v}_{i-1})$. Due to (51) and (39), we obtain for any i large enough and some $\delta_0 > 0$ that

$$\begin{aligned}
&\|\mathbf{P}\mathbf{u} - \mathbf{P}\mathbf{v}_i^{ex}\|_{\ell_2(\mathcal{N})} \leq \|\mathbf{H}(\mathbf{u}) - \mathbf{H}(\mathbf{v}_{i-1})\|_{\ell_2(\mathcal{N})} \\
&\leq L\|\mathbf{P}\mathbf{u} - \mathbf{P}\mathbf{v}_{i-1}\|_{\ell_2(\mathcal{N})} \leq L\left(\frac{\varepsilon_0^{i-1}(\varepsilon_0 - L(L/\varepsilon_0)^{i-2})}{\varepsilon_0 - L} + L^{i-2}r^*\right) \\
&\leq \frac{L}{\varepsilon_0}\left(\frac{\varepsilon_0^{i}(\varepsilon_0 - L(L/\varepsilon_0)^{i-2})}{\varepsilon_0 - L} + L^{i-2}\varepsilon_0 r^*\right) \leq \varepsilon_0^i\left(\frac{L}{\varepsilon_0 - L} + \delta_0\right).
\end{aligned}$$

Due to the stopping criterion of Algorithm 1 we have for all i that for the last $j-$th iteration $\varepsilon_0^i \leq \frac{\theta}{3\rho^K}\epsilon_j$. Therefore, for all i large enough, due to (51) we get

$$\|\mathbf{P}\mathbf{u} - \mathbf{P}\mathbf{v}_i^{ex}\|_{\ell_2(\mathcal{N})} \leq \frac{\theta}{6}\epsilon_j,$$

and

$$\left\|\mathbf{P}\mathbf{u} - (\mathrm{id} - \mathbf{P})\mathbf{v}_i^{(j-1)} - \mathbf{v}_i^{(j,K)}\right\|_{\ell_2(\mathcal{N})} \leq \frac{\theta\epsilon_j}{6} + \frac{2\theta\epsilon_j}{3}, \tag{54}$$

which implies that

$$\left\|(\mathbf{P}\mathbf{u})_{\epsilon_j} - (\mathrm{id} - \mathbf{P})\mathbf{v}_i^{(j-1)} - \mathbf{v}_i^{(j,K)}\right\|_{\ell_2(\mathcal{N})} \leq \theta\epsilon_j. \tag{55}$$

Due to (A1), (55), and $\|\cdot\|_{\mathcal{A}^{\tilde{s}}}$ being a quasi-norm, it follows that $\mathbf{v}_i^{(j)} :=$ $\mathbf{COARSE}[(1-\theta)\epsilon_j, \mathbf{v}_i^{(j,K)}]$, for i large enough, satisfies

$$\begin{aligned}
\|\mathbf{v}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} &\leq C_2(s)\|(\mathbf{P}\mathbf{u})_{\epsilon_j} - (\mathrm{id} - \mathbf{P})\mathbf{v}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}} \\
&\leq C_1(s)C_2(s)\left(\|(\mathbf{P}\mathbf{u})_{\epsilon_j}\|_{\mathcal{A}^{\tilde{s}}} + \|\mathrm{id} - \mathbf{P}\|\|\mathbf{v}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}\right),
\end{aligned}$$

so by (52) and $\epsilon_j = 3\rho^K/\theta\epsilon_{j-1}$ (see Algorithm 1),

$$\left(\epsilon_j^{\tilde{\tilde{s}}/s-1}\|\mathbf{v}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}}\right) \leq C'\|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s} + C_1(s)C_2(s)\|\mathrm{id} - \mathbf{P}\|(3\rho^K/\theta)^{\tilde{\tilde{s}}/s-1}\left(\epsilon_{j-1}^{\tilde{\tilde{s}}/s-1}\|\mathbf{v}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}\right).$$

We can conclude that for $K > 0$ large enough, and by the assumption (50), the solutions of the homogeneous part of this recursion converge to zero, and so

$$\epsilon_j^{\tilde{\tilde{s}}/s-1}\|\mathbf{v}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s}, \tag{56}$$

uniformly with respect to j. And, due to (A1) we also have

$$\begin{aligned}
\#\,\mathrm{supp}(\mathbf{v}_i^{(j)}) &\lesssim T(\#\,\mathrm{supp}(\mathbf{v}_i^{(j)})) \\
&\lesssim \epsilon_j^{-1/\tilde{s}}\|(\mathbf{P}\mathbf{u})_{\epsilon_j} - (\mathrm{id} - \mathbf{P})\mathbf{v}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}} \\
&\lesssim \epsilon_j^{-\frac{\tilde{\tilde{s}}/\tilde{s}}{s}}\left(\epsilon_j^{\frac{\tilde{\tilde{s}}}{s}-1}\left[\|(\mathbf{P}\mathbf{u})_{\epsilon_j}\|_{\mathcal{A}^{\tilde{s}}} + \|\mathrm{id} - \mathbf{P}\|\|\mathbf{v}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}\right]\right)^{1/\tilde{s}}.
\end{aligned}$$

Thus, by (52) and (56) we get with $\tilde{\tilde{s}}/\tilde{s} = 1+\delta$ that

$$\#\operatorname{supp}(\mathbf{v}_i^{(j)}) \lesssim T(\#\operatorname{supp}(\mathbf{v}_i^{(j)})) \lesssim \epsilon_j^{-\frac{\tilde{\tilde{s}}/\tilde{s}}{s}} \|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{\tilde{\tilde{s}}/\tilde{s}}{s}} = \epsilon_j^{-\frac{1+\delta}{s}} \|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}. \tag{57}$$

Next, recall that $0 < s < \tilde{\tilde{s}}$ and by Algorithm 1 we have $\varepsilon_i \lesssim \epsilon_j$ for all j. Then (56) implies that for all j and i large enough we have $\varepsilon_i^{\tilde{\tilde{s}}/s-1}\|\mathbf{v}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s}$. In particular, for i large enough,

$$\varepsilon_i^{\tilde{\tilde{s}}/s-1}\|\mathbf{v}_i\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s}. \tag{58}$$

By the same argument, (57) yields for i large enough

$$\#\operatorname{supp}(\mathbf{v}_i) \lesssim T(\#\operatorname{supp}(\mathbf{v}_i)) \lesssim \varepsilon_i^{-\frac{1+\delta}{s}} \|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}. \tag{59}$$

Note that the stopping criterion in Algorithm 2 implies that for i large enough we have $\varepsilon \lesssim \varepsilon_i$. Therefore, from (59) we also get that

$$\#\operatorname{supp}(\mathbf{u}_\varepsilon) \lesssim T(\#\operatorname{supp}(\mathbf{u}_\varepsilon)) \lesssim \varepsilon^{-\frac{1+\delta}{s}} \|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}. \tag{60}$$

To prove part c), since ε_i decreases geometrically, it is sufficient to show that the number of algebraic operations needed for each iterations, i.e., for the computation of $\mathbf{v}_i := \mathbf{SOLVE}[\varepsilon_i, \mathbf{A}, \mathbf{l} - \mathbf{A}1(\mathbf{v}_{i-1})]$, is $\lesssim \varepsilon_i^{-(1+\delta)/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s}$ for all i large enough. Due to assumption (A3) we have $T(\#\operatorname{supp}(\mathbf{g}_{i-1}^{(j)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$ and $\|\mathbf{g}_{i-1}^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$. Therefore by assumption (A2) we have $T(\#\operatorname{supp}(\mathbf{f}_{i-1}^{(j)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$ and $\|\mathbf{f}_{i-1}^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$. By induction assumption we have that $\|\mathbf{v}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$ and $T(\#\operatorname{supp}(\mathbf{v}_i^{(j-1)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$, that are trivially valid for $j=1$. Therefore, again by (A2) we have

$$\|\mathbf{v}_i^{(j,k)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}), \tag{61}$$

and

$$T(\#\operatorname{supp}(\mathbf{v}_i^{(j,k)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}, \tag{62}$$

for all $0 \le k \le K$. Therefore, by (A2) and (A3), the number of algebraic operations required to compute $\mathbf{v}_i^{(j,K)}$ is $\lesssim \epsilon_j^{-1/\tilde{s}}(1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}} + T(\#\operatorname{supp}(\mathbf{v}_{i-1}))$. Finally by assumption (A1) the application of $\mathbf{v}_i^{(j)} := \mathbf{COARSE}[(1-\theta)\epsilon_j, \mathbf{v}_i^{(j,K)}]$ costs $\lesssim T(\#\operatorname{supp}(\mathbf{v}_i^{(j,K)})) + o(\epsilon_j^{-1/\tilde{s}}\|\mathbf{v}_i^{(j,K)}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}) \lesssim \epsilon_j^{-1/\tilde{s}}(1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}}$. Thus, by (A1), (61) and (62) the induction hypothesis holds, i.e. $\|\mathbf{v}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$ and $T(\#\operatorname{supp}(\mathbf{v}_i^{(j)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$ for any $j \in \mathbb{N}_0$. This implies by induction that computing $\mathbf{v}_i^{(j)}$ from $\mathbf{v}_i^{(j-1)}$ takes a number of operations $\lesssim \epsilon_j^{-1/\tilde{s}}(1+\|\mathbf{v}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}} + T(\#\operatorname{supp}(\mathbf{v}_{i-1}))$. Since ϵ_j decreases geometrically and by formulas (58) and (59), the cost of the computation of $\mathbf{v}_i := \mathbf{SOLVE}[\varepsilon_i, \mathbf{A}, \mathbf{l} - \mathbf{A}1(\mathbf{v}_{i-1})]$ is a multiple of $\varepsilon_i^{-\frac{1+\delta}{s}}\|\mathbf{u}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}$.

Remark 7 1. The boundedness of $\mathbf{P}$ on $\mathcal{A}^t_{weak}$, $t \in (0, \tilde{s})$, has been verified numerically in [44,13] in case of wavelet frame discretization, by observing the optimal convergence of **SOLVE**. According to [35, Remark 3.13], the boundedness of $\mathbf{P}$ on $\mathcal{A}^t_{weak}$ for all $t \in (0, \tilde{s})$ is (almost) a necessary requirement for the scheme to behave optimally. There exist frames, for example time–frequency localized Gabor frames (and more generally all intrinsically polynomially localized frames [23,11]), for which the boundedness of the corresponding $\mathbf{P}$ has been proven rigorously, see [11, Theorem 7.1 in Section 7]. Therefore, for certain operator equations (for example, certain pseudodifferential equations appearing, e.g., in wireless communication modelling [26]) the optimal application of **SOLVE** based on Gabor frame discretizations is justified theoretically.

One can avoid requiring the boundedness of $\boldsymbol{P}$ and obtain a fully optimal scheme replacing **SOLVE** by its modified version **modSOLVE** (see [35]). We assume then, without loss of generality, that the number of algebraic operations needed to compute $\mathbf{g}_i^{(j)}$ in (A3) is bounded by $\lesssim \epsilon_j^{-1/s} \|\mathbf{v}_i\|^{1/s}_{\mathcal{A}^s}$. However the procedure **modSOLVE** requires the definition of an implementable alternative projector $\tilde{\boldsymbol{P}}$ onto $\mathrm{ran}(\mathbf{A})$ and it is possible if a suitable wavelet frame expansion is constructed.
If $\mathcal{F}$ is a Riesz basis, then the scheme is certainly optimal and our result confirms the one in [8, Theorem 7.5].

2. Due to Theorem 4, (51) holds if the physical parameters of the problem allow for choosing L, ε_0 and r^* such that $0 < L =: r^*\gamma < \varepsilon_0 < r^* < \gamma^{-1}$ for $\gamma = \gamma(\mathcal{F}, A, A_1) := 2\|\mathbf{A}^{-1}|_{\mathrm{ran}(\mathbf{A})}\|\|A_1\|\|F\|^3$. In particular, if $0 < \gamma < 1$ is small enough then (51) is satisfied. For example, for the magnetohydrodynamics problem in [5] and $\mathcal{F}$ being a Riesz basis, the dependence of γ on the viscosity η and the electric resistivity σ^{-1} of the fluid can be expressed explicitly. Note that

$$\langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle = \langle AF^* D_{\mathbf{V}}^{-1}\mathbf{u}, F^* D_{\mathbf{V}}^{-1}\mathbf{u} \rangle \geq \alpha \| \sum_n (D_{\mathbf{V}}^{-1}\mathbf{u})_n f_n \|_{\mathbf{V}} \sim \alpha \|\mathbf{u}\|_{\ell_2(\mathcal{N})}.$$

Therefore, if $\mathcal{F}$ is a Riesz basis, then $\mathrm{ran}(\mathbf{A}) = \ell_2(\mathcal{N})$ and $\|\mathbf{A}|^{-1}_{\mathrm{ran}(\mathbf{A})}\| = \|\mathbf{A}^{-1}\| \lesssim \alpha^{-1}$ with $\alpha \geq c(\Omega) \min\{\eta, \sigma^{-1}\} > 0$. This implies that, if η, σ^{-1} are large enough, then $\gamma < 1$ can be made sufficiently small. The norm equivalence $\| \sum_{n\in\mathcal{N}} (D_{\mathbf{V}}^{-1}\mathbf{u})_n f_n \|_{\mathbf{V}} \sim \|D_{\mathbf{V}}^{-1}\mathbf{u}\|_{\mathbf{V}_d}$ is valid only if $\mathcal{F}$ is a Riesz basis and it does not hold for pure frames.

5 Construction of Aggregated Wavelet Frames

To justify the theoretical assumptions of the previous sections and to emphasize the flexibility of frames, we present in this section a construction of suitable multiscale divergence-free frames on a bounded domain Ω, which are not necessarily affine images of $(0,1)^n$. Before we proceed with our frame construction, let us investigate the properties of the functions in the solution space $\mathbf{V}$. In case of fluid dynamics, the solution space, for example, is given by

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \int_\Omega (\nabla \cdot \mathbf{v}) q = 0 \text{ for all } q \in \dot{L}_2(\Omega)\}.$$

Due to

$$\int_\Omega (\nabla \cdot \mathbf{v}) q = 0 \quad \text{for all} \quad q \in \dot{L}_2(\Omega),$$

and [5, Lemma 1.a)] we get that the fluid velocity $\mathbf{v} \in \mathbf{V}(\mathrm{div}; \Omega) \cap \mathbf{H}_0^1(\Omega)$, where

$$\mathbf{V}(\mathrm{div}; \Omega) := \{\mathbf{v} \in \boldsymbol{L}_2(\Omega) : \ \nabla \cdot \mathbf{v} = 0\}.$$

A detailed discussion on constructions of wavelet bases for such $\mathbf{V}$ on Ω, which is an affine image of $(0,1)^n$, is given, for example, in [40–42]. For later convenience, in the notation of [42, (2.26)], we define $\mathbf{V}_0(\mathrm{div}; \Omega) := \mathbf{H}_0(\mathrm{div}; \Omega) \cap \mathbf{V}(\mathrm{div}; \Omega)$, where

$$\mathbf{H}_0(\mathrm{div}; \Omega) = \{\boldsymbol{K} \in \boldsymbol{L}_2(\Omega) : \ \nabla \cdot \boldsymbol{K} \in \boldsymbol{L}_2(\Omega), \ \boldsymbol{K} \cdot \mathbf{n}|_\Gamma = 0\}.$$

See [24, Theorem 2.6] for a characterization of the latter space.

In the following we present some results on constructing pure frames for $\mathbf{V}$ on more general domains Ω. To do that we modify the construction of pure frames for $\mathbf{H}^s(\Omega)$ given in [11] and based on the ODD (Overlapping Domain Decomposition) technique: We assume that $\{\Omega_i\}_{i=1}^M$, $M \in \mathbb{N}$, are overlapping subdomains such that $\Omega = \cup_{i=1}^M \Omega_i$. Such subdomains are assumed to be affine images of the reference domain $\square := (0,1)^n$ (see Figure 1). Moreover, we assume that for each $1 \le i \le M$ a divergence–free

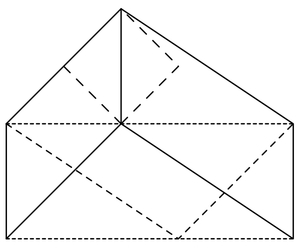

Fig. 1 Example of an Overlapping Domain Decomposition of a polygonal domain in 2D by means of patches which are affine images of $(0,1)^n$.

wavelet basis $\boldsymbol{\Psi}_i := \{\psi_{j,k}^i\}_{j \ge -1, k \in \mathcal{J}_j^i}$ is given for $\mathbf{V}(\mathrm{div}; \Omega_i) \cap \mathbf{H}_0^1(\Omega_i)$. We show that $\boldsymbol{\Psi} := \cup_{i=1}^M \boldsymbol{\Psi}_i = \{\psi_{j,k}^i\}_{j \ge -1, k \in \mathcal{J}_j^i, i=1,\ldots,M}$ is a Gelfand frame for $\mathbf{V}(\mathrm{div}; \Omega) \cap \mathbf{H}_0^1(\Omega)$.

These systems, called *aggregated divergence–free wavelet frames*, allow us to avoid dealing with interfacing patches used in disjoint domain decompositions (DDD) (see [17, 42]). DDD are usually rather complicated to implement and can yield ill–conditioned systems (see [42, p. 104, sec. More General Domains]).

Assume that $\mathcal{C} := \{\Omega_i\}_{i=1}^M$ is an overlapping, relatively compact covering of Ω such that

(C1) there exist affine (or even conformal if $n = 2$) maps $\kappa_i : \Box \to \Omega_i$, for all $i = 1, \ldots, M$.

The set of admissible domains Ω is restricted by raising condition (C1), e.g., the boundary of Ω has to be piecewise smooth. Nevertheless, the particularly attractive case of polyhedral domains is still covered.

We assume that the wavelets we use below have sufficient regularity and number of vanishing moments. We consider a template vector–valued wavelet divergence–free basis $\mathbf{\Psi}^\Box = \{\psi_{j,k}^\Box\}_{j\geq j_0, k\in\mathcal{J}_j^\Box}$ for $\mathbf{V}_\Box^s := \mathbf{V}(\mathrm{div};\Box) \cap \mathbf{H}_0^s(\Box)$, $s \geq 0$, and its biorthogonal dual $\tilde{\mathbf{\Psi}}^\Box = \{\tilde{\psi}_{j,k}^\Box\}_{j\geq j_0, k\in\mathcal{J}_j^\Box}$. It holds that $\mathbf{\Psi}^\Box$ is a frame for $\mathbf{V}_\Box^s$. Our aim is to show that the system

$$\mathbf{\Psi} := (\psi_{j,k}^i)_{(i,j,k)\in\Lambda}, \tag{63}$$

where

$$\psi_{j,k}^i(x) := \frac{\nabla\kappa_i\big(\kappa_i^{-1}(x)\big)\cdot\psi_{j,k}^\Box\big(\kappa_i^{-1}(x)\big)}{\big|\det\nabla\kappa_i\big(\kappa_i^{-1}(x)\big)\big|^{1/2}}, \quad \begin{array}{l}\text{for all } i = 1,...,M,\ j\geq j_0,\\ k\in\mathcal{J}_j^\Box,\ x\in\Omega_i,\end{array} \tag{64}$$

and $\psi_{j,k}^i(x) = \mathbf{0}$ for $x \in \Omega\setminus\Omega_i$, is a frame for $\mathbf{V}^s := \mathbf{V}(\mathrm{div};\Omega)\cap\mathbf{H}_0^s(\Omega)$. Analogously, we define its local duals by

$$\tilde{\psi}_{j,k}^i(x) := \frac{\nabla\kappa_i\big(\kappa_i^{-1}(x)\big)\cdot\tilde{\psi}_{j,k}^\Box\big(\kappa_i^{-1}(x)\big)}{\big|\det\nabla\kappa_i\big(\kappa_i^{-1}(x)\big)\big|^{1/2}}, \quad \begin{array}{l}\text{for all } i = 1,...,M,\ j\geq j_0,\\ k\in\mathcal{J}_j^\Box,\ x\in\Omega_i,\end{array} \tag{65}$$

and $\tilde{\psi}_{j,k}^i(x) = \mathbf{0}$ for $x \in \Omega\setminus\Omega_i$.

First of all observe that under assumption (C1) each $\mathbf{\Psi}^i := (\psi_{j,k}^i)_{j\geq j_0, k\in\mathcal{J}_j^\Box}$ is again a divergence-free wavelet frame for $\mathbf{V}_i^s := \mathbf{V}(\mathrm{div};\Omega_i)\cap\mathbf{H}_0^s(\Omega_i)$. Provided suitable zero-extensions, the space $\mathbf{V}_i^s$ can be identified with a closed subspace of $\mathbf{V}^s$. Moreover, by [40–42] for $\mathbf{u}_i = \sum_{j,k} c_{j,k}\psi_{j,k}^i$ the following norm equivalence holds

$$\|\mathbf{u}_i\|_{\mathbf{H}^s} \sim \left(\sum_{j,k} 2^{2sj}|c_{j,k}|^2\right)^{1/2}, \quad s\geq 0. \tag{66}$$

Let the projections of $\boldsymbol{L}_2(\Omega)$ onto $\mathbf{V}_i := \mathbf{V}(\mathrm{div},\Omega_i)$ be

$$P_{\mathbf{V}_i}(\mathbf{u}) := \sum_{j\geq j_0, k\in\mathcal{J}_j^\Box} \langle\mathbf{u},\tilde{\psi}_{j,k}^i\rangle\psi_{j,k}^i, \quad i = 1,\ldots,M. \tag{67}$$

We next present the frame construction that allows us to work with Ω, which are not necessarily affine images of $(0,1)^n$, and provides the characterization of $\mathbf{V}^s$ for any $s \geq 1$. Although it may seem intuitively clear that the union of local bases $\mathbf{\Psi} := \cup_{i=1}^{M} \mathbf{\Psi}_i = \{\psi^i_{j,k}\}_{j\ge -1, k\in \mathcal{J}^i_j, i=1,\dots,M}$ forms a global Gelfand frame, currently we can prove the frame property for divergence–free wavelets only under the following technical condition:

We require that $\mathcal{C}$ is such that

(C2) there exists suitable bounded operators $\pi_{\mathbf{V}^s_i} : \mathbf{V}^s \to \mathbf{V}^s_i$ for which it holds

$$\pi_{\mathbf{V}^s_i}(\mathbf{v})|_{\Omega_i \setminus \cup_{k=2}^{\tilde M} \Omega_{i_k}} \equiv \mathbf{v}|_{\Omega_i \setminus \cup_{k=2}^{\tilde M} \Omega_{i_k}}, \tag{68}$$

for any $\mathbf{v} \in \mathbf{V}^s$ such that $\Omega_i \subseteq \mathrm{supp}(\mathbf{v}) \subseteq \cup_{k=1}^{\tilde M} \Omega_{i_k}$ for $\{i_1, ..., i_{\tilde M}\} \subseteq \{1, ..., M\}$, without loss of generality $i := i_1$. The definition of $\pi_{\mathbf{V}^s_i}$ may depend on $\{\Omega_{i_1}, ..., \Omega_{i_{\tilde M}}\}$ only.

Remark 8 Note that the domain decomposition satisfying (C2) is necessarily overlapping. If it were not the case, then it would hold that $\Omega_i = \Omega_i \setminus \cup_{\{1,\dots,M\}\setminus\{i\}} \Omega_j$ and, by (68), $\pi_{\mathbf{V}^s_i}\mathbf{v} = \mathbf{v}|_{\Omega_i}$. This, would imply that $\mathbf{v}|_{\partial\Omega_i} = \pi_{\mathbf{V}^s_i}(\mathbf{v})|_{\partial\Omega_i} = 0$, which is not necessarily the case for an arbitrary $\mathbf{v} \in \mathbf{V}^s$.

To illustrate our frame construction we consider $\Omega = \Omega_1 \cup \Omega_2$, which is not an affine image of $(0,1)^2$, as in Figure 2. In this case we can choose $\pi_{\mathbf{V}^s_1}\mathbf{v} = \sum_{j\ge j_0, k\in\mathcal{J}^{\square,\circ}_j} \langle \mathbf{v}, \tilde\psi^1_{j,k}\rangle \psi^1_{j,k}$ where we exclude from the index set $\mathcal{J}^{\square,\circ}_j \subset \mathcal{J}^{\square}_j$ those wavelets adapted to the internal boundary of Ω_1. Both conditions (C1)–(C2) are indeed satisfied. Therefore it holds $\pi_{\mathbf{V}^s_1}\mathbf{v} = \mathbf{v}$ on $\Omega \setminus \Omega_2$ and $\pi_{\mathbf{V}^s_1}\mathbf{v} \in \mathbf{V}^s$, $s \ge 1$ and $P_{\mathbf{V}_1}(\pi_{\mathbf{V}^s_1}\mathbf{v}) = \pi_{\mathbf{V}^s_1}\mathbf{v}$. This, implies that $\mathbf{v} - P_{\mathbf{V}_1}(\pi_{\mathbf{V}^s_1}\mathbf{v}) \in \mathbf{V}^s$. Note that by construction $\mathrm{supp}(\mathbf{v} - P_{\mathbf{V}_1}(\pi_{\mathbf{V}^s_1}\mathbf{v})) \subseteq \Omega_2$ and $(\mathbf{v} - P_{\mathbf{V}_1}(\pi_{\mathbf{V}^s_1}\mathbf{v}))|_{\partial\Omega_2} = 0$. Therefore, we obtain the following decomposition $\mathbf{v} = P_{\mathbf{V}_1}(\pi_{\mathbf{V}^s_1}\mathbf{v}) + P_{\mathbf{V}_2}(\mathbf{v} - P_{\mathbf{V}_1}(\pi_{\mathbf{V}^s_1}\mathbf{v}))$.

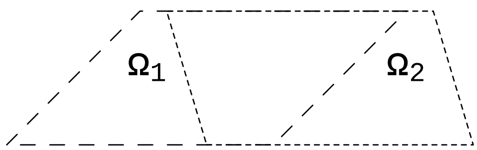

Fig. 2 An example of Overlapping Domain Decomposition satisfying (C1) and (C2).

The following theorem generalizes the argument above to domain decompositions consisting of more than two patches.

Theorem 7 *Let $\mathcal{C} := \{\Omega_i\}_{i=1}^M$ be an overlapping, relatively compact covering of Ω satisfying (C1)–(C2). Let $\Psi := (\psi^i_{j,k})_{(i,j,k)\in\Lambda}$ be as in* (63) *and $P_{\mathbf{V}_i}$ as in* (67). *For $\mathbf{u} \in \mathbf{V}^s$, $s \ge 1$, consider the functions and domains defined by the following recursion*

$$\mathbf{u}^{(0)} = \mathbf{u}, \quad \Omega^{(0)} = \Omega,$$

$$\begin{cases} \mathbf{u}_{n+1} = P_{\mathbf{V}_{n+1}}(\pi_{\mathbf{V}^s_{n+1}}\mathbf{u}^{(n)}) \\ \mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} - \mathbf{u}_{n+1} \\ \Omega^{(n+1)} = \mathrm{supp}(\mathbf{u}^{(n)}) \end{cases}, \quad n = 0, \dots, M-1. \tag{69}$$

Then

$$\mathbf{u} = \sum_{i=1}^{M} \mathbf{u}_i = \sum_{i=1}^{M} \sum_{j \geq j_0, k \in \mathcal{J}_j^{\square}} \langle \pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}, \tilde{\psi}_{j,k}^i \rangle \psi_{j,k}^i \tag{70}$$

with

$$\|\mathbf{u}\|_{\mathbf{H}^s} \sim \left(\sum_{i=1}^{M} \sum_{j \geq j_0, k \in \mathcal{J}_j^{\square}} 2^{2sj} |\langle \pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}, \tilde{\psi}_{j,k}^i \rangle|^2 \right)^{1/2} . \tag{71}$$

In particular, $\mathbf{\Psi}$ *is an* $\boldsymbol{L}_2$ *frame for* $\mathbf{V}_0(\mathrm{div};\Omega)$ *and a Gelfand frame for* $(\mathbf{V}^s, \mathbf{V}_0(\mathrm{div};\Omega), (\mathbf{V}^s)')$.

Proof By property (C2) and by induction we get $\mathbf{u}^{(n)} \in \mathbf{V}^s$ with $\|\mathbf{u}^{(n)}\|_{\mathbf{H}^s} \lesssim \|\mathbf{u}\|_{\mathbf{H}^s}$, $\mathbf{u}^{(n)}|_{\partial\Omega^{(n+1)}} = 0$ and $\Omega^{(n)} \subseteq \cup_{i=n+1}^{M} \Omega_i$. Thus, $\Omega^{(M)} = \emptyset$ and, therefore, in the recursive definition (69) we have $0 \leq n \leq M-1$. This implies also that $\mathrm{supp}(\mathbf{u}^{(M-1)}) \subseteq \Omega_M$ and $\mathbf{u}_M = P_{\mathbf{V}_M}(\pi_{\mathbf{V}_M^s} \mathbf{u}^{(M-1)}) = \mathbf{u}^{(M-1)}$. And by induction we get $\mathbf{u}^{(M-1)} = \mathbf{u} - \sum_{i=1}^{M-1} \mathbf{u}_i$. This implies $\mathbf{u} = \sum_{i=1}^{M-1} \mathbf{u}_i + \left(\mathbf{u} - \sum_{i=1}^{M-1} \mathbf{u}_i\right) = \sum_{i=1}^{M-1} \mathbf{u}_i + \mathbf{u}^{(M-1)} = \sum_{i=1}^{M} \mathbf{u}_i$, and, consequently, the wavelet decomposition of $\mathbf{u}$ is given by

$$\mathbf{u} = \sum_{i=1}^{M} \mathbf{u}_i = \sum_{i=1}^{M} P_{\mathbf{V}_i}(\pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}) = \sum_{i=1}^{M} \sum_{j \geq j_0, k \in \mathcal{J}_j^{\square}} \langle \pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}, \tilde{\psi}_{j,k}^i \rangle \psi_{j,k}^i .$$

By property (66) we have

$$\|\mathbf{u}\|_{\mathbf{H}^s} \lesssim \left(\sum_{i=1}^{M} \sum_{j \geq j_0, k \in \mathcal{J}_j^{\square}} 2^{2sj} |\langle \pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}, \tilde{\psi}_{j,k}^i \rangle|^2 \right)^{1/2} .$$

Conversely,

$$\begin{aligned} \left(\sum_{i=1}^{M} \sum_{j \geq j_0, k \in \mathcal{J}_j^{\square}} 2^{2sj} |\langle \pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}, \tilde{\psi}_{j,k}^i \rangle|^2 \right)^{1/2} &\lesssim \left(\sum_{i=1}^{M} \|\pi_{\mathbf{V}_i^s} \mathbf{u}^{(i-1)}\|_{\mathbf{H}^s}^2 \right)^{1/2} \\ &\lesssim \|\mathbf{u}\|_{\mathbf{H}^s} . \end{aligned}$$

Note that the norm equivalences above hold for also for $s = 0$ and, thus, by density of $\mathbf{V}^s$ in $\mathbf{V}_0(\mathrm{div};\Omega)$, $\mathbf{\Psi}$ is an $\boldsymbol{L}_2$ frame for $\mathbf{V}_0(\mathrm{div};\Omega)$ and a Gelfand frame for $(\mathbf{V}^s, \mathbf{V}_0(\mathrm{div};\Omega), (\mathbf{V}^s)')$.

Remark 9 1. The choice of the decomposition $\mathcal{C}$ is restricted by the technical condition (C2). The existence of operators $\pi_{\mathbf{V}_i^s}$ satisfying (C2) is crucial for showing the frame property for $\mathbf{\Psi} = (\psi_{j,k}^i)_{(i,j,k)\in\Lambda}$. Condition (C2) can indeed be verified, at least for $s = 1$ and for any overlapping decomposition constituted by Lipschitz subdomains. We construct maps $\pi_{\mathbf{V}_i^1}$ using the following result in [31] on suitable extensions of divergence-free functions with preassigned boundary conditions, compare also [28, Corollary 3.2].

Theorem 8 *Assume that D is a relatively compact Lipschitz domain in $\mathbb{R}^n$ which is the union of two non-overlapping relatively compact Lipschitz subdomains, i.e., $\overline{D} = \overline{D}_1 \cup \overline{D}_2$ and $D_1 \cap D_2 = \emptyset$. For all $\mathbf{v} \in \mathbf{V}(\mathrm{div}; D) \cap \mathbf{H}^1(D)$ such that $\mathbf{v}_{|\partial D_1 \cap \partial D} = 0$ (of course $\partial D_1 \cap \partial D$ can be empty), there exists $\tilde{\mathbf{v}} \in \mathbf{V}(\mathrm{div}; D) \cap \mathbf{H}^1_0(D)$ such that $\tilde{\mathbf{v}}_{|D_1} \equiv \mathbf{v}_{|D_1}$ and $\|\tilde{\mathbf{v}}\|_{\mathbf{H}^1} \lesssim \|\mathbf{v}\|_{\mathbf{H}^1}$.*

Assume $\mathbf{v} \in \mathbf{V}^1$ such that $\Omega_i \subseteq \mathrm{supp}(\mathbf{v}) \subseteq \cup_{k=1}^{\tilde{M}} \Omega_{i_k}$. Let us denote $D_1 = (\Omega_i \setminus \cup_{k=2}^{\tilde{M}} \Omega_{i_k})$ and $D_2 = (\Omega_i \cap (\cup_{k=2}^{\tilde{M}} \Omega_{i_k}))$, we have $\overline{D} = \overline{D}_1 \cup \overline{D}_2 = \overline{\Omega}_i$. Since $\mathbf{v}|_{\Omega_i} \in \mathbf{V}(\mathrm{div}; \Omega_i) \cap \mathbf{H}^1(\Omega_i)$, by Theorem 8 there exists $\tilde{\mathbf{v}} \in \mathbf{V}^1_i$ such that $\tilde{\mathbf{v}}|_{\Omega_i \setminus \cup_{k=2}^{\tilde{M}} \Omega_{i_k}} \equiv \mathbf{v}|_{\Omega_i \setminus \cup_{k=2}^{\tilde{M}} \Omega_{i_k}}$. Thus, we define $\pi_{\mathbf{V}^1_i}(\mathbf{v}) = \tilde{\mathbf{v}}$.

2. In [11,13,35] constructions of wavelet frames in $H^s_0(\Omega)$ are done by pointwise multiplication with partitions of unity. Unfortunately if $\mathbf{u} \in \mathbf{V}^s$ and γ is a smooth function supported in Ω_i, it is not true in general that $\gamma \mathbf{u} \in \mathbf{V}^s_i$. For this reason the arguments in [11,13,35] cannot be extended to the case of *divergence-free* wavelets.

3. The aggregate wavelet frames constructed in Theorem 7 are multiscale and produce coefficients that can be naturally structured into suitable trees, as the ones in [16,17]. Thus, the algorithms defined in [9], see (A3), for the evaluation of nonlinear functionals can also be applied in our case. The optimality also requires the characterization of Besov spaces in terms of norm equivalences. As soon as the local divergence–free bases on $\square$ provides such a characterization and provided the operator $\pi_{\mathbf{V}^s_i}$ are also bounded on Besov spaces, we can show similarly to Theorem 7 that the global divergence–free frame characterizes Besov spaces on the whole domain. We postpone the details to a forthcoming paper.

4. In case of magnetohydrodynamics the solution space is given by

$$\mathbf{V} := \{(\mathbf{v}, \boldsymbol{K}) \in \mathbf{H}^1_0(\Omega) \times \boldsymbol{L}_2(\Omega) : \int_\Omega (\nabla \cdot \mathbf{v}) q + \int_\Omega \boldsymbol{K} \cdot (\nabla \psi) = 0 \\ \text{for all } (q, \psi) \in \dot{L}_2(\Omega) \times \dot{H}^1(\Omega)\}.$$

Using the results in [24] we conclude that, as in case of fluid dynamics, the fluid velocity $\mathbf{v} \in \mathbf{V}(\mathrm{div}; \Omega) \cap \mathbf{H}^1_0(\Omega)$. The electric current $\boldsymbol{K} \in \mathbf{V}_0(\mathrm{div}; \Omega)$. The construction of frames for $\mathbf{V}$ follows again from Theorem 7.

6 Conclusion

We present the first *convergent and implementable adaptive scheme* based on *frame decompositions* for the numerical integration of nonlinear variational problems. Certainly, it is impossible to construct a universal scheme suitable for any physical problem. Thus, the convergence of the algorithm is ensured under certain assumptions on its physical parameters. Such assumptions are standard when dealing with nonlinear problems of fluid dynamics and magnetohydrodynamics and still lead to the analysis of physically realistic problems, see [30].

Divergence–free vector valued functions are crucial for applications in fluid dynamics and magnetohydrodynamics. The discretization of the problem is, therefore, realized by expanding the solution with respect to certain divergence–free wavelet frames constructed on Overlapping Domain Decompositions. The construction based on ODD is definitively an improvement in comparison with [40–42]. There divergence–free wavelet bases have been defined essentially only for domains that are affine images of cubes. Our construction of frames allows for more general polygonal domains and avoids the use of fictitious domain techniques, see, e.g., [14]. Of importance is that these aggregate divergence–free wavelet frames preserve multiscale properties of standard wavelet bases and their capability to characterize certain function spaces. These properties ensure the optimal convergence rates and complexity of the scheme we propose, if the solutions are in certain sparseness classes of functions. It has not yet been theoretically proven that the solutions belong to such function spaces (e.g., Besov spaces), numerical simulations [1] though support this assumption.

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