Fast solvers for hp-FEM using hexahedral elements

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Outline

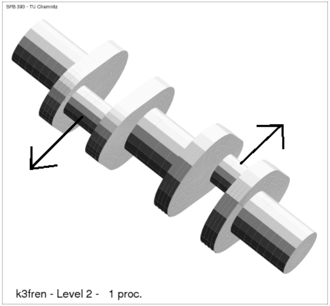
1. Setting of the problem
2. Definition of the preconditioner
3. Numerical experiments
   - Potential equation
   - Linear Elasticity
4. Conclusions

Motivation

Consider system of Lame equations for displacement
$u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^\top$

$$-\mu \triangle u - (\lambda + \mu) \nabla \cdot \nabla u = f \quad \text{in } \Omega \qquad (1)$$

of a crank shaft $(= \Omega)$



Model problem

- Solve

$$\begin{aligned} -\triangle u &= f, \\ u\,|_{\partial\Omega} &= 0, \end{aligned}$$

  in a domain $\Omega \subset \mathbb{R}^3$.
- weak formulation: find $u \in H_0^1(\Omega)$ such that

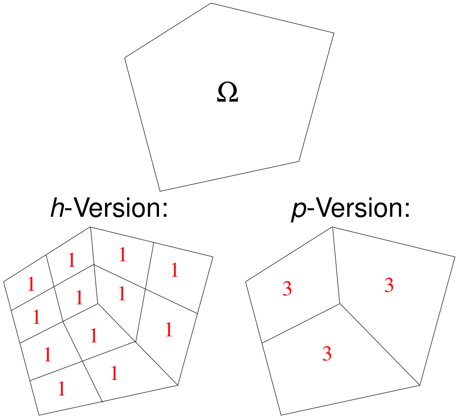
$$a_\triangle(u, v) = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \;\forall v \in H_0^1(\Omega). \qquad (2)$$

Finite elements

- Find a mesh of hexahedrons R_s, $s = 1, \ldots, n_e$
- Mapping: $\Phi_s : \mathcal{R} \to R_s$ with $\mathcal{R} = (-1, 1)^3$,
- Space: $\mathbb{M} = \{u \in H_0^1(\Omega), u\,|_{R_s} = \tilde{u}(\Phi_s^{-1}(x, y, z)), \tilde{u} \in \mathbb{Q}_p\}$, where $\mathbb{Q}_p$ polynomials of degree $\leq p$ in each variable
- Discrete problem: Find $u_p \in \mathbb{M}$ with

$$a_\triangle(u_p, v_p) = \int_\Omega \nabla u_p \cdot \nabla v_p = \int_\Omega f v_p \; \forall v_p \in \mathbb{M} \tag{3}$$

h-FEM vs. *p*-FEM (2D)



Local basis functions

- Basis on $\mathcal{R}$: tensor products of integrated Legendre polynomials: $\widehat{L}_{ijk}(x,y,z) = \widehat{L}_i(x)\widehat{L}_j(y)\widehat{L}_k(z)$ $(0 \leq i,j,k \leq p)$, where

$$\widehat{L}_i(x) = \frac{2i-1}{2} \int_{-1}^{x} L_{i-1}(s)\,\mathrm{d}s$$

with

$$L_i(x) = \frac{1}{2^i i!} \frac{\mathrm{d}^i}{\mathrm{d}x^i} (x^2-1)^i$$

for $i \geq 2$ and

$$\widehat{L}_{0/1}(x) = \frac{1 \mp x}{2}$$

Finite element space

- Define now global basis functions with $\mathbb{M} = \text{span}\{\zeta_1, \ldots, \zeta_{n_p}\}$ in the usual way
- Then, (3) is equivalent to solve

$$A\underline{x} = \underline{b},$$

where

$$A = \left[a_\triangle(\zeta_j, \zeta_i)\right]_{i,j=1}^{n_p}.$$

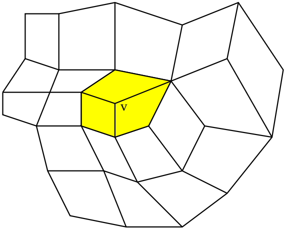
\+ A is weakly populated $\rightarrow$ iterative solvers

– A is ill-conditioned $\rightarrow$ pre-conditioners C^{-1}

Overlapping preconditioner

Consider the Additive Schwarz (ASM) splitting $\mathbb{M} = \mathbb{V}_0 \oplus \sum_v \mathbb{V}_v$, where

- $\mathbb{V}_0$ space of all piecewise linear basis functions on the FE-mesh
- $\mathbb{V}_v$ patch-space of a node v



Theorem (Pavarino 94)

The ASM-splitting $\mathbb{M} = \mathbb{V}_0 \oplus \sum_v \mathbb{V}_v$ is stable with respect to p and h.

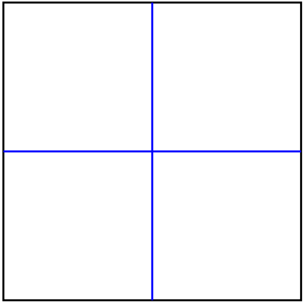
Inexact Additive Schwarz

Due to the ASM splitting $\mathbb{M} = \mathbb{V}_0 \oplus \sum_v \mathbb{V}_v$, solvers for $\mathbb{V}_0$ and $\mathbb{V}_v$ are required

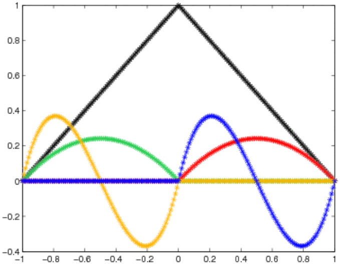
- Solvers for $\mathbb{V}_0$ are taken from the *h*-version (i.e. BPX-preconditioner)
- Solver for $\mathbb{V}_v$:
  1. direct solver applicable if $p \leq 7 \ldots 11$ $(\dim(\mathbb{V}_v) \leq 2000 \ldots 10000)$
  2. Special patch solver for $p \geq 7 \ldots 11$

Patch solver

Patch(2D)



1D-Functions ($p = 3$)



+ tensor product structure
- Mass matrix is weakly diagonal dominant

Patch Solver II

- Patch stiffness matrix (3D) can be expressed as

  $$K_{patch} = M_{1D} \otimes M_{1D} \otimes K_{1D} + M_{1D} \otimes K_{1D} \otimes M_{1D} + K_{1D} \otimes M_{1D} \otimes M_{1D},$$

  where K_{1D} and M_{1D} denote the stiffness and mass matrix on the 1D patch, respectively.
- Consider preconditioners for M_{1D} and K_{1D} so that
  - $c_{1,1} Q^{-\top} D_0 Q^{-1} \leq M_{1D} \leq c_{2,1} Q^{-\top} D_0 Q^{-1}$, ($L_2$-stability)
  - $c_{1,2} Q^{-\top} D_1 Q^{-1} \leq K_{1D} \leq c_{2,2} Q^{-\top} D_1 Q^{-1}$, ($H_1$-stability)
  - the operation $Q\underline{u}$ requires $\mathcal{O}(p)$ operations,
  - D_0 and D_1 are diagonal matrices.

Patch solver III

We have

$$K_{patch} = M_{1D} \otimes M_{1D} \otimes K_{1D} + M_{1D} \otimes K_{1D} \otimes M_{1D} + K_{1D} \otimes M_{1D} \otimes M_{1D}.$$

Using the 1D-preconditioners for M_{1D} and K_{1D}, one can conclude

$$\begin{aligned} K_{patch} \quad & \sim \quad Q^{-\top} D_0 Q^{-1} \otimes Q^{-\top} D_0 Q^{-1} \otimes Q^{-\top} D_1 Q^{-1} \\ & \quad + Q^{-\top} D_0 Q^{-1} \otimes Q^{-\top} D_1 Q^{-1} \otimes Q^{-\top} D_0 Q^{-1} \\ & \quad + Q^{-\top} D_1 Q^{-1} \otimes Q^{-\top} D_0 Q^{-1} \otimes Q^{-\top} D_0 Q^{-1} \\ & = \quad (Q \otimes Q \otimes Q)^{-\top} (D_0 \otimes D_0 \otimes D_1 + D_0 \otimes D_1 \otimes D_0 + D_1 \otimes D_0 \otimes D_0)^{-1} \\ & \quad (Q \otimes Q \otimes Q)^{-1}. \end{aligned}$$

This gives the preconditioner

$$C_{patch}^{-1} = (Q \otimes Q \otimes Q) D (Q \otimes Q \otimes Q)^{\top}$$

with the diagonal matrix

$$D = D_0 \otimes D_0 \otimes D_1 + D_0 \otimes D_1 \otimes D_0 + D_1 \otimes D_0 \otimes D_0.$$

Patch Solver IV

- Choose preconditioner of the form

$$C_{patch}^{-1} = (Q \otimes Q \otimes Q) D (Q \otimes Q \otimes Q)^\top$$

with a diagonal matrix D and a matrix Q such that $Q\underline{r}$ requires $\mathcal{O}(p)$ flops, cf. [BeuSchneiderSchwab04].

- The 1D preconditioners for M_{1D} and K_{1D} can be viewed as basis transformation into a stable basis
$\Psi = [\psi_1, \ldots, \psi_{2p-1}] = Q[\widehat{L}_1, \widehat{L}_{2,l}, \ldots, \widehat{L}_{p,l}, \widehat{L}_{2,r}, \ldots, \widehat{L}_{p,r}]$.

1D mass and 1D stiffness matrix

- Let M_1 and K_1 be the 1D mass and 1D stiffness matrix on one 1D element $(-1, 1)$ w.r.t. basis $\left\{ \widehat{L}_i \right\}_{i=1}^{p}$, respectively. Then,

$$4M_1 = \begin{bmatrix} & \ddots & & \ddots & & & & \vdots \\ & -\frac{2}{11} & \frac{2}{7}+\frac{2}{11} & -\frac{2}{7} & 0 & 0 & 0 & \dots \\ 0 & & -\frac{2}{7} & \frac{2}{7}+\frac{2}{3} & -\frac{2}{3} & 0 & 0 & \dots \\ & \dots & 0 & -\frac{2}{3} & \frac{8}{3} & -2 & 0 & \dots \\ 0 & & \dots & 0 & -2 & \frac{12}{5} & -\frac{2}{5} & \dots \\ 0 & & & \dots & 0 & -\frac{2}{5} & \frac{2}{9}+\frac{2}{5} & -\frac{2}{9} \\ \vdots & & & & & \ddots & & \ddots \end{bmatrix}$$

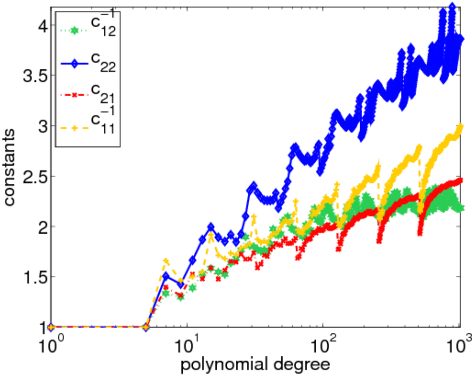
and

$$4K_1 = \operatorname{diag}\left[4p-2, 4p-10, \dots, 18, 10, 2, 6, 14, \dots, 4p-14, 4p-6\right]$$

- Consider h-FEM interpretations of the involved matrices,
- Use wavelets with block diagonal preconditioner for the boundary wavelet block of size $\log_2 p$ (Improves the constants)

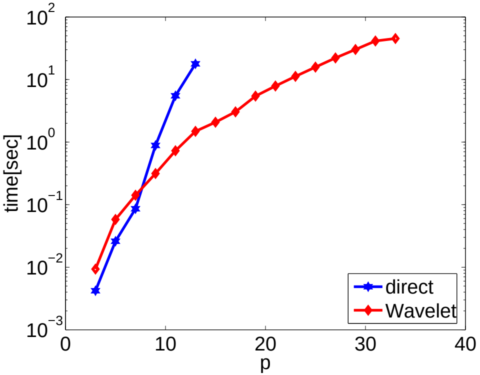
Quality of the 1D basis functions

- Eigenvalue bounds of the wavelet preconditioner for mass and stiffness matrix,
- one patch (two neighbouring elements 1d)
- Wavelet ψ_{22}, see [Beu09]



Comparison to direct solver

Solution time for different p, one patch of 8 cubes

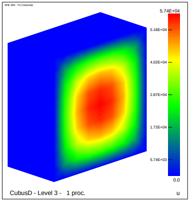


(rel. accuracy iterative solver is $10^{-10-p/2}$)

hp-preconditioner-Unit-cube

Solve

$$
\begin{aligned}
-\triangle u &= 10000 \quad \text{in } \Omega = (0,1)^3 \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$



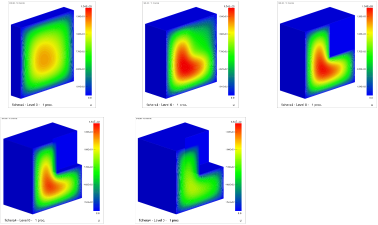
PCG-Iteration numbers ($\varepsilon = 10^{-5}$)

Lev vs. p	3	5	7	9	11	13	15
2	3	3	11	9	12	16	17
3	14	15	26	24	29	36	37
4	23	24	30	27	31	34	35
5	27	27	33				
6	28						

Fichera corner-uniform *hp*-FEM without preconditioner

Consider

$$-\triangle u = 10000 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$



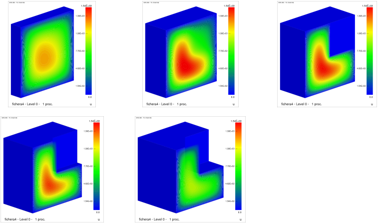
CG-Iteration numbers $\varepsilon = 10^{-5}$

Lev. vs. p	3	5	7	9	11	13
2	109	341	648	1160	1885	2689
3	146	455	862			
4	161					

hp-preconditioner-Fichera corner

Consider

$$
\begin{aligned}
-\triangle u &= 10000 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$



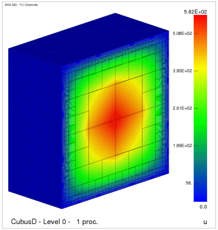
PCG-Iteration numbers using Pavarino+Wavelets/BPX ($\varepsilon = 10^{-5}$)

Lev vs. p	3	5	7	9	11	13	15	17	19
2	18	19	28	26	31	37	37	31	37
3	30	31	36	34	37	39	39		
4	35	35	39	37					
5	36	37							

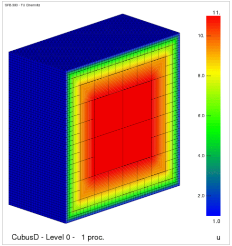
Example: unit square, BC-FEM

Consider

$$
\begin{aligned}
-\triangle u &= 10000 \quad \text{in } \Omega = (0,1)^3 \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$



Discretization with boundary concentrated FEM (incl. hanging nodes)
FE-mesh + polynomial degree



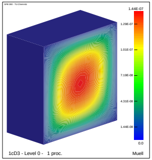
CG-Iteration numbers $\varepsilon = 10^{-5}$

Level	p_{max}	n_p	CG-It
2	1	27	2
3	3	249	15
4	5	5107	65
5	7	50793	99
6	9	337755	131
7	11	1806121	157

Unit cube, uniform *hp*-FEM without preconditioner

Consider

$$\begin{aligned} -\mu\triangle u-(\lambda+\mu)\nabla\cdot\nabla u &= 1 \quad \text{in}(0,1)^3 \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$



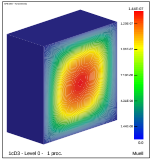
CG-Iteration numbers with $\varepsilon = 10^{-5}$

Lev. vs. p	1	3	5	7
2	2	53	149	250
3	8	97	217	344
4	16	129	105	428

hp-preconditioner-Unit-cube

Consider

$$\begin{aligned} -\mu \triangle u - (\lambda + \mu) \nabla \cdot \nabla u &= 1 \quad \text{in} (0,1)^3 \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$



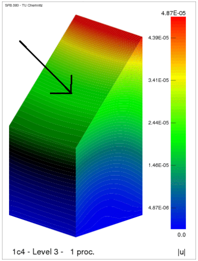
PCG-Iteration numbers using Pavarino+Wavelets/BPX with $\varepsilon = 10^{-5}$

Lev. vs. p	1	3	5	7	9	11	13	15	17	19
2	4	10	9	18	15	21	26	28	21	28
3	8	27	29	42	38	50	63	66	51	
4	16	28	28	41	36					
5	17	29	30							

Brick uniform *hp*-FEM without preconditioner

Consider

$$-\mu\triangle u - (\lambda+\mu)\nabla\cdot\nabla u = 0 \quad \text{in } \Omega$$
$$b.c. \quad \text{on } \partial\Omega$$



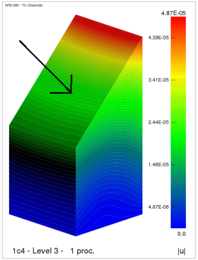
CG-Iteration with $\varepsilon = 10^{-5}$

Lev. vs. p	1	3	5	7	9
2	21	174	464	751	1349
3	48	284	801	1471	2129
4	101	413			

hp-preconditioner-Unit-cube

Consider

$$\begin{aligned} -\mu\triangle u - (\lambda+\mu)\nabla\cdot\nabla u &= 0 \quad \text{in } \Omega \\ b.c. & \quad\quad \text{on } \partial\Omega \end{aligned}$$



PCG-Iteration numbers using Pavarino+Wavelets/BPX with $\varepsilon = 10^{-5}$

Lev. vs. p	1	3	5	7	9
2	21	70	83	91	96
3	34	83	99	106	110
4	45	89	105	113	

Results:

- Wavelets
- Inexact tensor product based patch solver in 3D

To Do

- Implementation of the solver for more general problems (incl. hanging nodes) and Linear Elasticity

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