On the Cauchy Problem for the Boltzmann Equation in the Whole Space: Global Existence and Uniform Stability in $L^2_\xi(H^N_x)$

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Abstract

Based on a refined energy method, in this paper we prove the global existence and uniform-in-time stability of the solution in the space $L^2_\xi(H^N_x)$ to the Cauchy problem for the Boltzmann equation around a global Maxwellian in the whole space $\mathbb{R}^3$. Compared with the solution space used by the spectral analysis and the classical energy method, the velocity weight functions or time derivatives need not be included in the norms of $L^2_\xi(H^N_x)$, which is realized by introducing some temporal interactive energy functionals to estimate the macroscopic dissipation rate. The key proof is carried out in terms of the macroscopic equations together with the local conservation laws. It is also found that the perturbed macroscopic variables actually satisfy the linearized compressible Navier-Stokes equations with remaining terms only related to the microscopic part.

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Contents

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1 Introduction

The Boltzmann equation for the hard-sphere monatomic gas in the whole space $\mathbb{R}^3$ takes the form

$$\partial_t f + \xi \cdot \nabla_x f = Q(f,f). \tag{1.1}$$

Here, the unknown $f = f(t,x,\xi)$ is a non-negative function standing for the number density of gas particles which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t > 0$. Q is the bilinear collision operator defined by

$$\begin{aligned} Q(f,g) &= \int_{\mathbb{R}^3 \times S^2} (f' g'_* - f g_*) |(\xi - \xi_*) \cdot \omega| d\omega d\xi_*, \\ f = f(t,x,\xi), \quad f' &= f(t,x,\xi'), \quad g_* = g(t,x,\xi_*), \quad g'_* = g(t,x,\xi'_*), \\ \xi' = \xi - [(\xi - \xi_*)\cdot\omega]\omega, &\quad \xi'_* = \xi_* + [(\xi - \xi_*)\cdot\omega]\omega, \quad \omega \in S^2. \end{aligned}$$

We define the perturbation $u = u(t,x,\xi)$ by

$$f = \mathbf{M} + \sqrt{\mathbf{M}} u, \tag{1.2}$$

where the global Maxwellian

$$\mathbf{M} = \frac{1}{(2\pi)^{3/2}} \exp\left(-|\xi|^2/2\right)$$

is normalized to have zero bulk velocity and unit density and temperature. Then the equation for the perturbation u reads

$$\partial_t u + \xi \cdot \nabla_x u = \mathbf{L} u + \Gamma(u,u), \tag{1.3}$$

where

$$\begin{aligned} \mathbf{L} u &= \frac{1}{\sqrt{\mathbf{M}}} \left[Q(\mathbf{M}, \sqrt{\mathbf{M}} u) + Q(\sqrt{\mathbf{M}} u, \mathbf{M}) \right], \\ \Gamma(u,u) &= \frac{1}{\sqrt{\mathbf{M}}} Q(\sqrt{\mathbf{M}} u, \sqrt{\mathbf{M}} u). \end{aligned}$$

It is well-known that for the linearized collision operator $\mathbf{L}$, one has

$$\begin{aligned} (\mathbf{L} u)(\xi) &= -\nu(\xi) u(\xi) + (K u)(\xi), \\ \nu(\xi) &= \int_{\mathbb{R}^3 \times S^2} |(\xi - \xi_*) \cdot \omega| \mathbf{M}_* \, d\omega d\xi_*, \\ (K u)(\xi) &= \int_{\mathbb{R}^3 \times S^2} \left(-\sqrt{\mathbf{M}} u_* + \sqrt{\mathbf{M}'_*} u' + \sqrt{\mathbf{M}'} u'_* \right) |(\xi - \xi_*) \cdot \omega| \sqrt{\mathbf{M}_*} d\omega d\xi_* \\ &= \int_{\mathbb{R}^n} K(\xi, \xi_*) u(\xi_*) d\xi_*, \end{aligned}$$

where $\nu(\xi)$ is called the collision frequency and K is a self-adjoint compact operator on $L^2(\mathbb{R}^3_\xi)$ with a real symmetric integral kernel $K(\xi, \xi_*)$. The nullspace of the operator $\mathbf{L}$ is the five dimensional space of collision invariants

$$\mathcal{N} = Ker \mathbf{L} = \text{span} \left\{ \sqrt{\mathbf{M}};\ \xi_i \sqrt{\mathbf{M}}, i = 1,2,3;\ |\xi|^2 \sqrt{\mathbf{M}} \right\}. \tag{1.4}$$

From the Boltzmann's H-theorem, the linearized collision operator $\mathbf{L}$ is non-positive and furthermore, $-\mathbf{L}$ is locally coercive in the sense that there is a constant $\lambda > 0$ such that

$$-\int_{\mathbb{R}^3} u\mathbf{L}u\,d\xi \geq \lambda \int_{\mathbb{R}^3} \nu(\xi)\left(\{\mathbf{I}-\mathbf{P}\}u\right)^2 d\xi, \quad \forall\, u \in D(\mathbf{L}), \tag{1.5}$$

where for fixed (t,x), $\mathbf{P}$ denotes the velocity projection operator from $L^2(\mathbb{R}^3_\xi)$ to $\mathcal{N}$ and $D(\mathbf{L})$ is the domain of $\mathbf{L}$ given by

$$D(\mathbf{L}) = \left\{ u \in L^2(\mathbb{R}^3_\xi) \,\middle|\, \nu(\xi)u \in L^2(\mathbb{R}^3_\xi) \right\}.$$

Let's introduce some notations for the presentation throughout this paper. We use $\langle\cdot,\cdot\rangle$ to denote the inner product in the Hilbert space $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ or $L^2(\mathbb{R}^3_x)$ or $L^2(\mathbb{R}^3_\xi)$, and $\|\cdot\|$ to denote the corresponding L^2 norm. We also define

$$\langle u, v\rangle_\nu \equiv \langle \nu(\xi)u, v\rangle$$

for any functions $u = u(x,\xi)$ and $v = v(x,\xi)$ to be the weighted inner product in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$, and use $\|\cdot\|_\nu$ for the corresponding weighted L^2 norm. For the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, \quad \text{and} \quad |\alpha| = \sum_{i=1}^{3} \alpha_i.$$

For simplicity, we also use ∂_i to denote ∂_{x_i} for each $i = 1, 2, 3$. In addition, C always denotes a general constant and if the dependence need be specified, then the notations C_i, $i = 1, 2, \cdots$ are used. We define the temporal energy functional as

$$[[u(t)]]^2 \equiv \sum_{|\alpha|\leq N} \|\partial_x^\alpha u(t)\|^2,$$

and the dissipation rate as

$$[[u(t)]]_\nu^2 \equiv \|\{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2 + \sum_{0<|\alpha|\leq N} \|\partial_x^\alpha u(t)\|_\nu^2,$$

where $N \geq 4$ is an integer. For fixed t, we call

$$u(t) \in L^2_\xi(H^N_x) \equiv L^2(\mathbb{R}^3_\xi; H^N(\mathbb{R}^3_x))$$

if $[[u(t)]]$ is bounded. Notice that these norms as above include only the spatial derivatives but not the time or velocity derivatives.

Our main results about the global existence and the uniform stability are stated as follows.

Theorem 1.1. *Let $f_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \geq 0$. There exist constants $\delta_0 > 0$, $\lambda_0 > 0$ and $C_0 > 0$ such that if $[[u(0)]] \leq \delta_0$, then there exists a unique global solution $u(t,x,\xi)$ to* (1.3) *such that $f(t,x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u(t,x,\xi) \geq 0$, and*

$$[[u(t)]]^2 + \lambda_0 \int_0^t [[u(s)]]_\nu^2 ds \leq C_0[[u(0)]]^2, \tag{1.6}$$

for any $t \geq 0$.

Theorem 1.2. *Let* $f_0(x,\xi)=\mathbf{M}+\sqrt{\mathbf{M}}u_0(x,\xi)\geq 0$ *and* $g_0(x,\xi)=\mathbf{M}+\sqrt{\mathbf{M}}v_0(x,\xi)\geq 0$. *There exist constants* $\delta_1\in(0,\delta_0)$, $\lambda_1>0$ *and* $C_1>0$ *such that if*

$$\max\{[[u(0)]],[[v(0)]]\}\leq\delta_1,$$

then the solutions $u(t,x,\xi)$, $v(t,x,\xi)$ *obtained in Theorem 1.1 satisfy*

$$[[u(t)-v(t)]]^2+\lambda_1\int_0^t[[u(s)-v(s)]]_\nu^2ds\leq C_1[[u(0)-v(0)]]^2, \tag{1.7}$$

for any $t\geq 0$.

The global existence near Maxwellian as in Theorem 1.1 has been already shown in some other function spaces. The first global existence theorem was established in the space

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi;H^k(\mathbb{R}^3_x)),\quad \beta_1>\frac{5}{2},\ k\geq 2, \tag{1.8}$$

by using the spectral analysis [27, 33, 34], where

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi)\equiv\{u|(1+|\xi|)^{\beta_1}u\in L^\infty(\mathbb{R}^3_\xi)\}.$$

The same result was obtained in [28] for the torus case with the space

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi;C^k(\mathbb{T}^3_x)),\quad \beta_1>\frac{5}{2},\ k=0,1,\cdots. \tag{1.9}$$

Recently, [35] presented a function space

$$L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)\cap L^\infty_{\beta_2}(\mathbb{R}^3_\xi;L^\infty(\mathbb{R}^3_x)),\quad \beta_2>\frac{3}{2}, \tag{1.10}$$

in which the Cauchy problem is globally well-posed in a mild sense without any regularity conditions. Notice that if the spatial regularity is neglected for the moment, the solution space $L^2_\xi(H^N_x)$ in Theorem 1.1 is larger than (1.8), (1.9) and (1.10) in the sense that the velocity integrability in $L^2_\xi(H^N_x)$ is the lowest among them since the following strict inclusion relations hold

$$L^\infty_{\beta_1}(\mathbb{R}^3_\xi)\subsetneqq L^\infty_{\beta_2}(\mathbb{R}^3_\xi)\subsetneqq L^2(\mathbb{R}^3_\xi),$$

where it has been supposed that β_1 and β_2 are sufficiently close to $\frac{5}{2}$ and $\frac{3}{2}$, respectively. On the other hand, by means of the classical energy method [13, 24, 22], the well-posedness was also established in the Sobolev space

$$H^{N(n_1,n_2,n_3)}_{t,x,\xi}(\mathbb{R}^3_x\times\mathbb{R}^3_\xi), \tag{1.11}$$

which denotes a set of all functions whose derivatives of all variables t, x and ξ up to N order are integrable in $L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$, where

$$N=N(n_1,n_2,n_3)\equiv n_1+n_2+n_3\geq 4.$$

In particular, for the case without any external force, n_3 can be taken as zero, which means that the velocity derivatives need not be considered [13, 22, 24, 38], whereas they have to be included for the case with forcing [6, 7, 15, 16, 29, 36]. Compared with (1.11), the solution space $L^2_\xi(H^N_x)$ in Theorem 1.1 is again in the weak form

$$L^2_\xi(H^N_x)=H^{N(0,N,0)}_{t,x,\xi}(\mathbb{R}^3_x\times\mathbb{R}^3_\xi),$$

with $n_1 = n_3 = 0$. Finally, as pointed out in [35], it is a challenging problem to seek for other larger spaces such that the Cauchy problem becomes well-posed near Maxwellian, which is also the main motivation of this paper.

In Theorem 1.1 the construction of the global solution is based on the nonlinear energy method developed in [12, 13, 14, 15, 16]. In particular, for the Boltzmann equation in the whole space, [13] obtained the global existence in terms of the energy functional

$$|||u(t)|||^2 \equiv \sum_{\alpha_0+|\alpha|\leq N} \|\partial_t^{\alpha_0}\partial_x^{\alpha}u(t)\|^2, \tag{1.12}$$

and the dissipation rate

$$|||u(t)|||_\nu^2 \equiv \|\{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2 + \sum_{0<\alpha_0+|\alpha|\leq N} \|\partial_t^{\alpha_0}\partial_x^{\alpha}u(t)\|_\nu^2, \tag{1.13}$$

in which the time derivatives have to be included for the proof. In this case, the initial perturbation $u_0(x,\xi)$ could have an algebraic decay in the velocity ξ because of boundedness of the initial energy

$$|||u(0)|||^2 \equiv \sum_{\alpha_0+|\alpha|\leq N} \|\partial_t^{\alpha_0}\partial_x^{\alpha}u(0)\|^2,$$

where the time derivatives of $u_0(x,\xi)$ are naturally defined through the equation (1.3), for example, the first order time derivative is given by

$$\partial_t u(0) = -\xi\cdot\nabla_x u_0 + (-\nu(\xi)+K)u_0 + \Gamma(u_0,u_0).$$

Theorem 1.1 shows that one can remove the time derivatives from norms (1.12) and (1.13). Thus the well-posedness in the Sobolev space $L^2_\xi(H^N_x)$ for the Cauchy problem of the nonlinear Boltzmann equation in the whole space can be established only in terms of the energy method.

The proof of Theorem 1.1 is motivated by [13], but some technical modifications are needed in order to illuminate the time derivatives. Notice that the dissipation rate $[[u(t)]]_\nu^2$ is equivalent with the sum of the microscopic dissipation rate

$$\sum_{|\alpha|\leq N} \|\partial_x^{\alpha}\{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2$$

and the macroscopic dissipation rate

$$\sum_{0<|\alpha|\leq N} \|\partial_x^{\alpha}\mathbf{P}u(t)\|^2.$$

The microscopic dissipation rate is easily obtained by the coercive property (1.5) for $-\mathbf{L}$. In terms of the coefficients a,b,c of $\mathbf{P}u$ defined by (2.4), the macroscopic dissipation rate is further equivalent with

$$\sum_{0<|\alpha|\leq N} \|\partial_x^{\alpha}(a,b,c)\|^2. \tag{1.14}$$

It was observed in [13] that the macroscopic equations for a,b,c behave like an elliptic system, so that the estimates on (1.14) can be also obtained. But, the remaining terms in the macroscopic equations contain the time derivatives $-\partial_t\{\mathbf{I}-\mathbf{P}\}u$. In order to bound (1.14) by the microscopic dissipation rate without any time derivatives, we will introduce some temporal interactive energy functionals between the microscopic part $\{\mathbf{I}-\mathbf{P}\}u$ and the macroscopic part $\mathbf{P}u$:

$$\mathcal{I}^a_{\alpha,i}(u(t)),\ \mathcal{I}^b_{\alpha,i}(u(t)),\ \mathcal{I}^c_{\alpha,i}(u(t)),\ \ \mathcal{I}^{ab}_{\alpha,i}(u(t))$$

which are defined by (3.10)-(3.13). These interactive functionals are indeed the inner products in the spatial space between coefficients of the velocity projection for $\{\mathbf{I}-\mathbf{P}\}u$ and $\mathbf{P}u$. The key point of the proof is to use the local macroscopic conservation laws to replace the time derivatives of the macroscopic components a, b and c. Precisely, we will obtain the following Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + C\mathcal{D}(u(t)) \leq C\mathcal{E}_M(u(t))\mathcal{D}(u(t)), \tag{1.15}$$

where the constant $M>0$ is chosen to be large enough and

$$\begin{aligned}
\mathcal{E}_M(u(t)) &\equiv \frac{M}{2}[[u(t)]]^2 + 2\mathcal{I}(u(t)),\\
\mathcal{I}(u(t)) &\equiv \sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\left[\mathcal{I}^a_{\alpha,i}(u(t)) + \mathcal{I}^b_{\alpha,i}(u(t)) + \mathcal{I}^c_{\alpha,i}(u(t)) + \mathcal{I}^{ab}_{\alpha,i}(u(t))\right],\\
\mathcal{D}(u(t)) &\equiv \sum_{|\alpha|\leq N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2 + \sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2.
\end{aligned}$$

Notice that the interactive functional $\mathcal{I}(u(t))$ can be bounded by $C[[u(t)]]^2$ for some constant C. Then $\mathcal{E}(u(t))$ and $\mathcal{D}(u(t))$ are indeed the equivalent energy functional and dissipation rate respectively:

- $\frac{1}{C}[[u(t)]]^2 \leq \mathcal{E}_M(u(t)) \leq C[[u(t)]]^2$,
- $\frac{1}{C}[[u(t)]]_\nu^2 \leq \mathcal{D}(u(t)) \leq C[[u(t)]]_\nu^2$.

Thus (1.15) is enough to give the uniform-in-time a priori estimate (1.6) for the case of small initial data.

The proof of Theorem 1.2 about the stability of solutions is almost the same as one of Theorem 1.1 with more careful estimates on the difference

$$\Gamma(u,u) - \Gamma(v,v) = \Gamma(u-v,u) + \Gamma(v,u-v).$$

In terms of the equivalent functionals defined as above, one can also obtain the other Lyapunov-type inequality for the difference $w=u-v$ of two solutions in the form:

$$\frac{d}{dt}\mathcal{E}_M(w(t)) + C\mathcal{D}(w(t)) \leq C\{\mathcal{D}(u(t)) + \mathcal{D}(v(t))\}\mathcal{E}_M(w(t)).$$

Then, by using the time integrability

$$\int_0^\infty \{\mathcal{D}(u(s)) + \mathcal{D}(v(s))\}ds < \infty$$

and the Gronwall's inequality, the uniform-in-time stability estimate (1.7) in the solution space $L^2_\xi(H^N_x)$ follows. We mention the recent work [18] about the uniform stability in $L^2_{x,\xi}\equiv L^2_\xi(L^2_x)$ for solutions satisfying some general framework conditions.

Meanwhile, in this paper it is also found that the nonlinear Boltzmann equation can be exactly written as the linearized compressible Navier-Stokes equations with some remaining terms only generated by the microscopic part $\{\mathbf{I}-\mathbf{P}\}u$. Again, the key point is to combine the macroscopic equations for (a, b, c) with the local macroscopic conservation laws. Here we remark that this observation together with the refined energy method in this paper could be useful to deal with the study of the optimal decay-in-time estimates in the Sobolev space $H^N_{x,\xi} \equiv H^N(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ on the solution operator of the linearized Boltzmann equation with an external force $-\nabla_x\phi(x) + F(t, x)$, see the recent works [6, 7]. The solution space $H^N_{x,\xi}$ is different from that in [7], where the velocity weight function is included in norms in order to illuminate the time derivatives. This work is left to be considered in the future.

There are extensive literatures on the existence theory for the Cauchy problem of the Boltzmann equation, see the books [4, 8] or monographs [3, 37] and references therein. The well-known result is the global existence of the renormalized solution with large data proved by DiPerna-Lions [5] where the uniqueness problem remains open. On the other hand, the existence is established in the framework of small perturbation of a global Maxwellian [9, 27, 32] or an infinite vacuum [2, 17, 21], where uniqueness can be justified. In particular, so far there are two basic methods to deal with solutions near a global Maxwellian. One is based on the spectral analysis of the linearized Boltzmann equation and the bootstrap argument for the nonlinear equation [27, 32, 33, 34, 35], and the other one is based on the direct nonlinear energy method [13, 22, 24, 29, 38]. Finally, we also mention that [25] developed the theory of three-dimensional Green's function for the Boltzmann equation, where based on the pointwise estimates, the wave structure of the convergence of the solution to the global Maxwellian can be finely exposed when initial data exponentially decay in x.

The rest of this paper is arranged as follows. In Section 2, we use the macro-micro decomposition to obtain the macroscopic equations and the local macroscopic conservation laws. In addition, we give a further review of some literatures about the nonlinear energy method and explain our refined energy method in a brief way. Finally, we devote ourselves to prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4, respectively.

2 Macro-micro decomposition

2.1 Macroscopic equations

For fixed (t, x), any function $u(t, x, \xi)$ can be uniquely decomposed as

$$\begin{cases} u(t,x,\xi) = u_1 + u_2, \\ u_1 \equiv \mathbf{P}u \in \mathcal{N}, \\ u_2 \equiv \{\mathbf{I}-\mathbf{P}\}u \in \mathcal{N}^\perp, \end{cases} \tag{2.1}$$

where u_1 is called the macroscopic part, and u_2 the microscopic part. Plugging this decomposition into the perturbed Boltzmann equation (1.3), the time evolution of the macroscopic part u_1 is determined by the linear term generated by the microscopic part u_2 and the nonlinear term $\Gamma(u, u)$ as follows

$$\partial_t u_1 + \xi \cdot \nabla_x u_1 = -(\partial_t u_2 + \xi \cdot \nabla_x u_2 + \mathbf{L}u_2) + \Gamma(u, u), \tag{2.2}$$

where $\mathbf{L}u_1 = 0$ is used. In what follows, we rewrite the term on the right hand side of (2.2) as the sum of three terms r, ℓ and n defined by

$$\begin{aligned} r &\equiv -\partial_t u_2, \\ \ell &\equiv -\xi\cdot\nabla_x u_2 - \mathbf{L}u_2, \\ n &\equiv \Gamma(u,u). \end{aligned}$$

Then (2.2) becomes

$$\partial_t u_1 + \xi\cdot\nabla_x u_1 = r + \ell + n. \tag{2.3}$$

Furthermore, in order to precisely study the time evolution of u_1 in the finite dimensional space $\mathcal{N}$, $u_1 = \mathbf{P}u$ is expanded as

$$u_1 = \left\{ a(t,x) + \sum_{i=1}^{3} b_i(t,x)\xi_i + c(t,x)|\xi|^2 \right\}\sqrt{\mathbf{M}}. \tag{2.4}$$

For later use, if the dependence of a, b, c on u need be emphasized, then we write a^u, b^u, c^u instead of them. By putting the expansion (2.4) into (2.3) and collecting the coefficients with respect to the basis $\{e_k\}_{k=1}^{13}$ consisting of

$$\sqrt{\mathbf{M}},\ \left(\xi_i\sqrt{\mathbf{M}}\right)_{1\le i\le 3},\ \left(|\xi_i|^2\sqrt{\mathbf{M}}\right)_{1\le i\le 3},\ \left(\xi_i\xi_j\sqrt{\mathbf{M}}\right)_{1\le i<j\le 3},\ \left(|\xi|^2\xi_i\sqrt{\mathbf{M}}\right)_{1\le i\le 3}, \tag{2.5}$$

then one has the following macroscopic equations

$$\begin{cases} \sqrt{\mathbf{M}}: & \partial_t a = r^{(0)} + \ell^{(0)} + n^{(0)}, \\ \xi_i\sqrt{\mathbf{M}}: & \partial_t b_i + \partial_i a = r_i^{(1)} + \ell_i^{(1)} + n_i^{(1)}, \\ |\xi_i|^2\sqrt{\mathbf{M}}: & \partial_t c + \partial_i b_i = r_i^{(2)} + \ell_i^{(2)} + n_i^{(2)}, \\ \xi_i\xi_j\sqrt{\mathbf{M}}: & \partial_i b_j + \partial_j b_i = r_{ij}^{(2)} + \ell_{ij}^{(2)} + n_{ij}^{(2)}, \quad i\neq j, \\ |\xi|^2\xi_i\sqrt{\mathbf{M}}: & \partial_i c = r_i^{(3)} + \ell_i^{(3)} + n_i^{(3)}, \end{cases} \tag{2.6}$$

where $r^{(0)}$, $r_i^{(1)}$, $r_i^{(2)}$, $r_{ij}^{(2)}$ and $r_i^{(3)}$ are coefficients of r with respect to the corresponding elements in the basis (2.5) and similarly, $\ell^{(0)}$, $\ell_i^{(1)}$, $\ell_i^{(2)}$, $\ell_{ij}^{(2)}$, $\ell_i^{(3)}$ and $n^{(0)}$, $n_i^{(1)}$, $n_i^{(2)}$, $n_{ij}^{(2)}$, $n_i^{(3)}$ are the corresponding coefficients for ℓ and n, respectively. More precisely, the coefficients $r^{(0)}$, $r_i^{(1)}$, $r_i^{(2)}$, $r_{ij}^{(2)}$ and $r_i^{(3)}$ for $r = -\partial_t u_2$ can be written as

$$\begin{aligned} r^{(0)} &= \sum C_k^{(0)}\langle e_k, -\partial_t u_2\rangle = -\partial_t \tilde{r}^{(0)}, \quad \tilde{r}^{(0)} \equiv \sum C_k^{(0)}\langle e_k, u_2\rangle, \\ r_i^{(1)} &= \sum C_{i,k}^{(1)}\langle e_k, -\partial_t u_2\rangle = -\partial_t \tilde{r}_i^{(1)}, \quad \tilde{r}_i^{(1)} \equiv \sum C_{i,k}^{(1)}\langle e_k, u_2\rangle, \\ r_i^{(2)} &= \sum C_{i,k}^{(2)}\langle e_k, -\partial_t u_2\rangle = -\partial_t \tilde{r}_i^{(2)}, \quad \tilde{r}_i^{(2)} \equiv \sum C_{i,k}^{(2)}\langle e_k, u_2\rangle, \\ r_{ij}^{(2)} &= \sum C_{ij,k}^{(2)}\langle e_k, -\partial_t u_2\rangle = -\partial_t \tilde{r}_{ij}^{(2)}, \quad \tilde{r}_{ij}^{(2)} \equiv \sum C_{ij,k}^{(2)}\langle e_k, u_2\rangle, \quad i\neq j, \\ r_i^{(3)} &= \sum C_{i,k}^{(3)}\langle e_k, -\partial_t u_2\rangle = -\partial_t \tilde{r}_i^{(3)}, \quad \tilde{r}_i^{(3)} \equiv \sum C_{i,k}^{(3)}\langle e_k, u_2\rangle, \end{aligned}$$

where the summation is taken over $k \in \{1,2,\cdots,13\}$ and $C_k^{(0)}$, $C_{i,k}^{(1)}$, $C_{i,k}^{(2)}$, $C_{ij,k}^{(2)}$, $C_{i,k}^{(3)}$ are some constants for linear combinations. Thus, (2.6) becomes

$$\partial_t a = -\partial_t \tilde{r}^{(0)} + \ell^{(0)} + n^{(0)}, \tag{2.7}$$

$$\partial_t b_i + \partial_i a = -\partial_t \tilde{r}_i^{(1)} + \ell_i^{(1)} + n_i^{(1)}, \tag{2.8}$$

$$\partial_t c + \partial_i b_i = -\partial_t \tilde{r}_i^{(2)} + \ell_i^{(2)} + n_i^{(2)}, \tag{2.9}$$

$$\partial_i b_j + \partial_j b_i = -\partial_t \tilde{r}_{ij}^{(2)} + \ell_{ij}^{(2)} + n_{ij}^{(2)}, \quad i \neq j, \tag{2.10}$$

$$\partial_i c = -\partial_t \tilde{r}_i^{(3)} + \ell_i^{(3)} + n_i^{(3)}. \tag{2.11}$$

Remark 2.1. *All constant coefficients $C_k^{(0)}$ etc can be explicitly computed from the constant transform matrix between the basis $\{e_k\}_{k=1}^{13}$ and its orthonormality by applying the Gram-Schmidt process [10, 11]. Here we skip their accurate values since it is not necessary for the proof of our main results. But, it should be pointed out that some of them equal zero and hence some of terms on the right hand sid of* (2.7)-(2.11) *can vanish.*

An important fact observed in [13] is that only based on two macroscopic equations (2.9) and (2.10), the macroscopic component $b = (b_1, b_2, b_3)$ satisfies an elliptic-type equation as described in the following proposition. We give its proof for completeness.

Proposition 2.1. *For each $j = 1, 2, 3$, b_j satisfies the equation*

$$\begin{aligned} -\Delta_x b_j - \partial_j \partial_j b_j = & \sum_{i \neq j} \partial_j \left[-\partial_t \tilde{r}_i^{(2)} + \ell_i^{(2)} + n_i^{(2)} \right] \\ & - \sum_{i \neq j} \partial_i \left[-\partial_t \tilde{r}_{ij}^{(2)} + \ell_{ij}^{(2)} + n_{ij}^{(2)} \right] \\ & -2\partial_j \left[-\partial_t \tilde{r}_j^{(2)} + \ell_j^{(2)} + n_j^{(2)} \right]. \end{aligned} \tag{2.12}$$

Proof. Set

$$\begin{aligned} \gamma_i^{(2)} &= -\partial_t \tilde{r}_i^{(2)} + \ell_i^{(2)} + n_i^{(2)}, \\ \gamma_{ij}^{(2)} &= -\partial_t \tilde{r}_{ij}^{(2)} + \ell_{ij}^{(2)} + n_{ij}^{(2)}. \end{aligned}$$

By (2.9) and (2.10), we compute

$$\begin{aligned} & -\Delta_x b_j - \partial_j \partial_j b_j \\ & = -\sum_{i \neq j} \partial_i \partial_i b_j - 2\partial_j \partial_j b_j \\ & = -\sum_{i \neq j} \partial_i \left[-\partial_j b_i + \gamma_{ij}^{(2)} \right] + 2\partial_j \left[\partial_t c - \gamma_j^{(2)} \right] \\ & = \sum_{i \neq j} \partial_j \partial_i b_i + 2\partial_j \partial_t c - \sum_{i \neq j} \gamma_{ij}^{(2)} - 2\gamma_j^{(2)}. \end{aligned}$$

Again using (2.9) to replace $\partial_i b_i$ in the above equation, we have

$$\begin{aligned} & -\Delta_x b_j - \partial_j \partial_j b_j \\ & = \sum_{i \neq j} \partial_j \left[-\partial_t c + \gamma_i^{(2)} \right] + 2\partial_j \partial_t c - \sum_{i \neq j} \gamma_{ij}^{(2)} - 2\gamma_j^{(2)} \\ & = \sum_{i \neq j} \partial_j \gamma_i^{(2)} - \sum_{i \neq j} \gamma_{ij}^{(2)} - 2\gamma_j^{(2)}. \end{aligned}$$

This completes the proof of the proposition.

□

Later on, (2.12) will be used to give the viscosity terms in the linearized Navier-Stokes equations. This subsection ends with the following remark, which shows the key point of the proof of Theorem 1.1.

Remark 2.2. *Notice that the time derivative $r = -\partial_t u_2$ is separated from the linear part on the right hand side of the macroscopic equation (2.2). The aim that we do in this way is to obtain the bound of the macroscopic dissipation rate by using the microscopic rate containing only the spatial derivatives. Hence, we only need to carry out the elementary energy estimates without considering the time derivatives. This is different from [13], where the time derivatives have to be included in the microscopic dissipation rate to estimate (a, b, c), which in turn leads to the fact that the energy functional $[[u(t)]]^2$ must also include the time derivatives.*

2.2 Macroscopic conservation laws

On the other hand, a, $b = (b_1, b_2, b_3)$ and c also satisfy the local macroscopic conservation laws. In fact, multiplying (1.3) by the collision invariants in (1.4) and integrating them over $\mathbb{R}^3_\xi$, we have

$$\begin{aligned}\partial_t\langle\sqrt{\mathbf{M}}, u\rangle + \nabla_x \cdot \langle\xi\sqrt{\mathbf{M}}, u\rangle = 0,\\ \partial_t\langle\xi_i\sqrt{\mathbf{M}}, u\rangle + \nabla_x \cdot \langle\xi\xi_i\sqrt{\mathbf{M}}, u\rangle = 0,\\ \partial_t\langle|\xi|^2\sqrt{\mathbf{M}}, u\rangle + \nabla_x \cdot \langle|\xi|^2\xi\sqrt{\mathbf{M}}, u\rangle = 0.\end{aligned}$$

By using the decomposition (2.1) and the expansion (2.4), we compute that for the conservative quantities,

$$\begin{cases}\langle\sqrt{\mathbf{M}}, u\rangle = \langle\sqrt{\mathbf{M}}, u_1\rangle = a + 3c,\\ \langle\xi_i\sqrt{\mathbf{M}}, u\rangle = \langle\xi_i\sqrt{\mathbf{M}}, u_1\rangle = b_i,\\ \langle|\xi|^2\sqrt{\mathbf{M}}, u\rangle = \langle|\xi|^2\sqrt{\mathbf{M}}, u_1\rangle = 3a + 15c,\end{cases}$$

and for the flux functions,

$$\begin{cases}\langle\xi_i\xi_j\sqrt{\mathbf{M}}, u\rangle = \langle\xi_i\xi_j\sqrt{\mathbf{M}}, u_1 + u_2\rangle = (a + 5c)\delta_{ij} + \langle\xi_i\xi_j\sqrt{\mathbf{M}}, u_2\rangle,\\ \langle|\xi|^2\xi_i\sqrt{\mathbf{M}}, u\rangle = \langle|\xi|^2\xi_i\sqrt{\mathbf{M}}, u_1 + u_2\rangle = 5b_i + \langle|\xi|^2\xi_i\sqrt{\mathbf{M}}, u_2\rangle,\end{cases}$$

where we used the identities

$$\langle|\xi_i|^2, \mathbf{M}\rangle = 1, \quad \langle|\xi_i|^4, \mathbf{M}\rangle = 3, \quad \langle|\xi|^2|\xi_i|^2, \mathbf{M}\rangle = 5.$$

Hence we have the macroscopic conservation laws

$$\begin{aligned}\partial_t(a + 3c) + \nabla_x \cdot b = 0,\\ \partial_t b + \nabla_x(a + 5c) + \nabla_x \cdot \langle\xi \otimes \xi\sqrt{\mathbf{M}}, u_2\rangle = 0,\\ \partial_t(3a + 15c) + \nabla_x \cdot (15b) + \nabla_x \cdot \langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = 0.\end{aligned}$$

In terms of a, $b = (b_1, b_2, b_3)$ and c, the above equations can be rewritten as

$$\partial_t a - \frac{1}{2}\nabla_x \cdot \langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = 0, \tag{2.13}$$

$$\partial_t b_i + \partial_i(a + 5c) + \nabla_x \cdot \langle\xi\xi_i\sqrt{\mathbf{M}}, u_2\rangle = 0, \tag{2.14}$$

$$\partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{1}{6}\nabla_x \cdot \langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = 0. \tag{2.15}$$

It should be pointed out that the time derivative $\partial_t u_2$ does not appear in the equations (2.13)-(2.15). The comparison between (2.13)-(2.15) and (2.7)-(2.11) shows that as mentioned in Remark 2.1, some terms and their linear combinations on the right hand sid of (2.7)-(2.11) should vanish.

Furthermore, a little surprising thing is that combining the macroscopic equations (2.7)-(2.11) and the local macroscopic conservation laws (2.13)-(2.15), one can also write the Boltzmann equation (1.3) as the linearized viscous compressible Navier-Stokes equations with the remaining terms only related to the coefficients of the microscopic part u_2 under the basis $\{e_k\}_{k=1}^{13}$. Precisely, we have the following proposition.

Proposition 2.2. *$a = a(t,x), b = b(t,x)$ and $c = c(t,x)$ satisfy*

$$\partial_t(a+3c) + \nabla_x \cdot b = 0, \tag{2.16}$$

$$\partial_t b + \nabla_x(a+3c) + 2\nabla_x c - \Delta_x b - \frac{1}{3}\nabla_x \nabla_x \cdot b = R^b, \tag{2.17}$$

$$\partial_t c + \frac{1}{3}\nabla_x \cdot b - \Delta_x c = R^c, \tag{2.18}$$

where $R^b = (R^b_1, R^b_2, R^b_3)$ and R^c are defined by

$$\begin{aligned}
R^b_j = & -\nabla_x \cdot \langle \xi\xi_j \sqrt{\mathbf{M}}, u_2\rangle - \frac{1}{3}\nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle \\
& -\sum_{i\neq j}\partial_i\left[-\partial_t \tilde{r}^{(2)}_{ij} + \ell^{(2)}_{ij} + n^{(2)}_{ij}\right] - 2\partial_j\left[-\partial_t \tilde{r}^{(2)}_{j} + \ell^{(2)}_{j} + n^{(2)}_{j}\right], \\
R^c = & -\frac{1}{6}\nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle - \sum_i \partial_i\left[-\partial_t \tilde{r}^{(3)}_{i} + \ell^{(3)}_{i} + n^{(3)}_{i}\right].
\end{aligned}$$

Proof. (2.16) is just the conservation of mass. For (2.17), we first have from (2.9) and (2.15) that

$$\begin{aligned}
-\sum_{i\neq j}\partial_j\partial_j b_i &= 2\partial_j\partial_t c - \sum_{i\neq j}\partial_j\left[-\partial_t \tilde{r}^{(2)}_{i} + \ell^{(2)}_{i} + n^{(2)}_{i}\right] \\
&= 2\partial_j\left[-\frac{1}{3}\nabla_x \cdot b - \frac{1}{6}\nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle\right] \\
&\quad - \sum_{i\neq j}\partial_j\left[-\partial_t \tilde{r}^{(2)}_{i} + \ell^{(2)}_{i} + n^{(2)}_{i}\right]
\end{aligned}$$

Adding the above equation to (2.12), we have

$$-\Delta_x b_j - \partial_j \nabla_x \cdot b = -\frac{2}{3}\partial_j\nabla_x \cdot b + \nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle + R^b_j,$$

i.e.,

$$-\Delta_x b_j - \frac{1}{3}\partial_j\nabla_x \cdot b = \nabla_x \cdot \langle |\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle + R^b_j. \tag{2.19}$$

Again, adding (2.19) with (2.14) yields (2.17). Finally, (2.11) implies

$$-\Delta_x c = -\sum_i \partial_i\left[-\partial_t \tilde{r}^{(3)}_{i} + \ell^{(3)}_{i} + n^{(3)}_{i}\right].$$

Adding it to (2.15) gives (2.18). Hence this completes the proof of the proposition. □

From the fluid-type system (2.16)-(2.18), one can easily see the structure of the linearized compressible Navier-Stokes equations. For instance, if u_2 is set to be zero, then (2.16)-(2.18) are just the linearized Navier-Stokes equations [26]. It is well-known that the nonlinear Navier-Stokes equations can be derived as an approximation to the nonlinear Boltzmann equation through the Chapman-Enskog expansion. However, the macroscopic linear system (2.16)-(2.18) is part of the Boltzmann equation, where the remaining terms are generated by only the microscopic part u_2.

Finally, we mentioned that as in [22], the Boltzmann equation (1.1) can be also written as the nonlinear Navier-Stokes equations plus some higher order terms for the microscopic part. Precisely, the solution to the Boltzmann equation (1.1) is decomposed around a local Maxwellian as follows

$$f(t,x,\xi) = \mathbf{M}_{[\rho,v,\theta]} + \mathbf{G}, \tag{2.20}$$

where

$$\mathbf{M}_{[\rho,v,\theta]} = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left\{-\frac{|\xi - v|^2}{2\theta}\right\}$$

and

$$\begin{aligned} \rho &= \langle 1, f\rangle, \\ v &= \frac{1}{\rho}\langle \xi, f\rangle, \\ \theta &= \frac{1}{3\rho}\left\langle |\xi - v|^2, f\right\rangle. \end{aligned}$$

Then, the mass ρ, the momentum $m = \rho v$ and the energy $E = \frac{1}{2}\rho|v|^2 + \frac{3}{2}\rho\theta$ actually satisfy

$$\begin{cases} \partial_t \rho + \nabla_x \cdot m = 0, \\ \partial_t m_j + \sum\limits_{i=1}^3 \partial_i (v_i m_j) + \partial_j(\rho\theta) = \sum\limits_{i=1}^3 \partial_i \left[\mu(\theta)\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\delta_{ij}\nabla_x \cdot v\right)\right] \\ \qquad\qquad - \nabla_x \cdot \langle \xi\xi_j, \Theta\rangle, \\ \partial_t E + \sum\limits_{i=1}^3 \partial_i[(\rho E + \rho\theta)v_i] = \sum\limits_{i=1}^3 \partial_i[\kappa(\theta)\partial_i\theta] \\ \qquad + \sum\limits_{i,j=1}^3 \partial_i\left[\mu(\theta)v_j\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\delta_{ij}\nabla_x\cdot v\right)\right] - \nabla_x \cdot \left\langle \frac{1}{2}|\xi|^2\xi, \Theta\right\rangle. \end{cases} \tag{2.21}$$

Here the viscosity $\mu(\theta)$ and the heat conductivity $\kappa(\theta)$ can be explicitly represented with the help of the Burnett functions, and Θ is defined by

$$\Theta = \mathbf{L}^{-1}_{\mathbf{M}_{[\rho,v,\theta]}}\Big(\partial_t \mathbf{G} + \{\mathbf{I} - \mathbf{P}_{\mathbf{M}_{[\rho,v,\theta]}}\}\xi\cdot\nabla_x \mathbf{G} - Q(\mathbf{G},\mathbf{G})\Big),$$

where $\mathbf{L}_{\mathbf{M}_{[\rho,v,\theta]}}$ is the linearized collision operator corresponding to the decomposition (2.20), i.e.,

$$\mathbf{L}_{\mathbf{M}_{[\rho,v,\theta]}}(\cdot) \equiv 2\left[Q(\mathbf{M}_{[\rho,v,\theta]},\cdot) + Q(\cdot,\mathbf{M}_{[\rho,v,\theta]})\right],$$

and $\mathbf{P}_{\mathbf{M}_{[\rho,v,\theta]}}$ is the projector to the kernel space of $\mathbf{L}_{\mathbf{M}_{[\rho,v,\theta]}}$. Notice that by definitions of ρ, m and E, it holds that

$$\begin{cases} \rho = 1 + a + 3c, \\ m = b, \\ E = \frac{3}{2} + \frac{3}{2}a + \frac{15}{2}c, \end{cases} \quad \textit{i.e.}, \quad \begin{cases} a = \frac{5}{2}(\rho - 1) - (E - \frac{3}{2}), \\ b = m, \\ c = \frac{1}{3}(E - \frac{3}{2}) - \frac{1}{2}(\rho - 1). \end{cases}$$

This means that (a, b, c) is indeed the linearized hydrodynamical variables, which is essentially consistent with the perturbation (1.2) around the global Maxwellian. However, if one directly linearizes the nonlinear Navier-Stokes equations (2.21), the linearized system derived does not give the same structure as (2.16)-(2.18), since the remaining terms contain some higher order ones of (a, b, c) after linearization.

2.3 Refined energy method

For the Boltzmann equation in the whole space, so far there are two kinds of L^2 energy methods available. One is initiated by Liu-Yu [24] and developed by Liu-Yang-Yu [22] and Yang-Zhao [38] in terms of the macro-micro decomposition around a local Maxwellian as in (2.20). This method has the general applications not only in the study of the nonlinear stability of solutions [36, 39] and convergence rates [6] but also in the stability analysis of three well-known wave patterns for the Boltzmann equation, such as the shock wave [24], rarefaction wave [23] and contact discontinuity [19, 20]. The other is founded by Guo [13] and developed by Strain [29] and Strain-Guo [30, 31] due to the decomposition (1.2) around a global Maxwellian. The norms used by both methods include the time derivatives for the proof of the global classical solution. Here we will refine the second kind of method in the sense that the time derivatives can be excluded from the norms. In this subsection, we will give a brief sketch of our method.

Our goal is to obtain the dissipation rate $[[u(t)]]_\nu^2$, which as mentioned in Section 1, is equivalent with

$$\sum_{|\alpha|\leq N} \|\partial_x^\alpha u_2\|_\nu^2 + \sum_{0<|\alpha|\leq N} \|\partial_x^\alpha (a, b, c)\|^2.$$

The first part (microscopic dissipation rate) are directly derived from

$$-\sum_{|\alpha|\leq N} \langle \partial_x^\alpha \mathbf{L} u, \partial_x^\alpha u\rangle \geq \lambda \sum_{|\alpha|\leq N} \|\partial_x^\alpha u_2\|_\nu^2.$$

The analysis of the second part (macroscopic dissipation rate) is based on the macroscopic equations (2.7)-(2.11) together with the macroscopic conservation laws (2.13)-(2.15). In fact, the macroscopic dissipation rate can be bounded by the microscopic dissipation rate containing no time derivatives.

Next, let's explain the technical part in the proof. For this time, we consider the estimates only on b_j as an example. For simplicity, instead of (2.12), b_j is supposed to satisfy

$$-\Delta_x b_j - \partial_j \partial_j b_j = \partial_j \left[-\partial_t \tilde{r}^{(2)} + \ell^{(2)} + n^{(2)} \right],$$

where $\tilde{r}^{(2)}$, $\ell^{(2)}$ and $n^{(2)}$ denote the linear combinations of coefficients for u_2, ℓ and n, respectively, with respect to the basis $\{e_k\}_{k=1}^{13}$. The standard energy estimate gives that for each α,

$$\begin{aligned}\|\nabla_x \partial_x^\alpha b_j\|^2 + \|\partial_j \partial_x^\alpha b_j\|^2 = & -\frac{d}{dt}\langle \partial_x^\alpha \tilde{r}^{(2)}, -\partial_j \partial_x^\alpha b_j\rangle + \langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha \partial_t b_j\rangle \\ & - \left\langle \partial_x^\alpha \left[\ell^{(2)} + n^{(2)} \right], \partial_j \partial_x^\alpha b_j \right\rangle.\end{aligned}$$

Thus by using the conservation laws (2.14) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{d}{dt}\langle\partial_x^\alpha\tilde{r}^{(2)},-\partial_x^\alpha\partial_j b_j\rangle+\|\nabla_x\partial_x^\alpha b_j\|^2+\|\partial_j\partial_x^\alpha b_j\|^2\\&\leq\epsilon\|\partial_x^\alpha\partial_j(a,b_j,c)\|^2\\&\quad+\frac{C}{\epsilon}\left\{\|\partial_x^\alpha\nabla_x\cdot\langle\xi\xi_j\sqrt{\mathbf{M}},u_2\rangle\|^2+\|\partial_x^\alpha\partial_j\tilde{r}^{(2)}\|^2+\left\|\partial_x^\alpha\left[\ell^{(2)},n^{(2)}\right]\right\|^2\right\},\end{aligned}\tag{2.22}$$

for $\epsilon>0$ small enough to be determined later, where the spatial inner product

$$\langle\partial_x^\alpha\tilde{r}^{(2)},-\partial_x^\alpha\partial_j b_j\rangle$$

is called the temporal interactive energy functional between u_2 and b. As in Lemmas 3.6 and 3.7, the second term on the right hand side of (2.22) is bounded by

$$\frac{C}{\epsilon}\left\{\|\nabla_x\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\}.$$

Similar estimates hold for a and c. Thus, after taking summation over $|\alpha|\leq N-1$ and choosing some $\epsilon>0$ small enough, we can obtain the desired estimates on the macroscopic dissipation rate.

3 Global existence

3.1 Preliminaries

We list the following lemmas about some Sobolev inequalities and the basic estimates on the nonlinear term $\Gamma(u,u)$.

Lemma 3.1 ([1]). *Let $u\in H^2(\mathbb{R}^3)$. Then*
(i) $\|u\|_{L^\infty}\leq C\|\nabla u\|^{\frac12}\|\nabla^2 u\|^{\frac12}\leq C\|\nabla u\|_{H^1}$;
(ii) $\|u\|_{L^6}\leq C\|\nabla u\|$;
(iii) $\|u\|_{L^q}\leq C\|u\|_{H^1}$, $2\leq q\leq 6$.

Lemma 3.2 ([16]).

$$\begin{aligned}|\langle\Gamma(u,v),w\rangle|\leq C\Bigg\{&\int_{\mathbb{R}^3}\|\nu^{1/2}u\|_{L^2_\xi}\|v\|_{L^2_\xi}\|\nu^{1/2}w\|_{L^2_\xi}dx\\&+\int_{\mathbb{R}^3}\|\nu^{1/2}v\|_{L^2_\xi}\|u\|_{L^2_\xi}\|\nu^{1/2}w\|_{L^2_\xi}dx\Bigg\};\end{aligned}$$

$$\|\langle\Gamma(u,v),w\rangle\|_{L^2_x}+\|\langle\Gamma(v,u),w\rangle\|_{L^2_x}\leq C\|\nu^3 w\|_{L^\infty_{x,\xi}}\|u\|_{L^\infty_x(L^2_\xi)}\|v\|.$$

As usual, the global existence of the solution to (1.3) will be obtained by combining the local existence together with a priori estimates.

Proposition 3.1 (local existence). *There exist constants $\delta>0$ and $T^*>0$ such that if $[[u(0)]]\leq\delta$, then there is a unique solution $u(t,x,\xi)$ in $[0,T^*]\times\mathbb{R}^3\times\mathbb{R}^3$ to the Boltzmann equation* (1.3) *such that*

$$[[u(t)]]^2+\int_0^t\sum_{|\alpha|\leq N}\|\partial_x^\alpha u(s)\|_\nu^2 ds\leq 4[[u(0)]]^2,$$

for any $t \in [0, T^*]$. *Moreover,* $[[u(t)]] : [0, T^*] \to \mathbb{R}$ *is continuous. If* $f(0, x, \xi) = \mathbf{M} + \sqrt{\mathbf{M}} u(0, x, \xi) \geq 0$*, then*

$$f(t, x, \xi) = \mathbf{M} + \sqrt{\mathbf{M}} u(t, x, \xi) \geq 0.$$

Proposition 3.2 (a priori estimate)**.** *Let* $T > 0$*. Suppose that* $u(t, x, \xi)$ *is a solution in* $[0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ *to the Boltzmann equation* (1.3)*. There exist constants* $\delta_2 > 0$*,* $\lambda_2 > 0$ *and* $C_2 > 0$ *which are independent of* T*, such that if*

$$\sup_{0 \leq t \leq T} [[u(t)]] \leq \delta_2, \tag{3.1}$$

then

$$[[u(t)]]^2 + \lambda_2 \int_0^t [[u(s)]]_\nu^2 ds \leq C_2 [[u(0)]]^2, \tag{3.2}$$

for any $t \in [0, T]$*.*

Theorem 1.1 follows from Proposition 3.1 and Proposition 3.2 by the standard continuity argument. In order to prove Proposition 3.1, we apply ∂_x^α to (1.3) to obtain

$$\partial_t \partial_x^\alpha u + \xi \cdot \nabla_x \partial_x^\alpha = \mathbf{L} \partial_x^\alpha u + \sum_{\beta \leq \alpha} C_\beta^\alpha \Gamma(\partial_x^\beta u, \partial_x^{\alpha - \beta} u).$$

One can use Lemmas 3.1 and 3.2 to deal with the nonlinear term. For the linear term, it holds that

$$\langle \mathbf{L} \partial_x^\alpha u, \partial_x^\alpha u \rangle = -\|\partial_x^\alpha u\|_\nu^2 + \langle K \partial_x^\alpha u, \partial_x^\alpha u \rangle,$$

where K is compact and hence bounded on L_ξ^2. Thus the Gronwall's inequality will give the desired estimate if the time span T^* is small enough. The uniqueness is proved in the similar way. For the more details, see [12]. Here notice that the time derivatives need not be considered.

The proof of Proposition 3.2 will be divided into two parts: one part is to obtain the microscopic dissipation rate, given in the rest of this subsection, and the other part is to deal with the macroscopic dissipation rate, which consists the main issue of this paper and thus left to the next subsection.

Based on the equation (1.3), one can carry out the elementary energy estimates to obtain the microscopic dissipation rate. The following two lemmas can be proved in the similar way as in Section 6 of [13] but by considering the spatial derivatives only. For brevity, we omit details. The point is to use Lemmas 3.1 and 3.2 to carefully estimate the nonlinear term $\Gamma(u, u)$ in the following way

$$\begin{aligned} \langle \Gamma(u, u), u \rangle &= \langle \Gamma(u, u), u_2 \rangle \\ &= \langle \Gamma(u_1, u_1), u_2 \rangle + \langle \Gamma(u_2, u_1), u_2 \rangle + \langle \Gamma(u, u_2), u_2 \rangle. \end{aligned} \tag{3.3}$$

Lemma 3.3. *There exists a constant* $C > 0$ *such that*

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \lambda \|u_2\|_\nu^2 \leq C [[u(t)]] [[u(t)]]_\nu^2, \tag{3.4}$$

for any $t \in [0, T]$*.*

Lemma 3.4. *There exists a constant* $C > 0$ *such that*

$$\frac{1}{2} \frac{d}{dt} \sum_{0 < |\alpha| \leq N} \|\partial_x^\alpha u(t)\|^2 + \lambda \sum_{0 < |\alpha| \leq N} \|\partial_x^\alpha u_2\|_\nu^2 \leq C [[u(t)]] [[u(t)]]_\nu^2, \tag{3.5}$$

for any $t \in [0, T]$*.*

3.2 Energy estimates on the macroscopic part

In this subsection, we devote ourselves to obtain the macroscopic dissipation rate

$$\sum_{0<|\alpha|\leq N} \|\nabla_x^\alpha u_1\|^2.$$

Equivalently, in terms of the macroscopic coefficients a, b and c for $u_1 = \mathbf{P}u$, it suffices to obtain the estimates on

$$\sum_{0<|\alpha|\leq N} \|\nabla_x^\alpha (a,b,c)\|^2.$$

This comes from the following lemma.

Lemma 3.5. *For any α, $\partial_x^\alpha \mathbf{P}u = \mathbf{P}\partial_x^\alpha u$, and*

$$\|\partial_x^\alpha u_1\|^2 + \|\partial_x^\alpha u_2\|^2 = \|\partial_x^\alpha u\|^2.$$

For any k, there exists a constant $C > 1$ such that

$$\frac{1}{C}\|\nu^k \partial_x^\alpha u_1\|^2 \leq \|\partial_x^\alpha (a,b,c)\|^2 \leq C\|\partial_x^\alpha u_1\|^2.$$

Now we focus on the macroscopic equations (2.7)-(2.11) and the conservation laws (2.13)-(2.13) to estimate the higher order derivatives of the macroscopic coefficients (a, b, c) in L^2 norm. For this purpose, we first give two lemmas without proofs. Roughly speaking, the idea is just based on the fact that the velocity-coordinate projector is bounded uniformly in t and x and the velocity polynomials can be absorbed by the global Maxwellian $\mathbf{M}$ which exponentially decays in ξ.

The first lemma shows that among those terms on the right hand side of the macroscopic equations (2.7)-(2.11), the coefficients of the separated part $\tilde{r}$, the linear part ℓ and the nonlinear part n can be bounded by the microscopic dissipation rate.

Lemma 3.6. *It holds that*

$$\sum_{|\alpha|\leq N-1}\sum_{ij} \left\|\partial_x^\alpha \left[\tilde{r}^{(0)}, \tilde{r}_i^{(1)}, \tilde{r}_i^{(2)}, \tilde{r}_{ij}^{(2)}, \tilde{r}_i^{(3)}\right]\right\|^2 \leq C \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha u_2\|^2, \tag{3.6}$$

$$\sum_{|\alpha|\leq N-1}\sum_{ij} \left\|\partial_x^\alpha \left[\ell^{(0)}, \ell_i^{(1)}, \ell_i^{(2)}, \ell_{ij}^{(2)}, \ell_i^{(3)}\right]\right\|^2 \leq C \sum_{|\alpha|\leq N} \|\partial_x^\alpha u_2\|^2, \tag{3.7}$$

and

$$\sum_{|\alpha|\leq N}\sum_{ij} \left\|\partial_x^\alpha \left[n^{(0)}, n_i^{(1)}, n_i^{(2)}, n_{ij}^{(2)}, n_i^{(3)}\right]\right\|^2 \leq C[[u(t)]]^2[[u(t)]]_\nu^2. \tag{3.8}$$

Remark 3.1. *Since $\tilde{r}$ is generated by u_2, no differentiation is added on the right hand side of* (3.6)*.* (3.7) *is true because ℓ contains the first order derivatives $\nabla_x u_2$ and the zero order term $-\nu u_2 + K u_2$.* (3.8) *follows from the careful analysis on the nonlinear term as in Lemmas 3.3 and 3.4.*

The second lemma similarly shows that in the conservation laws (2.13)-(2.13), those terms containing the microscopic part u_2 can be also bounded by the microscopic dissipation rate.

Lemma 3.7. *It holds that*

$$\sum_{|\alpha|\leq N-1}\left\|\partial_x^\alpha\nabla_x\cdot\left[\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle,\left\langle\xi\otimes\xi\sqrt{\mathbf{M}},u_2\right\rangle\right]\right\|^2\leq C\sum_{0<|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2.$$

Next we state the key estimates on the macroscopic dissipation rate in the following theorem.

Theorem 3.1. *There exists a constant $C_3>0$ such that*

$$\begin{aligned}2\frac{d}{dt}&\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\left[\mathcal{I}^a_{\alpha,i}(u(t))+\mathcal{I}^b_{\alpha,i}(u(t))+\mathcal{I}^c_{\alpha,i}(u(t))+\mathcal{I}^{ab}_{\alpha,i}(u(t))\right]\\&\quad+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2\\&\qquad\leq C_3\left\{\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\},\end{aligned}\tag{3.9}$$

for any $t\in[0,T]$, where $\mathcal{I}^a_{\alpha,i}(u(t))$, $\mathcal{I}^b_{\alpha,i}(u(t))$, $\mathcal{I}^c_{\alpha,i}(u(t))$ and $\mathcal{I}^{ab}_{\alpha,i}(u(t))$ are the temporal interactive energy functionals defined by

$$\mathcal{I}^a_{\alpha,i}(u(t))=\left\langle\partial_x^\alpha\tilde r_i^{(1)},\partial_i\partial_x^\alpha a\right\rangle,\tag{3.10}$$

$$\begin{aligned}\mathcal{I}^b_{\alpha,i}(u(t))=&-\sum_{j\neq i}\left\langle\partial_x^\alpha\tilde r_j^{(2)},\partial_i\partial_x^\alpha b_i\right\rangle+\sum_{j\neq i}\left\langle\partial_x^\alpha\tilde r_{ji}^{(2)},\partial_j\partial_x^\alpha b_i\right\rangle\\&+2\left\langle\partial_x^\alpha\tilde r_i^{(2)},\partial_i\partial_x^\alpha b_i\right\rangle,\end{aligned}\tag{3.11}$$

$$\mathcal{I}^c_{\alpha,i}(u(t))=\left\langle\partial_x^\alpha\tilde r_i^{(3)},\partial_i\partial_x^\alpha c\right\rangle,\tag{3.12}$$

$$\mathcal{I}^{ab}_{\alpha,i}(u(t))=\langle\partial_i\partial_x^\alpha a,\partial_x^\alpha b_i\rangle.\tag{3.13}$$

Proof. We can control the higher order derivatives of (a,b,c) as follows. In what follows, we fix a constant $\epsilon\in(0,1)$ to be determined later.

Estimates on b. Applying ∂_x^α with $|\alpha|\leq N-1$ to the macroscopic equation (2.12) satisfied by b_j for each $j=1,2,3$, multiplying it by $\partial_x^\alpha b_j$ and then integrating it over $\mathbb{R}^3$, we have

$$\begin{aligned}&\|\nabla_x\partial_x^\alpha b_j\|^2+\|\partial_j\partial_x^\alpha b_j\|^2\\&=-\left\langle\partial_t\left[\sum_{i\neq j}\partial_j\partial_x^\alpha\tilde r_i^{(2)}-\sum_{i\neq j}\partial_i\partial_x^\alpha\tilde r_{ij}^{(2)}-2\partial_j\partial_x^\alpha\tilde r_j^{(2)}\right],\partial_x^\alpha b_j\right\rangle\\&\quad+\sum_{i\neq j}\left\langle\partial_j\partial_x^\alpha\left[\ell_i^{(2)}+n_i^{(2)}\right],\partial_x^\alpha b_j\right\rangle-\sum_{i\neq j}\left\langle\partial_i\partial_x^\alpha\left[\ell_{ij}^{(2)}+n_{ij}^{(2)}\right],\partial_x^\alpha b_j\right\rangle\\&\quad-2\left\langle\partial_j\partial_x^\alpha\left[\ell_j^{(2)}+n_j^{(2)}\right],\partial_x^\alpha b_j\right\rangle.\end{aligned}$$

Integrating by parts yields

$$
\begin{aligned}
&\|\nabla_x\partial_x^\alpha b_j\|^2+\|\partial_j\partial_x^\alpha b_j\|^2\\
&=-\frac{d}{dt}\left\langle\sum_{i\neq j}\partial_j\partial_x^\alpha\tilde r_i^{(2)}-\sum_{i\neq j}\partial_i\partial_x^\alpha\tilde r_{ij}^{(2)}-2\partial_j\partial_x^\alpha\tilde r_j^{(2)},\partial_x^\alpha b_j\right\rangle\\
&\quad+\left\langle\sum_{i\neq j}\partial_j\partial_x^\alpha\tilde r_i^{(2)}-\sum_{i\neq j}\partial_i\partial_x^\alpha\tilde r_{ij}^{(2)}-2\partial_j\partial_x^\alpha\tilde r_j^{(2)},\partial_x^\alpha\partial_t b_j\right\rangle\\
&\quad-\sum_{i\neq j}\left\langle\partial_x^\alpha\left[\ell_i^{(2)}+n_i^{(2)}\right],\partial_j\partial_x^\alpha b_j\right\rangle+\sum_{i\neq j}\left\langle\partial_x^\alpha\left[\ell_{ij}^{(2)}+n_{ij}^{(2)}\right],\partial_i\partial_x^\alpha b_j\right\rangle\\
&\quad+2\left\langle\partial_x^\alpha\left[\ell_j^{(2)}+n_j^{(2)}\right],\partial_j\partial_x^\alpha b_j\right\rangle. \qquad (3.14)
\end{aligned}
$$

The five terms on the right hand of (3.14) can be estimated as follows. The first term is just $-\frac{d}{dt}\mathcal{I}_{\alpha,j}^b(t)$, where $\mathcal{I}_{\alpha,j}^b(t)$ defined by (3.11) is the interactive energy functional between the microscopic part u_2 and the macroscopic part b. From the conservation laws (2.14), the second term is bounded by

$$
\begin{aligned}
&\epsilon\,\|\partial_x^\alpha\partial_t b_j\|^2+\frac{1}{4\epsilon}\left\|\sum_{i\neq j}\partial_j\partial_x^\alpha\tilde r_i^{(2)}-\sum_{i\neq j}\partial_i\partial_x^\alpha\tilde r_{ij}^{(2)}-2\partial_j\partial_x^\alpha\tilde r_j^{(2)}\right\|^2\\
&\leq\epsilon\left\|\partial_x^\alpha\partial_j(a+5c)+\partial_x^\alpha\nabla_x\cdot\left\langle\xi\xi_j\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2\\
&\quad+\frac{C}{\epsilon}\left[\sum_{i\neq j}\left\|\partial_j\partial_x^\alpha\tilde r_i^{(2)}\right\|^2+\sum_{i\neq j}\left\|\partial_i\partial_x^\alpha\tilde r_{ij}^{(2)}\right\|^2+\left\|\partial_j\partial_x^\alpha\tilde r_j^{(2)}\right\|^2\right]\\
&\leq C\epsilon\left[\|\partial_j\partial_x^\alpha(a,c)\|^2+\left\|\partial_x^\alpha\nabla_x\cdot\left\langle\xi\xi_j\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2\right]\\
&\quad+\frac{C}{\epsilon}\sum_{ij}\left\|\nabla_x\partial_x^\alpha\left[\tilde r_i^{(2)},\tilde r_{ij}^{(2)},\tilde r_j^{(2)}\right]\right\|^2.
\end{aligned}
$$

The sum of the final three terms is bounded by

$$
\begin{aligned}
&\epsilon\,\|\nabla_x\partial_x^\alpha b_j\|^2+\frac{C}{\epsilon}\left\{\sum_{i\neq j}\left\|\partial_x^\alpha\left[\ell_i^{(2)}+n_i^{(2)}\right]\right\|^2\right.\\
&\quad\left.+\sum_{i\neq j}\left\|\partial_x^\alpha\left[\ell_{ij}^{(2)}+n_{ij}^{(2)}\right]\right\|^2+\left\|\partial_x^\alpha\left[\ell_j^{(2)}+n_j^{(2)}\right]\right\|^2\right\}\\
&\qquad\qquad\leq\epsilon\,\|\nabla_x\partial_x^\alpha b_j\|^2+\frac{C}{\epsilon}\sum_{ij}\left\|\partial_x^\alpha\left[\ell_i^{(2)},\ell_{ij}^{(2)},n_i^{(2)},n_{ij}^{(2)}\right]\right\|^2.
\end{aligned}
$$

Putting all estimates into (3.14) and taking summation for α over $|\alpha|\leq N-1$ and j over

$j \in \{1,2,3\}$, we have

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le N-1}\sum_{j=1}^{3}\mathcal{I}^b_{\alpha,j}(t)+\sum_{|\alpha|\le N-1}\|\nabla_x\partial^\alpha_x b\|^2\\&\le C\epsilon\sum_{|\alpha|\le N-1}\left[\|\nabla_x\partial^\alpha_x(a,b,c)\|^2+\left\|\partial^\alpha_x\nabla_x\cdot\left\langle\xi\otimes\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2\right]\\&\quad+\frac{C}{\epsilon}\sum_{|\alpha|\le N-1}\sum_{ij}\left\|\nabla_x\partial^\alpha_x\left[\tilde r^{(2)}_i,\tilde r^{(2)}_{ij},\tilde r^{(2)}_j\right]\right\|^2\\&\quad+\frac{C}{\epsilon}\sum_{|\alpha|\le N-1}\sum_{ij}\left\|\partial^\alpha_x\left[\ell^{(2)}_i,\ell^{(2)}_{ij},n^{(2)}_i,n^{(2)}_{ij}\right]\right\|^2.\end{aligned}$$

By Lemmas 3.6 and 3.7, the above inequality implies

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le N-1}\sum_{j=1}^{3}\mathcal{I}^b_{\alpha,j}(t)+\sum_{|\alpha|\le N-1}\|\nabla_x\partial^\alpha_x b\|^2\\&\le C\epsilon\sum_{|\alpha|\le N-1}\|\nabla_x\partial^\alpha_x(a,b,c)\|^2\\&\quad+\frac{C}{\epsilon}\left\{\sum_{|\alpha|\le N}\|\partial^\alpha_x u_2\|^2+[[u(t)]]^2[[u(t)]]^2_\nu\right\}.\end{aligned}\tag{3.15}$$

Estimates on c. For any α with $|\alpha|\le N-1$ and each $i=1,2,3$, it follows from (2.11) that

$$\begin{aligned}\|\partial_i\partial^\alpha_x c\|^2&=\langle\partial_i\partial^\alpha_x c,\partial_i\partial^\alpha_x c\rangle\\&=-\left\langle\partial_t\partial^\alpha_x\tilde r^{(3)}_i,\partial_i\partial^\alpha_x c\right\rangle+\left\langle\partial^\alpha_x\left[\ell^{(3)}_i+n^{(3)}_i\right],\partial_i\partial^\alpha_x c\right\rangle\\&=-\frac{d}{dt}\left\langle\partial^\alpha_x\tilde r^{(3)}_i,\partial_i\partial^\alpha_x c\right\rangle+\left\langle\partial^\alpha_x\tilde r^{(3)}_i,\partial_i\partial^\alpha_x\partial_t c\right\rangle\\&\quad+\left\langle\partial^\alpha_x\left[\ell^{(3)}_i+n^{(3)}_i\right],\partial_i\partial^\alpha_x c\right\rangle.\end{aligned}\tag{3.16}$$

The three terms on the right hand side of (3.16) can be estimated as follows. The first term is just $-\frac{d}{dt}\mathcal{I}^c_{\alpha,i}(u(t))$, where $\mathcal{I}^c_{\alpha,i}(u(t))$ defined by (3.12) is the interactive energy functional between the microscopic part u_2 and the macroscopic part c. By the conservation laws (2.15), the second term is bounded by

$$\begin{aligned}&-\left\langle\partial_i\partial^\alpha_x\tilde r^{(3)}_i,\partial^\alpha_x\partial_t c\right\rangle\\&\le\epsilon\|\partial_x\partial_t c\|^2+\frac{1}{4\epsilon}\left\|\partial_i\partial^\alpha_x\tilde r^{(3)}_i\right\|^2\\&\le\epsilon\left\|\frac13\partial^\alpha_x\nabla_x\cdot b+\frac16\partial^\alpha_x\nabla_x\cdot\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2+\frac{1}{4\epsilon}\left\|\partial_i\partial^\alpha_x\tilde r^{(3)}_i\right\|^2\\&\le C\epsilon\|\partial^\alpha_x\nabla_x\cdot b\|^2+C\epsilon\left\|\partial^\alpha_x\nabla_x\cdot\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2+\frac{1}{4\epsilon}\left\|\partial_i\partial^\alpha_x\tilde r^{(3)}_i\right\|^2.\end{aligned}$$

The third term is bounded by

$$\epsilon\|\partial_i\partial^\alpha_x c\|^2+\frac{1}{4\epsilon}\left\|\partial^\alpha_x\left[\ell^{(3)}_i,n^{(3)}_i\right]\right\|^2.$$

Plugging all the above estimates into (3.16) and taking summation for α over $|\alpha|\leq N-1$ and i over $i\in\{1,2,3\}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\mathcal{I}^{c}_{\alpha,i}(u(t))+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha c\|^2\\
&\leq C\epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(b,c)\|^2+C\epsilon\sum_{|\alpha|\leq N-1}\left\|\partial_x^\alpha\nabla_x\cdot\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2\\
&\quad+\frac{1}{4\epsilon}\sum_{|\alpha|\leq N-1}\sum_{i}\left\{\left\|\partial_i\partial_x^\alpha\tilde{r}_i^{(3)}\right\|^2+\left\|\partial_x^\alpha\left[\ell_i^{(3)},n_i^{(3)}\right]\right\|^2\right\},
\end{aligned}
$$

which together with Lemmas 3.6 and 3.7 implies

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\mathcal{I}^{c}_{\alpha,i}(u(t))+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha c\|^2\\
&\leq C\epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(b,c)\|^2\\
&\quad+\frac{C}{\epsilon}\left\{\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\}.
\end{aligned}\tag{3.17}
$$

Estimates on a. For any α with $|\alpha|\leq N-1$ and each $i=1,2,3$, it follows from (2.8) that

$$
\begin{aligned}
\|\partial_i\partial_x^\alpha a\|^2&=\langle\partial_i\partial_x^\alpha a,\partial_i\partial_x^\alpha a\rangle\\
&=\left\langle-\partial_x^\alpha\partial_t b_i-\partial_x^\alpha\partial_t\tilde{r}_i^{(1)}+\partial_x^\alpha\left[\ell_i^{(1)}+n_i^{(1)}\right],\partial_i\partial_x^\alpha a\right\rangle\\
&=-\frac{d}{dt}\left[\langle\partial_x^\alpha b_i,\partial_i\partial_x^\alpha a\rangle+\left\langle\partial_x^\alpha\tilde{r}_i^{(1)},\partial_i\partial_x^\alpha a\right\rangle\right]\\
&\quad+\langle\partial_x^\alpha b_i,\partial_i\partial_x^\alpha\partial_t a\rangle+\left\langle\partial_x^\alpha\tilde{r}_i^{(1)},\partial_i\partial_x^\alpha\partial_t a\right\rangle\\
&\quad+\left\langle\partial_x^\alpha\left[\ell_i^{(1)}+n_i^{(1)}\right],\partial_i\partial_x^\alpha a\right\rangle.
\end{aligned}\tag{3.18}
$$

Similarly as before, we estimates the four terms on the right hand side of (3.18) as follows. The first term is $-\frac{d}{dt}[\mathcal{I}^{ab}_{\alpha,i}(u(t))+\mathcal{I}^{a}_{\alpha,i}(u(t))]$, where $\mathcal{I}^{a}_{\alpha,i}(u(t))$ defined by (3.10) is the interactive energy functional between the microscopic part u_2 and the macroscopic part a while $\mathcal{I}^{ab}_{\alpha,i}(u(t))$ defined by (3.13) is the one between only the macroscopic parts a and b. From the conservation laws (2.13), the second and third terms are bounded by

$$
\begin{aligned}
&-\langle\partial_x^\alpha\partial_i b_i,\partial_x^\alpha\partial_t a\rangle-\left\langle\partial_x^\alpha\partial_i\tilde{r}_i^{(1)},\partial_x^\alpha\partial_t a\right\rangle\\
&=-\left\langle\partial_x^\alpha\partial_i b_i,\frac{1}{2}\partial_x^\alpha\nabla_x\cdot\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\rangle-\left\langle\partial_x^\alpha\partial_i\tilde{r}_i^{(1)},\frac{1}{2}\partial_x^\alpha\nabla_x\cdot\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\rangle\\
&\leq\epsilon\|\partial_x^\alpha\partial_i b_i\|^2+\frac{C}{\epsilon}\left\|\partial_x^\alpha\nabla_x\cdot\left\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\right\rangle\right\|^2+C\left\|\partial_x^\alpha\partial_i\tilde{r}_i^{(1)}\right\|^2.
\end{aligned}
$$

The final term is bounded by

$$
\epsilon\|\partial_i\partial_x^\alpha a\|^2+\frac{C}{\epsilon}\left\|\partial_x^\alpha\left[\ell_i^{(1)},n_i^{(1)}\right]\right\|^2.
$$

Plugging all the above estimates into (3.18) and taking summation for α over $|\alpha| \leq N-1$ and i over $i \in \{1,2,3\}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\Big[\mathcal{I}^{a}_{\alpha,i}(u(t))+\mathcal{I}^{ab}_{\alpha,i}(u(t))\Big]+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha a\|^2\\
&\leq \epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b)\|^2+\frac{C}{\epsilon}\sum_{|\alpha|\leq N-1}\Big\|\partial_x^\alpha\nabla_x\cdot\Big\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\Big\rangle\Big\|^2\\
&\quad +C\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\Big\|\partial_x^\alpha\partial_i\tilde{r}_i^{(1)}\Big\|^2+\frac{C}{\epsilon}\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\Big\|\partial_x^\alpha\Big[\ell_i^{(1)},n_i^{(1)}\Big]\Big\|^2.
\end{aligned}
$$

which by Lemmas 3.6 and 3.7 implies

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\Big[\mathcal{I}^{a}_{\alpha,i}(u(t))+\mathcal{I}^{ab}_{\alpha,i}(u(t))\Big]+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha a\|^2\\
&\leq \epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b)\|^2\\
&\quad +\frac{C}{\epsilon}\left\{\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\}.
\end{aligned}
\tag{3.19}
$$

Finally we add up the inequalities (3.15), (3.17) and (3.19) to obtain

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\Big[\mathcal{I}^{a}_{\alpha,i}(u(t))+\mathcal{I}^{b}_{\alpha,i}(u(t))+\mathcal{I}^{c}_{\alpha,i}(u(t))+\mathcal{I}^{ab}_{\alpha,i}(u(t))\Big]\\
&+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2\\
&\leq C\epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2\\
&\quad +\frac{C}{\epsilon}\left\{\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\}.
\end{aligned}
\tag{3.20}
$$

We choose $\epsilon \in (0,1)$ such that $C\epsilon = \frac{1}{2}$. Then (3.20) leads to (3.9). Hence this completes the proof of the lemma.

□

Remark 3.2. *If the analysis is made on the linearized Navier-Stokes equations* (2.16)-(2.18) *and the local macroscopic conservation laws* (2.13)-(2.15)*, then an energy inequality similar to* (3.9) *can be obtained with the energy functional in* (3.9) *replaced by the sum of the interactive energy functional and the macroscopic energy functional*

$$
\sum_{|\alpha|\leq N}\|\partial_x^\alpha(a+3c,b,c)\|^2,
$$

which has been considered in the elementary energy estimates (3.4) *and* (3.5)*. In fact, by writing* (2.16)-(2.18) *in the skew symmetrization, the standard energy estimates from [26] can apply.*

3.3 Proof of global existence

In this subsection we are in a position to prove Proposition 3.2.

Proof of Proposition 3.2. Multiplying (3.4) and (3.5) by $M > 0$ suitably large to be determined later, taking summation for them and then adding it to (3.9), we have

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + \lambda_3\mathcal{D}(u(t)) \leq C\{[[u(t)]] + [[u(t)]]^2\}[[u(t)]]_\nu^2, \tag{3.21}$$

where

$$\lambda_3 = \min\{M\lambda - C_3, 1\}$$

and

$$\mathcal{E}_M(u(t)) \equiv \frac{M}{2}[[u(t)]]^2 + 2I(t), \tag{3.22}$$

$$\mathcal{I}(u(t)) \equiv \sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\left[\mathcal{I}^a_{\alpha,i}(u(t)) + \mathcal{I}^b_{\alpha,i}(u(t)) + \mathcal{I}^c_{\alpha,i}(u(t)) + \mathcal{I}^{ab}_{\alpha,i}(u(t))\right], \tag{3.23}$$

$$\mathcal{D}(u(t)) \equiv \sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|_\nu^2 + \sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2. \tag{3.24}$$

Notice that by Lemma 3.5, $\mathcal{D}(u(t))$ is equivalent with the dissipation rate $[[u(t)]]_\nu^2$, i.e., there exists a constant $C > 1$ such that

$$\frac{1}{C}[[u(t)]]_\nu^2 \leq \mathcal{D}(u(t)) \leq C[[u(t)]]_\nu^2.$$

Next, we claim that there exists a constant $C > 0$ such that

$$I(t) \leq C[[u(t)]]^2.$$

In fact, by the definitions (3.10)-(3.13) of $\mathcal{I}^a_{\alpha,i}(u(t))$, $\mathcal{I}^b_{\alpha,i}(u(t))$, $\mathcal{I}^c_{\alpha,i}(u(t))$ and $\mathcal{I}^{ab}_{\alpha,i}(u(t))$, we have

$$\begin{aligned}
|\mathcal{I}(u(t))| \leq &\sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}\left[|\mathcal{I}^a_{\alpha,i}(u(t))| + |\mathcal{I}^b_{\alpha,i}(u(t))| + |\mathcal{I}^c_{\alpha,i}(u(t))| + |\mathcal{I}^{ab}_{\alpha,i}(u(t))|\right]\\
&\leq C\sum_{|\alpha|\leq N-1}\sum_{i,j=1}^{3}\left\|\partial_x^\alpha\left[\tilde{r}_i^{(1)}, \tilde{r}_i^{(2)}, \tilde{r}_{ij}^{(2)}, \tilde{r}_i^{(3)}\right]\right\|^2\\
&\quad + C\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2 + C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha b\|^2\\
&\leq C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha u_2\|^2 + C\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_1\|^2\\
&\leq C[[u(t)]]^2.
\end{aligned}$$

Thus we can choose $M > 0$ suitably large so that $M\lambda - C_3 > 0$, hence $\lambda_3 > 0$ and there is a constant $C > 1$ such that

$$\frac{1}{C}[[u(t)]]^2 \leq \mathcal{E}_M(u(t)) \leq C[[u(t)]]^2. \tag{3.25}$$

In terms of the equivalent energy functional $\mathcal{E}_M(u(t))$ and dissipation rate $\mathcal{D}(u(t))$, (3.21) becomes

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + \lambda_3\mathcal{D}(u(t)) \le C\left\{\mathcal{E}_M(u(t))^{1/2} + \mathcal{E}_M(u(t))\right\}\mathcal{D}(u(t)),$$

which after using the Gronwall's inequality, yields the Lyapunov-type inequality

$$\frac{d}{dt}\mathcal{E}_M(u(t)) + \frac{\lambda_3}{2}\mathcal{D}(u(t)) \le C\mathcal{E}_M(u(t))\mathcal{D}(u(t)). \tag{3.26}$$

Integrating (3.26) over $[0,t]$ with $0 \le t \le T$ and using (3.25) and the assumption (3.1), we have

$$\mathcal{E}_M(u(t)) + (\lambda_3/2 - C\delta_2^2)\int_0^t \mathcal{D}(u(s))ds \le C\mathcal{E}_M(u(0)).$$

Thus if we can choose $\delta_2 > 0$ such that $C\delta_2^2 = \lambda_3/4$, then (3.2) follows. This completes the proof of Proposition 3.2.

Remark 3.3. *The above proof also shows that $-\mathbf{L}$ is positive in the sense that for the unique solution $u(t,x,\xi)$ to the nonlinear Boltzmann equation (1.3) with $[[u(t)]]$ being sufficiently small, it holds that*

$$-\sum_{|\alpha|\le N}\langle \mathbf{L}\partial_x^\alpha u, \partial_x^\alpha u\rangle \ge \lambda_4[[u(t)]]_\nu^2 + C_4\frac{d}{dt}\mathcal{I}(u(t)),$$

for any $t \ge 0$, where $\lambda_4 > 0$ and $C_4 > 0$ are some constants.

4 Uniform stability

In this section, we are concerned with the uniform stability of the unique solution obtained in Theorem 1.1. For this purpose, to the end it is assumed that there exist solutions $u = u(t,x,\xi)$, $v = v(t,x,\xi)$ to the Boltzmann equation (1.3) corresponding to given data $u_0(x,\xi)$, $v_0(x,\xi)$ with

$$\max\{[[u(0)]], [[v(0)]]\} \le \delta_1, \tag{4.1}$$

where $\delta_1 \in (0,\delta_0)$ is to be determined later. Since $\delta_1 \le \delta_0$, both u and v satisfy the inequality (1.6), i.e.

$$[[u(t)]]^2 + \lambda_0\int_0^t [[u(s)]]_\nu^2 ds \le C_0[[u(0)]]^2, \tag{4.2}$$

$$[[v(t)]]^2 + \lambda_0\int_0^t [[v(s)]]_\nu^2 ds \le C_0[[v(0)]]^2, \tag{4.3}$$

for any $t \ge 0$. To prove the uniform-in-time stability estimate (1.7), we set

$$w(t,x,\xi) = u(t,x,\xi) - v(t,x,\xi).$$

Then $w = w(t,x,\xi)$ satisfies

$$\partial_t w + \xi\cdot\nabla_x w = \mathbf{L}w + \Gamma(w,u) + \Gamma(v,w). \tag{4.4}$$

First we give a remark about the equation (4.4).

Remark 4.1. *Due to the analysis as in Section 2, $a^w(t,x)$, $b^w(t,x)$ and $c^w(t,x)$ which are coefficients of the macroscopic part $w_1 = \mathbf{P}w$ satisfy the same macroscopic equations* (2.7)-(2.11) *with the nonlinear term $n = \Gamma(u,u)$ replaced by $\Gamma(w,u)+\Gamma(v,w)$. Furthermore, since*

$$\langle \psi, \Gamma(w,u)+\Gamma(v,w)\rangle = 0, \quad \forall\ \psi \in \mathcal{N},$$

one can also obtain the local macroscopic conservation laws (2.13)-(2.15) *for $a^w(t,x)$, $b^w(t,x)$ and $c^w(t,x)$ with u_2 replaced by $w_2 = \{\mathbf{I}-\mathbf{P}\}w$.*

Next, similar to Lemmas 3.3 and 3.4, we have the following lemma about the estimates on the microscopic dissipation rate of w.

Lemma 4.1. *For any $\epsilon \in (0,1)$ and any $t \geq 0$, it holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\leq N}\|\partial_x^\alpha w\|^2 + \lambda\sum_{|\alpha|\leq N}\|\partial_x^\alpha w_2\|_\nu^2\\
&\quad\leq C\epsilon[[w(t)]]_\nu^2 + \frac{C}{\epsilon}\left\{[[u(t)]]^2 + [[v(t)]]^2\right\}[[w(t)]]_\nu^2\\
&\qquad + \frac{C}{\epsilon}\left\{[[u(t)]]_\nu^2 + [[v(t)]]_\nu^2\right\}[[w(t)]]^2,
\end{aligned} \tag{4.5}$$

where the constant $C>0$ is independent of ϵ and t.

Proof. From (4.4), one has

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha w(t)\|^2 + \lambda\|\partial_x^\alpha w_2\|_\nu^2 \leq \langle \partial_x^\alpha[\Gamma(w,u)+\Gamma(v,w)], \partial_x^\alpha w_2\rangle. \tag{4.6}$$

By symmetry, it suffices to consider the estimates on $\Gamma(w,u)$.

When $|\alpha| = 0$, as in (3.3), we write

$$\begin{aligned}
\langle \Gamma(w,u), w_2\rangle &= \langle \Gamma(w_1,u_1), w_2\rangle + \langle \Gamma(w_1,u_2), w_2\rangle\\
&\quad + \langle \Gamma(w_2,u_1), w_2\rangle + \langle \Gamma(w_2,u_2), w_2\rangle.
\end{aligned}$$

By using Lemmas 3.1 and 3.2, one can estimate each term as follows:

$$\begin{aligned}
\langle \Gamma(w_1,u_1), w_2\rangle &\leq C\|(a^w,b^w,c^w)\|\|(a^u,b^u,c^u)\|_{L_x^\infty}\|w_2\|_\nu\\
&\leq C\|\nabla_x(a^u,b^u,c^u)\|_{H_x^1}\|w_1\|\|w_2\|_\nu\\
&\leq \epsilon\|w_2\|_\nu^2 + \frac{C}{\epsilon}\|\nabla_x u_1\|_{L_\xi^2(H_x^1)}^2\|w_1\|^2,
\end{aligned}$$

$$\begin{aligned}
\langle \Gamma(w_1,u_2), w_2\rangle &\leq C\|\nu^{1/2}u_2\|_{L_x^\infty(L_\xi^2)}\|(a^w,b^w,c^w)\|\|w_2\|_\nu\\
&\leq C\|\nu^{1/2}\nabla_x u_2\|_{L_\xi^2(H_x^1)}\|w_1\|\|w_2\|_\nu\\
&\leq \epsilon\|w_2\|_\nu^2 + \frac{C}{\epsilon}\|\nu^{1/2}\nabla_x u_2\|_{L_\xi^2(H_x^1)}^2\|w_1\|^2,
\end{aligned}$$

$$\begin{aligned}
\langle \Gamma(w_2,u_1), w_2\rangle &\leq C\|(a^u,b^u,c^u)\|_{L_x^\infty}\|w_2\|_\nu^2\\
&\leq \epsilon\|w_2\|_\nu^2 + \frac{C}{\epsilon}\|\nabla_x u_1\|_{L_\xi^2(H_x^1)}^2\|w_2\|_\nu^2,
\end{aligned}$$

and

$$
\begin{aligned}
&\langle \Gamma(w_2, u_2), w_2 \rangle \\
&\le C\|u_2\|_{L^\infty_x(L^2_\xi)}\|w_2\|^2_\nu + C\|\nu^{1/2}u_2\|_{L^\infty_x(L^2_\xi)}\|w_2\|\|w_2\|_\nu \\
&\le \epsilon\|w_2\|^2_\nu + \frac{C}{\epsilon}\|\nabla_x u_2\|^2_{L^2_\xi(H^1_x)}\|w_2\|^2_\nu + \frac{C}{\epsilon}\|\nu^{1/2}\nabla_x u_2\|^2_{L^2_\xi(H^1_x)}\|w_2\|^2.
\end{aligned}
$$

Notice that all terms on the right hand side of the above four inequalities can be bounded in the same way as in (4.5). Hence, the case of $|\alpha| = 0$ in (4.5) is proved.

When $0 < |\alpha| \le N$, we write

$$
\langle \partial^\alpha_x \Gamma(w, u), \partial^\alpha_x w_2 \rangle = \sum_{|\beta|\le|\alpha|} C^\alpha_\beta \langle \Gamma(\partial^\beta_x w, \partial^\beta_x u), \partial^\alpha_x w_2 \rangle. \tag{4.7}
$$

From Lemma 3.2, for each β it holds that

$$
\begin{aligned}
\langle \Gamma(\partial^\beta_x w, \partial^\beta_x u), \partial^\alpha_x w_2 \rangle \le\ & C\int_{\mathbb{R}^3} \|\nu^{1/2}\partial^\beta_x w\|\|\partial^{\alpha-\beta}_x u\|\|\nu^{1/2}\partial^\alpha_x w_2\| dx \\
&+ C\int_{\mathbb{R}^3} \|\partial^\beta_x w\|\|\nu^{1/2}\partial^{\alpha-\beta}_x u\|\|\nu^{1/2}\partial^\alpha_x w_2\| dx.
\end{aligned} \tag{4.8}
$$

Without loss of generality, one can suppose $|\beta| \le |\alpha|/2$. Then the first and second terms of (4.8) are respectively bounded by

$$
\begin{aligned}
&\|\nu^{1/2}\partial^\beta_x w\|_{L^\infty_x(L^2_\xi)}\|\partial^{\alpha-\beta}_x u\|\|\partial^\alpha_x w_2\|_\nu \\
&\le \epsilon\|\partial^\alpha_x w_2\|^2_\nu + \frac{C}{\epsilon}\|\partial^{\alpha-\beta}_x u\|^2\|\nu^{1/2}\nabla_x\partial^\beta_x w\|^2_{L^2_\xi(H^1_x)} \\
&\le \epsilon\|\partial^\alpha_x w_2\|^2_\nu + \frac{C}{\epsilon}[[u(t)]]^2[[w(t)]]^2_\nu
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\partial^\beta_x w\|_{L^\infty_x(L^2_\xi)}\|\partial^{\alpha-\beta}_x u\|_\nu\|\partial^\alpha_x w_2\|_\nu \\
&\le \epsilon\|\partial^\alpha_x w_2\|^2_\nu + \frac{C}{\epsilon}\|\partial^{\alpha-\beta}_x u\|^2_\nu\|\nabla_x\partial^\beta_x w\|^2_{L^2_\xi(H^1_x)} \\
&\le \epsilon\|\partial^\alpha_x w_2\|^2_\nu + \frac{C}{\epsilon}[[u(t)]]^2_\nu[[w(t)]]^2.
\end{aligned}
$$

Thus the right hand side of (4.7) can be also bounded in the same way as in (4.5). Hence, the case of $0 < |\alpha| \le N$ in (4.5) is proved. By putting all estimates into (4.6), we get (4.5). This completes the proof of the lemma.

□

Moreover, similar to Theorem 3.1, the macroscopic dissipation rate of w is given in the following lemma.

Lemma 4.2. *There exists a constant $C_5 > 0$ such that for any $t \ge 0$,*

$$
\begin{aligned}
&2\frac{d}{dt}\mathcal{I}(w(t)) + \sum_{|\alpha|\le N-1} \|\nabla_x\partial^\alpha_x(a^w, b^w, c^w)\| \\
&\le C_5 \sum_{0<|\alpha|\le N} \|\partial^\alpha_x w_2\|^2 + C_5\left\{[[u(t)]]^2_\nu + [[v(t)]]^2_\nu\right\}[[w(t)]]^2.
\end{aligned} \tag{4.9}
$$

Proof. Recall Remark 4.1. To prove (4.9), it suffices to consider the estimates on the nonlinear term $\bar{n} \equiv \Gamma(w,u) + \Gamma(v,w)$. Similar to (3.8), it holds that

$$\sum_{|\alpha|\leq N}\sum_{ij}\left\|\partial_x^\alpha\left[\bar{n}^{(0)},\bar{n}_i^{(1)},\bar{n}_i^{(2)},\bar{n}_{ij}^{(2)},\bar{n}_i^{(3)}\right]\right\|^2 \leq C\left\{[[u(t)]]_\nu^2 + [[v(t)]]_\nu^2\right\}[[w(t)]]^2,$$

where $\bar{n}^{(0)},\bar{n}_i^{(1)},\bar{n}_i^{(2)},\bar{n}_{ij}^{(2)},\bar{n}_i^{(3)}$ are the coefficients of $\bar{n}$ with respect to the basis $\{e_k\}_{k=1}^{13}$. Then (4.9) follows from almost the same proof of (3.9). The details are omitted for brevity. This completes the proof of the lemma.

□

Finally we use the method of equivalent functionals to prove the uniform-in-time stability estimate (1.7).

Proof of Theorem 1.2. Similar to obtain (3.21), Lemmas 4.1 and 4.2 yield

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}_{M_1}(w(t)) + \lambda_5\mathcal{D}(w(t))\\ &\leq C\epsilon[[w(t)]]_\nu^2 + \frac{C}{\epsilon}\left\{[[u(t)]]^2 + [[v(t)]]^2\right\}[[w(t)]]_\nu^2\\ &+\frac{C}{\epsilon}\left\{[[u(t)]]_\nu^2 + [[v(t)]]_\nu^2\right\}[[w(t)]]^2,\end{aligned} \tag{4.10}$$

for $\epsilon\in(0,1)$ to be determined later, where $M_1 > M$ is fixed large enough such that

$$\lambda_5 \equiv \min\{M_1\lambda - C_4, 1\} > 0,$$

and $\mathcal{E}_{M_1}(w(t))$, $\mathcal{D}(w(t))$ defined by (3.22), (3.23) and (3.24) are the equivalent energy functional $[[w(t)]]^2$ and the dissipation rate $[[w(t)]]_\nu^2$ corresponding to w. In terms of $\mathcal{E}_{M_1}(w(t))$ and $\mathcal{D}(w(t))$, (4.10) gives the following Lyapunov-type inequality

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}_{M_1}(w(t)) + \left\{\lambda_5 - C\epsilon - \frac{CC_0\delta_1^2}{\epsilon}\right\}\mathcal{D}(w(t))\\ &\leq \frac{C}{\epsilon}\left\{[[u(t)]]_\nu^2 + [[v(t)]]_\nu^2\right\}\mathcal{E}_{M_1}(w(t)),\end{aligned} \tag{4.11}$$

where (4.1), (4.2) and (4.3) have been used. Then one can fix some $\epsilon\in(0,1)$ and $\delta_1\in(0,\delta_0)$ small enough such that

$$\lambda_6 \equiv \lambda_5 - C\epsilon - \frac{CC_0\delta_1^2}{\epsilon} > 0.$$

By the Gronwall's inequality and (4.1)-(4.3), (4.11) implies that there exists a constant $C_6 > 1$ such that

$$\mathcal{E}_{M_1}(w(t)) + \lambda_6\int_0^t \mathcal{D}(w(s))ds \leq C_6\mathcal{E}_{M_1}(w(0)),$$

for $t \geq 0$. Using the original energy functional and dissipation rate, (1.7) with $w = u - v$ follows. This completes the proof of Theorem 1.2.

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