

Abstract: A Quantitative Pólya's Theorem with Zeros

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1 Introduction

Let $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_n]$ and let $\mathbb{R}^+[X]$ denote polynomials in $\mathbb{R}[X]$ with non-negative coefficients.

Pólya's Theorem [5] says that if p is a homogeneous polynomial in n variables which is positive on the standard n -simplex, then for a sufficiently large exponent N , $(x_1 + \dots + x_n)^N p \in \mathbb{R}^+[X]$. In [6], the second and third authors gave an explicit bound for the exponent N in terms of the size of the coefficients and the minimum value of p on the simplex. This result has been used by other authors in applications; for example M. Schweighofer [8] used this quantitative Pólya's Theorem to give an algorithmic proof of Schmüdgen's Positivstellensatz, and de Klerk and Pasechnik [2] used it to give results on approximating the stability number of a graph.

This paper is part of an ongoing project, begun in [7], to understand exactly when Pólya's Theorem holds if the condition "positive on Δ_n " is relaxed to "non-negative on Δ_n ", and to give bounds in this case. Such results should have wide application as the original quantitative Pólya's Theorem did. In this work, we give a quantitative version of a result of M. Schweighofer which is a "localized" version of Pólya's Theorem. We then use this result to characterize, and give a bound for, forms which are positive on Δ_n apart from a zero at one vertex and satisfy the conclusion of Pólya's Theorem. This extends the results in [7].

D. Handelman [3, 4] has studied a related question, namely, for which pairs (q, f) of polynomials does there exist $N \in \mathbb{N}$ so that $q^N * f$ has nonnegative coefficients? (See also de Angelis and Tuncel [1].) The results in Section 3 without the bound can most likely can be deduced from Handelman's work.

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2 A localized Pólya's Theorem

In [9, Lemma 7], M. Schweighofer gives a condition which implies that the conclusion of Pólya's Theorem holds for (not necessarily homogeneous) $f \in \mathbb{R}[X]$. In this section, we give a computational version of this result. The idea is to find a representation of f , which depends on $x \in \Delta_n$, and which implies the conclusion of Pólya's Theorem for coefficients corresponding to X^α where $\frac{\alpha}{|\alpha|}$ is contained in a neighborhood around x . Our version of this result replaces neighborhoods of x by closed subsets of Δ_n containing x , which allows us to give an explicit bound for the exponent N needed.

We recall the notation of [6]. If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Suppose $p \in \mathbb{R}[X]$ is homogeneous of degree d , then write

$$p(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L(f) := \max_{|\alpha|=d} |b_\alpha|$.

Lemma 1. *Suppose $S \subseteq \Delta_n$ is closed and $p \in \mathbb{R}[X]$ is homogeneous of degree d such that $p(x) > 0$ for all $x \in S$. Let λ be the minimum of p on S . Then for*

$$N > \frac{d(d-1)}{2} \frac{L(f)}{\lambda} - d$$

and $\alpha \in \mathbb{N}^n$ such that $\frac{\alpha}{|\alpha|} \in S$, the coefficient of X^α in $(X_1 + \dots + X_n)^N p$ is nonnegative.

Proof. This follows from the proof of Theorem 1 in [6]. □

Proposition 1. *Given $p \in \mathbb{R}[X]$ and suppose there exists closed $S \subseteq \Delta_n$ such that there are homogeneous $g_1, \dots, g_m \in \mathbb{R}[X]$, and $h_1, \dots, h_m \in \mathbb{R}[X]^+$ with*

1. $p = g_1 h_1 + \dots + g_m h_m$
2. $g_i(x) > 0$ for all $x \in S$.

Suppose further that there exists a closed set T with $T \subseteq S \subseteq \Delta_n$ and $B \in \mathbb{N}$ with the following property: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \text{supp}(h_i)$ for some i , and $|\beta| \geq B$, then $\frac{\beta}{|\beta|} \in S$. Then there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, the coefficient of X^α in $(X_1 + \dots + X_n)^N p$ is nonnegative.

In particular, for each i , let $k(i)$ be the bound from Lemma 1 for g_i on S , i.e.,

$$k(i) = \frac{d_i(d_i-1)}{2} \frac{L(g_i)}{\lambda_i} - d_i,$$

where λ_i is the minimum of g_i on S and $d_i = \deg g_i$. Then we can take

$$N = \max\{k(g_1), \dots, k(g_m), B\}.$$

Outline of proof. Clearly, it suffices to show that for any $1 \leq j \leq m$, the coefficient of X^α in $(x_1 + \dots + x_n)^N g_j h_j$ is nonnegative. Suppose $\beta, \gamma \in \mathbb{N}^n$ are such that $\beta + \gamma = \alpha$ and the coefficients of X^β in $(X_1 + \dots + X_n)^N g_j$ and X^γ in h_j are non-zero. Since $h_j \in \mathbb{R}[X]^+$, the coefficient of X^γ in h_j is positive. Then since we have $|\beta| > N \geq B$ and $\alpha = \beta + \gamma$ for $\gamma \in \text{supp}(h_j)$, $\frac{\beta}{|\beta|} \in S$, hence by the choice of $k(j)$, it follows that the coefficient of X^β in $(X_1 + \dots + X_n)^{k(j)} g_j$ is nonnegative and we are done. \square

We now obtain Schweighofer's result [9, Lemma 7] as a corollary:

Corollary 1. *Let $f \in \mathbb{R}[X]$. Suppose that for every $x \in \Delta_n$ there are homogeneous $g_1, \dots, g_m \in \mathbb{R}[X]$, and $h_1, \dots, h_m \in \mathbb{R}[X]^+$ such that*

- 1) $f = g_1 h_1 + \dots + g_m h_m$
- 2) $g_i(x) > 0$ for $i = 1, \dots, m$

Then there exists $N \in \mathbb{N}$ such that the coefficients of $(X_1 + \dots + X_n)^N f$ are nonnegative.

Outline of proof. By continuity of the g_i 's and compactness, we can assume that we have a finite number of $B_{\epsilon_x}(x)$'s covering Δ_n so that for each x , we have g_i, h_i as above with $g_i(x) > 0$ for all $x \in B_{2\epsilon}(x)$. It is then enough to show that there is an $N \in \mathbb{N}$ such that the coefficients of X^α in $(X_1 + \dots + X_n)^N f$ for $\frac{\alpha}{|\alpha|} \in B_{\epsilon_x}(x)$ are nonnegative.

Fix $x \in \Delta_n$ and let $S = \overline{B_{2\epsilon_x}} \cap \Delta_n$ and $T = \overline{B_{\epsilon_x}(x)} \cap \Delta_n$. Clearly, S and T satisfy the conditions of the Proposition, and so we need only find $B \geq 1 \in \mathbb{N}$ so that the following property holds: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T, \beta + \gamma = \alpha, \gamma \in \text{supp}(h_i)$, for some i , and $|\beta| \geq B$, then $\frac{\beta}{|\beta|} \in S$. Let $M = \max\{\deg(h_i)\}$ and choose B large enough such that $\frac{2M}{B} \leq \epsilon_x$. Then a series of inequalities, exactly as in the proof of [9, Lemma 7] shows that the property holds. \square

3 Pólya's Theorem with zeros

If a nonzero form p satisfies the conclusion of Pólya's Theorem, then p cannot have interior zeros. Furthermore, if such a p is zero on the interior of a face of Δ_n , then p must be zero on the entire face, see [7, Section 3]. In this section, we apply Proposition 1 to give a quantitative version of Pólya's Theorem for forms which have a zero at a corner of Δ_n . This generalizes the main result from [7].

Write v_1, \dots, v_n for the vertices of Δ_n , i.e., $v_1 = (1, 0, \dots, 0), \dots, v_n = (0, \dots, 0, 1)$. For $r \in \mathbb{R}, 0 < r < 1$ and $j = 1, \dots, n$, let $\Delta_n(j, r)$ be the corner simplex with vertices

$$\{v_j\} \cup \{v_j + r(v_i - v_j) \mid i \neq j\} = (1-r)v_j + r\{v_i\}$$

In other words, $\Delta_n(j, r)$ is the scaled simplex $r \cdot \Delta_n$ translated by $(1-r)v_j$ and nestled in the v_j corner of Δ_n .

Lemma 2. Given $f = cx_1^e + \phi \in \mathbb{R}[X]$ where $c > 0$ and the degree of ϕ in x_1 is less than e . Then there are $r, s > 0$ so that $f \geq s$ on $\Delta_n(1, r)$.

In particular, suppose U is the sum of the absolute values of the coefficient of f , then this holds for

$$r = \frac{c}{c + 2U}, \quad s = \frac{c}{2} \left(\frac{c}{c + 2U} \right)^e$$

Proof. For $i = 2, \dots, n$, let $y_i = \frac{x_i}{x_1}$, then we have $f(x_1, \dots, x_n) = x_1^e f(1, y_2, \dots, y_n)$. Let r be as given and suppose $(x_1, \dots, x_n) \in \Delta_n(1, r)$. Then for each i , $y_i = \frac{x_i}{x_1} \leq \frac{r}{1-r} = \frac{c}{2U}$. Since the degree of ϕ in x_1 is less than e , $\phi(1, y_2, \dots, y_n)$ has no constant term and thus $|\phi(1, y_2, \dots, y_n)| \leq \left(\frac{c}{2U}\right)(U) = \frac{c}{2}$. Since $(x_1, \dots, x_n) \in \Delta_n(1, r)$, we have $x_1 \geq r$ and thus

$$\begin{aligned} f(1, y_2, \dots, y_n) &= x_1^e (c + \phi(1, y_2, \dots, y_n)) \\ &\geq \left(\frac{c}{c + 2U} \right)^e \left(c - \frac{c}{2} \right) =: s. \end{aligned}$$

□

Suppose p is positive on Δ_n except for a zero at one v_i ; for ease of exposition we may as well assume $i = 1$. Let $p_d(x_2, \dots, x_n)$ be the leading coefficient of p as a polynomial in x_1 and $e = \deg_{x_1} p$, so that

$$p = p_d(x_2, \dots, x_n)x_1^e + q(x_1, \dots, x_n)$$

where $\deg_{x_1} q < e$. Note that our assumptions on p imply that p_d is positive definite. Let $\mathbb{R}[\tilde{X}]$ denote $\mathbb{R}[x_2, \dots, x_n]$ and for $\alpha = (\alpha_2, \dots, \alpha_n) \in \mathbb{N}^{n-1}$, let \tilde{X}^α denote $x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Theorem 1. Given p as above, suppose that $\text{supp}(p_d)$ contains a multiple of every monomial in $\text{supp}(q)$. Then p satisfies the conclusion of Pólya's Theorem if and only if every coefficient in p_d is nonnegative.

In particular, suppose every coefficient of p_d is nonnegative. Then we can find an expression

$$p = \sum_{i=1}^m \tilde{X}^{\gamma_i} (c_i x_1^e + \phi_i),$$

with $\gamma_i \in \mathbb{N}^{n-1}$, $0 < c_i \in \mathbb{R}$, and $\phi_i \in \mathbb{R}[X]$ with $\deg_{x_1} \phi_i < e$. Let $c = \min_i \{c_i\}$ and U the sum of the absolute values of the coefficients of p . Now let r and s be as in Lemma 2 (using this c and U) and let λ be the minimum of p on the closure of $\Delta_n \setminus \Delta_n(1, r)$. Then for

$$N > \max \left\{ \frac{e(e-1)}{2} \frac{L(p)}{s}, \frac{d(d-1)}{2} \frac{L(p)}{\lambda} \right\},$$

all coefficients of $(x_1 + \cdots + x_n)^N p$ are nonnegative.

Outline of proof. If the conclusion of Pólya's Theorem holds, say with exponent N , then the coefficients of $x_1^{n-d+N} \cdot p_d$ are nonnegative which implies that the coefficients of p_d are nonnegative.

Now suppose that every coefficient of p_d is nonnegative and let $\text{supp}(p_d) = \{\gamma_1, \dots, \gamma_m\} \subseteq \mathbb{R}[\tilde{X}]$. Then we have $p = (\sum_{i=1}^m c_i \tilde{X}^{\gamma_i} x_1^e) + q$ for $q \in \mathbb{R}[X]$ and $c_i > 0$. Then, by assumption, we can write this as

$$p = \sum_{i=1}^m \tilde{X}^{\gamma_i} (c_i x_1^e + \phi_i),$$

for some $\phi_i \in \mathbb{R}[X]$ with $\deg_{x_1} \phi_i < e$. Set $g_i := c_i x_1^e + \phi_i$ and $h_i := \tilde{X}^{\gamma_i}$ and apply Lemma 2 to the g_i 's. Note that the coefficients of the g_i 's are a subset of the coefficients of p and hence we can use U for the bound in the lemma.

We now apply Proposition 1 to p with g_i, h_i as above, $S = T = \Delta_n(1, r)$, and $B = 1$. It is straightforward to check that the assumptions of the proposition hold in this case. Applying the proposition, and noting that the coefficients of each g_i are a subset of the coefficients of p , we can use the bound $\frac{e(e-1)}{2} \frac{L(p)}{s}$ for each g_i in Lemma 1.

Finally, we note that $p > 0$ on $\Delta_n \setminus \Delta_n(1, r)$ and hence we can apply Lemma 1 in the case where $\frac{\alpha}{|\alpha|} \in \Delta_n \setminus \Delta_n(1, r)$. Therefore we obtain the bound on N as given. \square

Remark 1. Suppose the assumption that $\text{supp}(p_d)$ contains a multiple of every monomial in $\text{supp}(q)$ doesn't hold. Also suppose that there is a monomial in $\text{supp}(q)$ which has a negative coefficient and is not divisible by any monomial in $\text{supp}(p_d)$. Then it is not too hard to see that the conclusion of Pólya's Theorem can't hold in this case.

We obtain as a corollary the main result in [7]. As in [7], we say that a form p which is nonnegative on Δ_n has a *simple zero* at v_j if the coefficient of x_j^d in p is zero, but the coefficient of $x_j^{d-1} x_i$ is non-zero (and necessarily positive) for each $i \neq j$. In other words, $\text{supp}(p)$ contains $(d-1) \cdot v_j + v_i$ for $i \neq j$, but not $d \cdot v_j$.

Corollary 2. *Suppose p is positive on Δ_n except for simple zeros at some v_j 's. Then Pólya's Theorem holds for p and there is a bound for the exponent N in terms of the degree and coefficients of p , and the minimum of p on Δ_n minus corner simplices $\Delta_n(j, r_j)$'s.*

Proof. If p has a simple zero at, say, v_1 , then in the notation of the theorem we have $e = d - 1$ and $p_d = x_2 + \dots + x_n$. Hence this follows immediately from the theorem. \square

Example 1. Consider

$$q(x_1, x_2, x_3, x_4) := x_1^2(x_2^2 + x_3^2 + x_4^2) - ax_1x_2x_3x_4 + x_2^4 + x_3^4 + x_4^4,$$

where $0 < a < 6$, then q is positive on Δ_n except for a zero at $(1, 0, 0, 0)$, however the term $-ax_1x_2x_3x_4$ is not divisible by any monomial in the leading coefficient of q as a polynomial in x_1 . It is not too hard to see that for any $N \in \mathbb{N}$, the coefficient of $x_1^{N+1}x_2x_3x_4$ will always be $-a$ and hence the conclusion of Pólya's Theorem does not hold in this case.

On the other hand, let $p = q + x_2^2x_2x_3$, then all assumptions of the theorem are satisfied. Indeed, we have

$$p = x_2^2(x_1^2 + x_2^2) + x_3^2(x_1^2 + x_3^2) + x_4^2(x_1^2 + x_4^2) + x_2x_3(x_1^2 - ax_1x_4)$$

and we can apply Lemma 2 to $x_1^2 - ax_1x_4$ and hence obtain an explicit N for which $(x_1 + x_2 + x_3 + x_4)^N p$ has nonnegative coefficients.

Remark 2. The theorem surely has a generalization to forms p which are positive on Δ_n except for zeros on a lower dimensional face F of Δ_n , replacing x_1^e by the leading form of p as a polynomial in the variables corresponding to the face F . The statement of such a generalization would be complicated, however for particular examples it should be possible to construct an explicit bound on the exponent N in Pólya's Theorem, using the ideas in Lemma 2 and the theorem. This will be the subject of the Ph.D thesis of the first author.

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