

Fast solvers for hp-FEM using hexahedral elements

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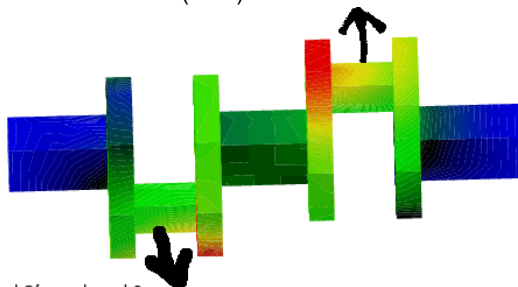
Crank shaft

Consider system of Lamé equations for displacement

$$u = [u_1 \quad u_2 \quad u_3]^T$$

$$-\mu \Delta u - (\lambda + \mu) \nabla \cdot \nabla u = f \quad \text{in } \Omega \quad (1)$$

of a crank shaft ($= \Omega$)



k3fren - Level 0 - 1 proc.

Formulation of the problem

- Solve

$$\begin{aligned}-\Delta u &= f, \\ u|_{\partial\Omega} &= 0,\end{aligned}$$

in a domain $\Omega \subset \mathbb{R}^d$.

- weak formulation: find $u \in H_0^1(\Omega)$ such that

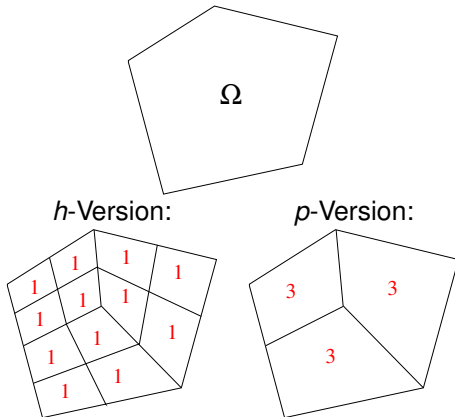
$$a_{\Delta}(u, v) = \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v \quad \forall v \in H_0^1(\Omega). \quad (2)$$

Finite elements

- Find a mesh of quadrilaterals/hexahedrals R_s , $s = 1, \dots, n_e$
- Mapping: $\Phi_s : \mathcal{R} \rightarrow R_s$ with $\mathcal{R} = (-1, 1)^d$, $d = 2, 3$
- Space: $\mathbb{M} = \{u \in H_0^1(\Omega), u|_{R_s} = \tilde{u}(\Phi_s^{-1}(x, y)), \tilde{u} \in \mathbb{Q}_p\}$, where \mathbb{Q}_p polynomials of degree $\leq p$ in each variable
- Discrete problem: Find $u_p \in \mathbb{M}$ with

$$a_\Delta(u_p, v_p) = \int_{\Omega} \nabla u_p \cdot \nabla v_p = \int_{\Omega} f v_p \quad \forall v_p \in \mathbb{M} \quad (3)$$

h -FEM vs. p -FEM



Local basis functions

- Basis on \mathcal{R} : tensor products of integrated Legendre polynomials:
 $\widehat{L}_{ijk}(x, y, z) = \widehat{L}_i(x)\widehat{L}_j(y)\widehat{L}_k(z)$ ($0 \leq i, j, k \leq p$), where

$$\widehat{L}_i(x) = \sqrt{\frac{(2i-3)(2i-1)(2i+1)}{4}} \int_{-1}^x L_{i-1}(s) ds$$

with

$$L_i(x) = \frac{1}{2^i i!} \frac{d^i}{dx^i} (x^2 - 1)^i$$

for $i \geq 2$ and

$$\widehat{L}_{0/1}(x) = \frac{1 \mp x}{2}$$

Finite element space

- Define now global basis functions with $\mathbb{M} = \text{span}\{\zeta_1, \dots, \zeta_{n_p}\}$ in the usual way
- Then, (3) is equivalent to solve

$$A\underline{x} = \underline{b},$$

where

$$A = [a_{\Delta}(\zeta_j, \zeta_i)]_{i,j=1}^{n_p}.$$

- + A is weakly populated \rightarrow iterative solvers
- A is ill-conditioned \rightarrow pre-conditioners C^{-1}

Topic of this talk

in this talk: Use tensor product arguments of
Beuchler/Schneider/Schwab in order to find preconditioners and
extension, i.e. investigate the 1D problem

1D mass and stiffness matrix

Let

$$M_{1,0} = \left[\langle \widehat{L}_i, \widehat{L}_j \rangle \right]_{i,j=2}^p$$

$$K_{1,0} = \left[\langle \widehat{L}'_i, \widehat{L}'_j \rangle \right]_{i,j=2}^p$$

be the 1D mass and 1D stiffness matrix w.r.t. basis $\{\widehat{L}_i\}_{i=2}^p$.
 Then, element stiffness matrix (in 2D) w.r.t. interior bubbles is

$$K_2 = M_{1,0} \otimes K_{1,0} + K_{1,0} \otimes M_{1,0}$$

Result of Beuchler/Schneider/Schwab 04

Theorem

There is a basis $\Psi = [\psi_1, \dots, \psi_{p-1}] = Q[\widehat{L}_2, \dots, \widehat{L}_p]$ which is stable in L_2 and H^1 , i.e. there are Q and diagonal matrices D_0 and D_1 with

- $c_{1,1}Q^{-T}D_0Q^{-1} \leq M_{1,0} \leq c_{2,1}Q^{-T}D_0Q^{-1}$, (L_2 -stability)
- $c_{1,2}Q^{-T}D_1Q^{-1} \leq K_{1,0} \leq c_{2,2}Q^{-T}D_1Q^{-1}$, (H_1 -stability)
- the operation Qu requires $\mathcal{O}(p)$ operations.

Remark

- the matrix Q can be constructed,

Wavelet preconditioner

Wavelet-Preconditioner for K_2 :

$$\begin{aligned} K_2 &= M_{1,0} \otimes K_{1,0} + K_{1,0} \otimes M_{1,0} \\ &\sim Q^{-T} D_0 Q^{-1} \otimes Q^{-T} D_1 Q^{-1} + Q^{-T} D_1 Q^{-1} \otimes Q^{-T} D_0 Q^{-1} \\ &= (Q^{-T} \otimes Q^{-T})(D_0 \otimes D_1 + D_1 \otimes D_0)(Q^{-1} \otimes Q^{-1}). \end{aligned}$$

This gives

$$C^{-1} = (Q \otimes Q)(D_0 \otimes D_1 + D_1 \otimes D_0)^{-1}(Q^T \otimes Q^T)$$

on each element,

(cf. Beuchler, Schneider, Schwab)

original result with $\log p$ -term, removed by a refined estimate in
 Beuchler/Braess

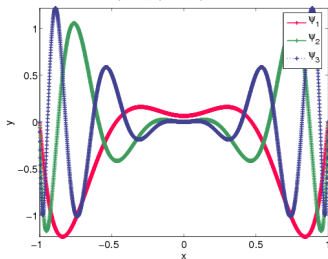
Construction of Q

- Compute the nonzero entries of $M_{1,0}$ and $K_{1,0}$,
- Ignore origin of $M_{1,0}$ and $K_{1,0}$ and interpret both matrices as h -FEM mass and stiffness matrix on $(0, 1)$,
- Use wavelets

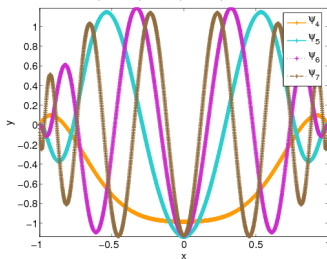
Plots of the p -wavelets

even shape functions (for $p = 15$):

ψ_1, ψ_2, ψ_3



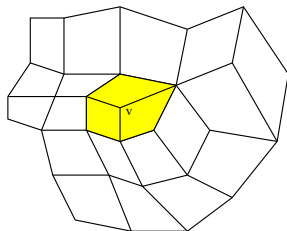
$\psi_4, \psi_5, \psi_6, \psi_7$



Overlapping preconditioner

Overlapping ASM Preconditioner of Pavarino requires

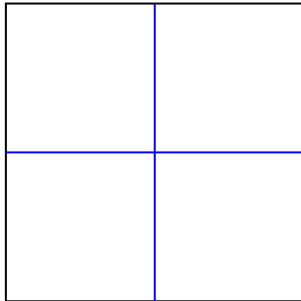
- solver for $p = 1$ required (h -version)
- for each node v : let p_v be the corresponding patch
- need solver on this patch if $p > 7 \dots 10$



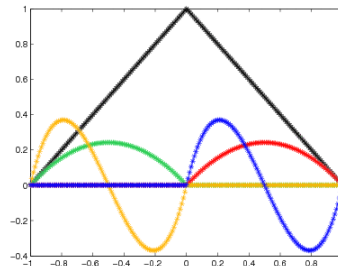
Consider patch solver for high p :

Patch solver

Patch



1D-Functions ($p = 3$)



- + tensor product structure
- Mass matrix is not diagonally dominant

1D mass and 1D stiffness matrix

- Consider new scaling of basis functions
- Let $M_{1,1}$ and $K_{1,1}$ be the 1D mass and 1D stiffness matrix w.r.t. basis $\{\tilde{L}_i\}_{i=0}^p$, respectively,
- Consider again h -FEM interpretations of the involved matrices,
- Use wavelets with block diagonal preconditioner for the boundary wavelet block of size $\log_2 p$ (Improves the constants)

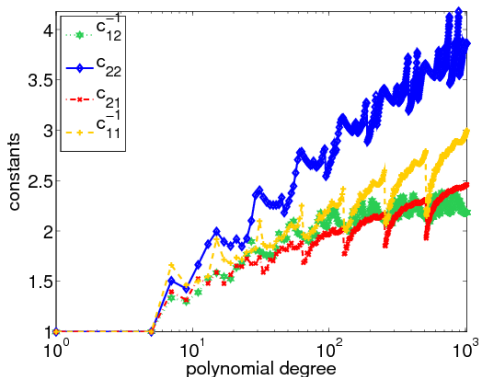
Comparison diagonal vs. block diagonal wavelet preconditioning, $p = 2^k$

Let Q_N be wavelet trafo

n	$Q_N^T M_{1,e} Q_N$				$Q_N^T (K_{1,e} + M_{1,e}) Q_N$			
	block-diagonal		diagonal		block-diagonal		diagonal	
	λ_{max}	λ_{min}^{-1}	λ_{max}	λ_{min}^{-1}	λ_{max}	λ_{min}^{-1}	λ_{max}	λ_{min}^{-1}
16	1.5	1.7	3.6	96	1.6	1.9	1.8	3.0
32	1.6	1.9	4.5	108	1.8	2.4	2.0	5.3
64	1.7	2.0	5.4	120	2.0	2.8	2.2	8.3
128	1.8	2.0	6.2	131	2.1	3.1	2.2	11.7
256	1.9	2.1	7.1	142	2.1	3.4	2.3	15.5
512	2.0	2.2	8.0	153	2.2	3.7	2.3	19.8

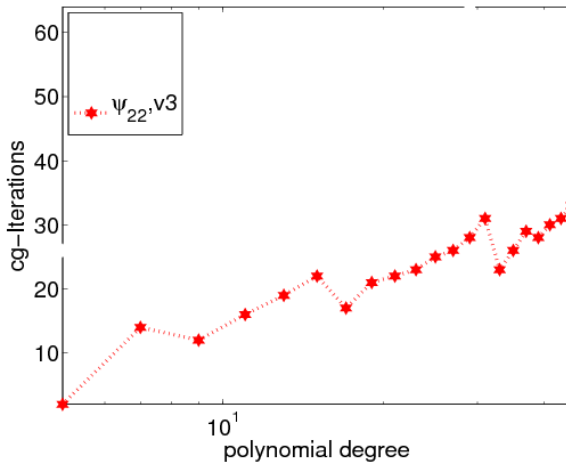
1D general p

- Eigenvalue bounds of the wavelet preconditioner for mass and stiffness matrix,
- one patch (two neighbouring elements 1d)
- Wavelet ψ_{22}



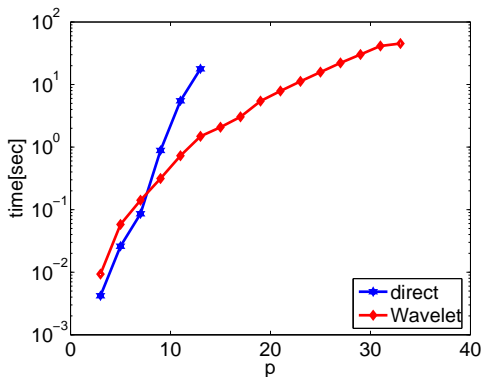
PCG- Iterations 3D, 1 patch of 8 cubes

relative accuracy 10^{-6}



Comparison to direct solver

Solution time for different p



(rel. accuracy iterative solver is $10^{-10-p/2}$)

Results:

- Generalization of the wavelets
- Inexact tensor product based patch solver in 2D and 3D

To Do

- Implementation in general 3D solver

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