The Automatic Computation of the Complete Root Classification for a Parametric Polynomial

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Abstract

A new algorithm for the automatic computation of the complete root classification of a real parametric polynomial is described. The improvement lies in the fact that previous algorithms required 'revised sign lists' derived from the discriminant sequence. This is replaced here by the direct use of 'sign lists' derived from the discriminant sequence without the need for revision of the lists. As well as offering greater efficiency, the new algorithm offers a better filter for eliminating non-realizable conditions.

The new algorithm is applied to the problem of determining the conditions for the positive definiteness of a polynomial. For a class of sparse polynomials, the number of conditions remains surprisingly small.

Key words: Complete root classification, parametric polynomial, real quantifier elimination, root classification.

1. Introduction

Counting and classifying the roots of a polynomial is a well-established problem area; see Basu, Pollack & Roy (2003) for references. One approach that has been successful used in the past is based on the Complete Root Classification (CRC) of a polynomial, as defined in Yang, Hou & Zeng (1996) and Yang (1999). Algorithms suitable for manual computation were given by those authors, while algorithms suitable for automatic computation were described in Liang & Zhang (1999) and Liang & Jeffrey (2006). In this paper, a new algorithm which allows significant efficiency improvements is presented and its implementation in MAPLE is used to solve a series of problems in quantifier elimination, specifically in positive-definiteness testing. It is shown that for a class of sparse polynomials, remarkably compact conditions are possible.

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Complete root classifications are based on Root Classifications (RC). For a polynomial $p \in \mathbb{R}[x]$, a Root Classification is a list giving the number of distinct real roots, including multiplicities, and the number of complex conjugate pairs, including multiplicities. The roots themselves are not computed. For a polynomial with parametric coefficients, a Complete Root Classification (CRC) is a collection of all the possible root classifications, together with conditions on the polynomial coefficients such that the RC is realized. The CRC has been applied in studies of ordinary differential equations, of integral equations, of mechanics problems, and to real quantifier elimination; for references see Liang & Jeffrey (2006).

The CRC of a real parametric quartic polynomial was found by Arnon (1988), but the first method for establishing the CRC of a real parametric polynomial of any degree was given by Yang, Hou & Zeng (1996). They illustrated their method by computing the CRC of a reduced sextic polynomial. Liang & Zhang (1999) proposed and implemented an algorithm for the automatic generation of CRCs, and also extended the approach to complex parametric polynomials. Further improvements to the algorithm were proposed by Liang & Jeffrey (2006).

This paper presents several advances on the above works. The automatic generation of CRCs is made more efficient, and new methods are used to filter extraneous cases from the results. In addition, the algorithm of Yang, Hou & Zeng (1996) defines a 'discrimination matrix' and a 'discriminant sequence' which are close to, but not identical with, similar quantities used elsewhere in real algebraic geometry. Here we connect the method of Yang, Hou & Zeng (1996) with standard definitions. For example, we relate their discriminant sequence to the signed subresultant coefficient sequence. These identifications help with the more efficient algorithm just referred to.

The new algorithm has been programmed in MAPLE, and as an application, some well-known benchmark problems are considered: the positive definiteness of polynomials. However, it should be emphasized that the CRC of a polynomial contains more information than is needed for these problems, and consequently it has more potential applications than the examples given here. As with many symbolic calculations, solving general cases of high degree polynomials will introduce large numbers of unknowns and consequently very large solutions. Therefore we have concentrated on a class of sparse polynomials, for which surprisingly compact results are possible.

2. Previous work

In this section, we review the existing algorithms due to Yang, Hou & Zeng (1996) and Liang & Jeffrey (2006). For convenience, we recall some definitions.

2.1. Definitions

Definition 1. Let $p \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. The $2n \times 2n$ matrix

$$M = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_0 & & & & \\ 0 & na_n & (n-1)a_{n-1} & \dots & a_1 & & & & \\ & a_n & a_{n-1} & \dots & a_1 & a_0 & & & \\ & 0 & na_n & \dots & 2a_2 & a_1 & & & \\ & & & \vdots & \vdots & & & & \\ & & & a_n & a_{n-1} & a_{n-2} & \dots & a_0 \\ & & & 0 & na_n & (n-1)a_{n-1} & \dots & a_1 \end{pmatrix} \quad (1)$$

is called the discrimination matrix of p.

Definition 2. For $1 \le k \le 2n$, let M_k be the kth principal minor of M, and let $D_k = M_{2k}$. The n-tuple $D = [D_1, D_2, \dots, D_n]$ is called the discriminant sequence of p.

Definition 3. If $\operatorname{sgn} x$ is the signum function, with $\operatorname{sgn} 0 = 0$, then the list

$$[\operatorname{sgn} D_1, \operatorname{sgn} D_2, \dots, \operatorname{sgn} D_n]$$

is called the sign list of p.

Definition 4. The revised sign list $[e_1, e_2, \dots, e_n]$ of p is constructed from the sign list $[s_1, s_2, \dots, s_n]$ of p as follows. If $[s_i, s_{i+1}, \dots, s_{i+j}]$ is a section of the given list, where $s_i \ne 0$, $s_{i+1} = s_{i+2} = \dots = s_{i+j-1} = 0$ and $s_{i+j} \ne 0$, then we replace the subsection $[s_{i+1}, \dots, s_{i+j-1}]$ by $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \dots]$.

The revised sign list of p is denoted $\operatorname{rsl}(p)$.

Definition 5. Let $\Delta(p)$ denote $\gcd(p(x), p'(x))$. Let $\Delta^0(p) = p(x)$, $\Delta^j(p) = \Delta(\Delta^{j-1}(p))$, for $j = 1, 2, \dots$. Then $\Delta^0(p), \Delta^1(p), \Delta^2(p), \dots$ is called the Δ-sequence of p.

2.2. Polynomials with Known Coefficients

For polynomials $p \in \mathbb{R}[x]$, Yang, Hou & Zeng (1996) gave an algorithm for obtaining the Root Classification (RC) based on the following theorem.

Theorem 6. *Suppose a polynomial $p \in \mathbb{R}[x]$ has revised sign list $\operatorname{rsl}(p)$, and Δ-sequence $\Delta^j(p), j = 1, 2, \dots$. If the number of non-vanishing members of $\operatorname{rsl}(p)$ is s, and the number of sign changes in $\operatorname{rsl}(p)$ is v, then $p(x)$ has v pairs of distinct complex-conjugate roots and $s-2v$ distinct real roots. Further, if $\Delta^j(p)$ has k distinct roots with respective multiplicities $n_1, n_2, \dots, n_k$, and $\Delta^{j-1}(p)$ has m distinct roots, then $m \ge k$, and the multiplicities of these m distinct roots are $n_1 + 1, n_2 + 1, \dots, n_k + 1, 1, \dots, 1$ respectively.*

Proof. Proofs were given in Yang, Hou & Zeng (1996) and Gantmacher (1959). □

The algorithm uses this theorem to count the number of roots of p, and then if necessary counts the roots of each relevant $\Delta^j(p)$ (until there are no multiplicities). When a polynomial has symbolic coefficients, however, the algorithm must be modified.

2.3. *Polynomial with Symbolic Coefficients*

For a parametric polynomial $p \in \mathbb{R}[a_0, a_1, \ldots, a_n][x]$, the published descriptions of the algorithm are very brief. On this point Yang (1999) says only

> "Using Theorem 3.3 [in Yang (1999)], by a thorough and detailed analysis on the sign list of the sequence ... we conclude that ... ".

From this we have concluded that the algorithm they use for a parametric polynomial proceeds by examining each possible case of the sign sequence; for each case, a revised sign list is constructed and finally the root classification (without multiplicities) is found. If there are multiplicities, then Definition 5 must be replaced, because if a standard GCD function is applied to a parametric polynomial the function will mostly conclude $\gcd(p, p') = 1$. However, the $\gcd(p, p')$ might not equal 1 because of the relations between the coefficients implied by particular cases of the sign sequence.

Definition 7. The Multiple Factor Sequence $\Delta_0(p), \Delta_1(p), \ldots, \Delta_{n-1}(p)$ of p is defined using M, the discrimination matrix defined in (1). Let M_{ki} denote the submatrix defined using Matlab matrix notation as `[1:2k;1:2k-1,2k+i]`, then

$$\Delta_k(p) = \sum_{i=0}^{k} \det M_{n-k,i} x^{k-i}$$

Lemma 8. *If* $\mathrm{rsl}(p)$ *contains* k *zeros, then* $\Delta_k(p) = \gcd(p(x), p'(x))$.

Proof. See Yang (1999). □

2.4. *An Example*

The following example will be used at several places in the exposition below.

Example 9. We consider the real parametric polynomial

$$p(x) = x^6 + ax^2 + bx + c \ . \tag{2}$$

Its sign list is

$$[1, 0, 0, \operatorname{sgn} D_4, \operatorname{sgn} D_5, \operatorname{sgn} D_6] \tag{3}$$

where

$$D_4 = a^3 \ , \qquad D_5 = 256a^5 + 1728a^2c^2 - 5400ab^2c + 1875b^4 \ ,$$
$$D_6 = -1024a^6c + 256a^5b^2 - 13824a^3c^3 + 43200a^2b^2c^2 - 22500ab^4c + 3125b^6 - 46656c^5 \ .$$

Since each of D_4, D_5, D_6 can be positive, zero or negative, there are 27 possible realizations of this sign list to examine. Let us take one example sign list: consider the case $D_4 < 0, D_5 = 0, D_6 = 0$. For this case, the sign list becomes $[1, 0, 0, -1, 0, 0]$ and $\mathrm{rsl}(p)$ becomes $[1, -1, -1, -1, 0, 0]$. Then by Theorem 6, $p(x)$ has one pair of distinct complex conjugate roots and two distinct real roots. For finding the multiplicities, the GCD is obtained from the multiple factor sequence as $\Delta^1(p) = \Delta_2(p) = 4a^3x^2 + 5a^2bx + 6a^2c$. The discriminant sequence for this polynomial is $[1, E_2]$, where $E_2 = 25a^4b^2 - 96a^5c$. If

$E_2 > 0$, then again using Theorem 6, $\Delta^1(p)$ has two distinct real roots. Therefore, we conclude that for this case $p(x)$ has one pair of complex conjugate roots of multiplicity 1 and two real roots of multiplicities 2. If $E_2 < 0$, then $\Delta^1(p)$ has a pair of complex conjugate roots. So we conclude that for this case $p(x)$ has one pair of complex conjugate roots of multiplicity 2 and two real roots of multiplicities 1. Similarly, if $E_2 = 0$, then $p(x)$ has one pair of complex conjugate roots of multiplicity 1 and two real roots of respective multiplicities 1 and 3. This analysis would be repeated for each case.

There are 27 possible cases of the sign list in this example, but we show below that there are only 10 cases of RC in the CRC (further details of this problem are given below as Example 25). Therefore, there remains the work of condensing the 27 cases of sign lists into just 10 RCs. This is done by *ad hoc* analysis.

2.5. Automatic Computation of CRC

An algorithm for the automatic computation of CRCs was described in Liang & Zhang (1999) and Liang & Jeffrey (2006). Although based on the theorems above, it followed a different direction. We first notice the following facts. For a general parametric polynomial of degree n (i.e. one in which all coefficients are present and symbolic), the initial number of cases in its sign list that must be examined is 3^{n-1} (the first entry of a sign list is always equivalent to 1). Some of these cases will have subcases, but we do not need to estimate their number. In contrast, the number of entries in the CRC of a polynomial of degree n is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \mathfrak{P}(n-2k)\,\mathfrak{P}(k) < \exp\left(\pi\sqrt{2n/3}\right)/4\sqrt{3} .$$

where $\mathfrak{P}(k)$ denotes the number of partitions of the integer k. The upper bound uses the well-known asymptotic result for $\mathfrak{P}$.

Not only is the number of members in a CRC less than the number of cases of a sign list, but in many applications, not all members of a CRC are needed. For example, in the applications to positive-definiteness given below, only those RCs in a CRC with no real roots are needed. Therefore it is more efficient computationally to approach the calculations differently. The approach of Liang & Zhang (1999) and Liang & Jeffrey (2006) starts by generating all required RC members of the CRC. Then for each RC, the conditions that the coefficients of the polynomial should satisfy were found.

2.6. Need for Improvement

We identify the points in the existing algorithms where improvements are made here. The first point is the use of revised sign lists. This is a major source of inefficiency, because conditions expressed in terms of revised sign lists have until now been transferred to conditions in terms of sign lists manually. Definition 4 is equivalent to defining a mapping Φ from a sign list to a revised sign list. Therefore the previous algorithms required the inverse mapping Φ^{-1}. However, Φ is not injective, and therefore Φ^{-1} is multivalued, and more importantly is difficult to compute.

As an example, consider the polynomial $p_6 := x^6 + ax^2 + bx + c$, whose discriminant sequence was given above. One condition (among many) for $p6$ having no real roots is that its revised sign list be $[1, -1, -1, 1, -1, -1]$. According to the special structure of the discriminant sequence of p_6 given in (3), we have

$$\Phi^{-1}[1, -1, -1, 1, -1, -1] = \{[1, 0, 0, 1, -1, -1], [1, 0, 0, 1, 0, -1], [1, 0, 0, 0, -1, -1]\} .$$

Therefore, the given condition is transferred to the following:

$$[D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0] \vee [D_4 > 0 \wedge D_5 = 0 \wedge D_6 < 0] \vee [D_4 = 0 \wedge D_5 < 0 \wedge D_6 < 0].$$

This case, already cumbersome, is none the less relatively simple because of the nature of the polynomial. However, if the polynomial were a general parametric polynomial, it would be very difficult to find $\Phi^{-1}[1, -1, -1, 1, -1, -1]$, and of course more so for higher degrees. Consequently, it would be a great improvement to avoid revised sign lists.

The second point concerns the realizability of the conditions obtained by the inverse mapping Φ^{-1}. We continue with the example of $x^6 + ax^2 + bx + c$.

Example 10. For the case of no real roots, one condition given above is

$$D_4 = 0 \wedge D_5 < 0 \wedge D_6 < 0 \; , \tag{4}$$

with the D_k given in (3).

We assert that this condition is not realizable. Since $D_4 = 0$, then $a = 0$, and then $D_5 = 1875b^4 \geq 0$. This is a contradiction. So non-realizable conditions are included in the output of the existing algorithm, and no mechanism was offered to detect them.

In summary, although the old CRC algorithms give correct results, the computations are difficult because we have to transfer conditions on revised sign lists to conditions on sign lists, and the results may contain non-realizable conditions. These weaknesses provide the motivation of the current paper.

3. Basic Definitions and Theorems

We begin this section by connecting the discriminant sequence in Definition 2 to the signed subresultant coefficient sequence defined in Basu, Pollack & Roy (2003).

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ be real polynomials with $n = \deg(p) > m = \deg(q)$.

Definition 11 (Sylvester-Habicht matrix)**.** For $0 \leq j \leq m$, the jth Sylvester-Habicht matrix $\mathrm{SyHa}_j(p, q)$ of $p(x)$ and $q(x)$ is the $(n+m-2j)$-by-$(n+m-j)$ matrix whose rows are $x^{m-j-1}p, \ldots, xp, p, q, xq, \ldots, x^{n-j-1}q$ expressed in the basis $x^{n+m-j-1}, \ldots, x, 1$.

$$\mathrm{SyHa}_j(p,q) = \begin{pmatrix} a_n & \cdots & \cdots & \cdots & \cdots & a_0 & 0 & 0 \\ 0 & \ddots & & & & & \ddots & 0 \\ \vdots & \ddots & a_n & \cdots & \cdots & \cdots & \cdots & a_0 \\ \vdots & & 0 & b_m & \cdots & \cdots & \cdots & b_0 \\ \vdots & \iddots & \iddots & & & & \iddots & 0 \\ 0 & \iddots & & & & \iddots & \iddots & \vdots \\ b_m & \cdots & \cdots & \cdots & b_0 & 0 & \cdots & 0 \end{pmatrix}.$$

Definition 12 (Signed subresultant coefficient)**.** The jth signed subresultant coefficient $\mathrm{sRes}_j(p, q)$ is the determinant of $\mathrm{SyHa}_{j,j}(p, q)$ obtained by taking the first $n + m - 2j$ columns of $\mathrm{SyHa}_j(p, q)$. By convention, we extend these definitions for $m < j < n$ by $\mathrm{sRes}_j(p, q) = 0$ and $\mathrm{sRes}_n(p, q) = \operatorname{sgn} a_n$.

We can now connect the different definitions.

Lemma 13. *Let $D = [D_1, \ldots, D_n]$ be the discriminant sequence of $p(x)$. Then for $1 \leq j \leq n$, $D_j = a_n \cdot \mathrm{sRes}_{n-j}(p, p')$.*

Proof. For $1 \leq j \leq n$, the $(n-j)$th Sylvester-Habicht matrix of p and p', $\mathrm{SyHa}_{n-j}(p, p')$, has size $(2j-1) \times (n+j-1)$ and its rows are $x^{j-2}p, \ldots, p, p', \ldots, x^{j-1}p'$ considered as vectors in the basis $x^{n+j-2}, \ldots, x, 1$. Transform $\mathrm{SyHa}_{n-j}(p, p')$ into a matrix $\widetilde{M}$ by the following $j-1$ steps:

(1) Set $\widetilde{M} = \mathrm{SyHa}_{n-j}(p, p')$ and $k = 1$.
(2) Move the last row of $\widetilde{M}$ to row $2k-1$ by $2(j-k)$ successive swaps and reset $\widetilde{M}$ to the new matrix and increment k.
(3) Repeat step 2 until $k = j-1$.

The final matrix $\widetilde{M}$ can also be obtained by taking the second to $(2j)$th rows of the discrimination matrix M (see Definition 1) of p, and then deleting the first column of the obtained matrix. Since each step uses an even number of swaps, the result follows. □

Remark 14. In Yang, Zhang & Hou (1996, § 49), it is pointed out that the discriminant sequence can also be obtained from a Bézout matrix. The matrix called Bézout by Yang, Zhang & Hou (1996) is different from the Cayley form implemented in Maple, but the differences are no longer important, since the signed subresultant coefficient is a more standard way of presenting the material. By Lemma 13, the multiple factor sequence of p and the signed subresultant sequence of p and p' are the same up to a constant factor. Therefore, the discriminant sequence and the multiple factor sequence of p can be obtained by computing the signed subresultant coefficient sequence and the signed subresultant sequence of p and p'. These can be done by Algorithm 8.21 in Basu, Pollack & Roy (2003).

We now continue to the basis of the new algorithm. The following definitions come from Basu, Pollack & Roy (2003).

Definition 15 (TaQ)**.** Let $p(x), q(x)$ be two polynomials in $\mathbb{R}[x]$. The Tarski Query of q for p in $\mathbb{R}$ is the number

$$\mathrm{TaQ}(q, p) = \#\{x \in \mathbb{R} | p(x) = 0 \wedge q(x) > 0\} - \#\{x \in \mathbb{R} | p(x) = 0 \wedge q(x) < 0\} .$$

Definition 16 (PmV)**.** Let $s = [s_n, \ldots, s_0]$ be a finite list of elements in $\mathbb{R}$ such that $s_n \neq 0$. Let $m < n$ such that $s_{n-1} = \cdots = s_{m+1} = 0$, and $s_m \neq 0$, and $s' = s_m, \ldots, s_0$. If there exists no such m, then s' is the empty list. We define inductively

$$\mathrm{PmV}(s) = \begin{cases} 0 & \text{if } s' = \emptyset, \\ \mathrm{PmV}(s') + \epsilon_{n-m} \operatorname{sgn}(s_n s_m) & \text{if } n-m \text{ odd}, \\ \mathrm{PmV}(s') & \text{if } n-m \text{ even}. \end{cases}$$

Where $\epsilon_{n-m} = (-1)^{(n-m)(n-m-1)/2}$.

The main theorem for the improved CRC algorithm requires the following lemmas. The first two lemmas are from Basu, Pollack & Roy (2003), where they are Proposition 2.51 and Theorem 4.30 respectively. Let $\mathrm{Ind}(q/p)$ be the Cauchy index of q/p on $\mathbb{R}$.

Lemma 17. *Given two polynomials $p(x), q(x)$ in $\mathbb{R}[x]$, we have* $\mathrm{TaQ}(q,p) = \mathrm{Ind}(p'q/p)$.

Lemma 18. *Let $p(x), q(x)$ be the two polynomials in Definition 11. We have*

$$\mathrm{PmV}([\mathrm{sRes}_n(p,q), \mathrm{sRes}_{n-1}(p,q), \ldots, \mathrm{sRes}_0(p,q)]) = \mathrm{Ind}(q/p) \ .$$

Lemma 19. *Let $D = [D_1, \ldots, D_n]$ be the discriminant sequence of $p(x)$. Then*

$$\#\{x \in \mathbb{R} | p(x) = 0\} = \mathrm{PmV}([D_1, \ldots, D_n]) + 1 \ .$$

Proof. Observe that $\#\{x \in \mathbb{R} | p(x) = 0\} = \mathrm{TaQ}(1,p)$. Then from Lemma 17, we have $\mathrm{TaQ}(1,p) = \mathrm{Ind}(p'/p)$. By Lemma 18,

$$\mathrm{Ind}(p'/p) = \mathrm{PmV}([\mathrm{sRes}_n(p,p'), \mathrm{sRes}_{n-1}(p,p'), \ldots, \mathrm{sRes}_0(p,p')]) \ .$$

By Lemma 13,

$$\begin{aligned}&\mathrm{PmV}([\mathrm{sRes}_n(p,p'), \mathrm{sRes}_{n-1}(p,p'), \ldots, \mathrm{sRes}_0(p,p')]) \\ &\quad = \mathrm{PmV}([\mathrm{sgn}(a_n), D_1/a_n, \ldots, D_n/a_n]) = 1 + \mathrm{PmV}([D_1/a_n, \ldots, D_n/a_n]) \ ,\end{aligned}$$

since $\mathrm{sgn}(a_n)$ and $D_1/a_n = na_n$ have the same sign. Finally,

$$1 + \mathrm{PmV}([D_1/a_n, \ldots, D_n/a_n]) = 1 + \mathrm{PmV}([D_1, \ldots, D_n]) \ .$$

□

The main theorem is now

Theorem 20. *Let $D = [D_1, \ldots, D_n]$ be the discriminant sequence of a real polynomial $p(x)$ of degree n, and let ℓ be the maximal subscript such that $D_\ell \neq 0$. If $\mathrm{PmV}(D) = r$, then $p(x)$ has $r+1$ distinct real roots and $\frac{1}{2}(\ell - r - 1)$ pairs of distinct complex conjugate roots.*

Proof. By Lemma 8, we have $\deg(\gcd(p,p')) = n - \ell$. Therefore, $p(x)$ has ℓ distinct roots. Then by Lemma 19, we obtain the desired results. □

From Theorem 20, we obtain the following important corollary, which is necessary for detecting the non-realizable sign lists in the output conditions.

Corollary 21. *Let $S = [s_1, \ldots, s_n]$ and $R = [r_1, \ldots, r_n]$ be the sign list and the revised sign list of $p(x)$ respectively. Then* $\mathrm{PmV}(S) = \mathrm{PmV}(R)$.

Proof. Let ℓ be the maximal subscript of S such that $s_\ell \neq 0$. Let m and ν be the number of sign permanences and the number of sign changes of R. By Theorem 20, we have $\mathrm{PmV}(S) = \#\{x \in \mathbb{R} | p(x) = 0\} - 1$. By Theorem 6, $\#\{x \in \mathbb{R} | p(x) = 0\} = \ell - 2\nu$. So

$$\mathrm{PmV}(S) = \ell - 2\nu - 1 \ . \tag{5}$$

On the other hand, it is easy to see that $\mathrm{PmV}(R) = m - \nu$ and $\ell - 1 = m + \nu$. So

$$2\nu = \ell - 1 - \mathrm{PmV}(R) \ . \tag{6}$$

Therefore, from (5) and (6), we get $\mathrm{PmV}(S) = \ell - (\ell - 1 - \mathrm{PmV}(R)) - 1 = \mathrm{PmV}(R)$. □

Example 22. We give an example of the use of the above corollary, by proving the non-realizability of condition (4) from a different point of view. The condition is equivalent to the sign list $[1, 0, 0, 0, -1, -1]$, which has the revised sign list $[1, -1, -1, 1, -1, -1]$. Since

$$\mathrm{PmV}([1, 0, 0, 0, -1, -1]) = 1 \neq -1 = \mathrm{PmV}([1, -1, -1, 1, -1, -1]),$$

then Corollary 21 states that the sign list $[1, 0, 0, 0, -1, -1]$ is not realizable for p_6. Similarly we can prove that the sign list $[1, 0, 0, -1, 0, -1]$ is not realizable for p_6, because

$$\mathrm{PmV}([1, 0, 0, -1, 0, -1]) = 1 \neq -1 = \mathrm{PmV}([1, -1, -1, -1, 1, -1]) \ .$$

Observe that, by Theorem 6, $\mathrm{rsl}(p_6) = [1, -1, -1, -1, 1, -1]$ is one condition for p_6 having no real roots, and $[1, 0, 0, -1, 0, -1] \in \Phi^{-1}[1, -1, -1, -1, 1, -1]$.

4. An Improved CRC Algorithm

We use the following notation. A Root Classification (RC) is denoted by the list L

$$L = [L_1, L_2] = [\,[n_1, n_2, \ldots], [m_1, -m_1, m_2, -m_2, \ldots]\,] \ , \tag{7}$$

where the $n_k \in \mathbb{Z}$ are the multiplicities of the real roots, and the $m_k \in \mathbb{Z}$ are the multiplicities of complex conjugate pairs of roots. The entries $-m_k$ in L_2 are mnemonic reminders that pairs of roots are being described.

Our aim is to find, for each RC of a real polynomial $p(x)$, the conditions on its coefficients that correspond to the RC. The algorithm therefore starts from a list $\mathcal{L}$ of RCs. If a CRC is required (i.e. complete information) then all possible RCs are included in $\mathcal{L}$ (using, for example, the Maple command `combinat:-partition`. See Liang & Jeffrey (2006) for details), otherwise, a partial list of RCs can be selected for $\mathcal{L}$. In what follows, we use the functions in our Maple implementation of the algorithm. This makes the presentation simpler and allows the text to be related to the code (which, however, is not given).

At each stage of the algorithm, we compute Sign Lists (SLs) by filtering a SL-realization, obtained using the following function.

Algorithm 1. SLReal
Input: A real polynomial $g(x)$ with parametric coefficients.
Output: A set of sign lists.
Procedure:

- Compute the discriminant sequence $D = [D_1, \ldots]$ for $g(x)$.
- If $g(x)$ contains 3 or fewer parameters, find a sample of the D-invariant decomposition for the parametric space by partial CAD (Collins & Hong, 1991), then substitute the sample points into D one by one.
- If $g(x)$ contains more than 3 parameters, assign the values -1, 0 and 1 to each D_k in D that contains symbols.

To filter the output of `SLReal`, we first define the function `RCInfo`. Let L be the RC of a polynomial $g(x)$. Then `RCInfo`(L) is the list $[d, \ell, r]$, the degree of $g(x)$, the number of distinct roots and the number of distinct real roots specified by the RC. In the notation of (7),

$$d = \sum n_i + 2 \sum m_j \ , \quad \ell = \#L_1 + \#L_2 \ , \quad r = \#L_1 \ .$$

The filtering of an SL-realization is done by `GenAllSL` and is the key part of the improved algorithm. It is based on Theorem 20 and Corollary 21.

Algorithm 2. GenAllSL.
Input: A real parametric polynomial $g(x)$ and a root classification L.
Output: All sign lists of $g(x)$ corresponding to L.
Procedure:

- Compute $[d, \ell, r] = \texttt{RCInfo}(L)$.
- Compute $\texttt{SLReal}(g(x))$.
- From the output of `SLReal`, collect those lists for which the maximal subscript of non-vanishing members is ℓ, the PmV is $r-1$ and the PmV of the revised sign list is $r-1$.

Remark 23. Compared with the corresponding algorithm *GenRealRSL* in the old CRC algorithm, `GenAllSL` has four advantages. First, it uses sign lists instead of revised sign lists, which makes the computation much easier. Secondly, it uses Corollary 21 to detect and delete non-realizable conditions. Thirdly, it takes advantage of any sparsity in $g(x)$. For example, the polynomial $x^{10} + ax^2 + bx + c$ in Example 28 has the discriminant sequence $D = [1, 0, 0, 0, 0, 0, 0, a^7, D_9, D_{10}]$. By using its sparsity, one only needs to consider $3^3 = 27$ sign lists, while one has to consider $3^9 = 19683$ sign lists initially otherwise. Notice, finally, that if the output of `GenAllSL` is an empty sequence, then the corresponding RC is not realizable.

Only in the following cases will one application of `GenAllSL` be sufficient for finding all the conditions on the coefficients of a polynomial. If the RC of a polynomial $p(x)$ is L and is such that $[d, \ell, r] = \texttt{RCInfo}(L)$, then the cases are

(1) $d = \ell$,
(2) $\ell = 1$.
(3) $\ell = 2$ and $r = 0$.
(4) $d - \ell = 1$.
(5) $r = 0$ and $d - \ell = 2$.

For other cases, by Lemma 8, $\Delta^1(p) = \Delta_{d-\ell}(p)$, and by Proposition 7 in Liang & Jeffrey (2006), the RC of $\Delta^1(p)$ should be $L_1 = \texttt{MinusOne}(L)$, where the function `MinusOne` takes a root classification L and returns an RC generated from L by decreasing the absolute values of all numbers in L by 1, and then erasing all numbers of value 0, and keeping empty lists unchanged. We can then call `GenAllSL` recursively. This is the basis of the following function which generates the conditions for $p(x)$ having L as its root classification.

Algorithm 3. Cond
Input: a real parametric polynomial $p(x)$; an RC L.
Output: a sequence of mixed lists. Each list consists of a polynomial in the Δ-sequence of $p(x)$, followed by its all possible sign lists (if the output is NULL, then L is not realizable).
Procedure:
$sl := \texttt{GenAllSL}(p(x), L)$
if $sl = \emptyset$
    return NULL
else if $[d, \ell, r]$ meets one of the five cases
    return $[p, sl]$
else
    $C := \texttt{Cond}(\Delta^1(p), \texttt{MinusOne}(L))$
    if C = NULL

return NULL
else
return $[p, sl], C$

Proof. [The correctness of algorithm `Cond`] It is easy to see that the number of recursions in the algorithm is bounded by $\deg(p)-1$, so the algorithm will terminate in finite steps.

Now suppose that $L_0 := L$ is the RC of $\Delta^0(p) := p(x)$, and the number of recursions is m. Let $[d, \ell, r] =$ `RCInfo`(L_0), then $\Delta^0(p)$ should be a polynomial of degree d, with ℓ distinct roots and r distinct real roots. So by Theorem 20, the sign list of $\Delta^0(p)$ should be one of $sl_0 := GenAllSL(\Delta^0(p), L_0)$. Let $L_1 = MinusOne(L_0)$. Then, as we pointed out above, the RC of $\Delta^1(p)$ is L_1. So similarly, we can conclude that the sign list of $\Delta^1(p)$ should be one of $sl_1 := GenAllSL(\Delta^1(p), L_1)\dots.$ The sign list of $\Delta^m(p)$ should be one of $sl_m := GenAllSL(\Delta^m(p), L_m)$. For any $k(0 \le k \le m)$, if $sl_k = \emptyset$, then L_k is not realizable for $\Delta^k(p)$, so L is not realizable for p.

On the other hand, if the sign list of $\Delta^j(p)$ is one of $sl_j := GenAllSL(\Delta^j(p), L_j)$ $(0 \le j \le m)$, then by Theorem 6 and Theorem 20, using similar discussion as above, we can conclude that the RC of $p(x)$ is L. □

In summary, the improved algorithm for generating the CRC of a real parametric polynomial is the following.

Algorithm 4. CRC
Input: A real parametric polynomial $p(x)$ and a complete list $\mathcal{L}$ of RCs of $p(x)$.
Output: the CRC of $p(x)$.
for L in $\mathcal{L}$ do
if `Cond`$(p, L) \neq$ NULL
output L and $Cond(p, L)$

Remark 24. As we pointed out in Liang & Jeffrey (2006), if the parametric polynomial $p(x)$ has a general form, or if the output of `SLReal` is obtained via partial CAD, then all the RCs in the CRC are realizable. Otherwise, an RC in the CRC is realizable if and only if the semi-algebraic set defined by the associated conditions is non-empty.

5. Implementation and Applications

The algorithm has been implemented in Maple, and in this section we give some examples of its use. We first demonstrate the output of the Maple program, and then we show applications to some problems in real quantifier elimination. All the computations were performed with Maple 10 running on a 1.6 GHz Pentium CPU. All times were less than 2 seconds and therefore are not reported.

Example 25. Find the CRC of the polynomial $p_6 = x^6 + ax^2 + bx + c$.

Here is the Maple output.

```
(*) p6:=x^6+a*x^2+b*x+c
The Complete Root Classification of p6 is:
(The condition format is: [poly, all possible sign lists])
(1) [[6],[]], if and only if
      [p6, [1,0,0,0,0,0]]
(2) [[1,1,1,1],[1,-1]], if and only if
      [p6, [1,0,0,-1,-1,-1]]
(3) [[1,1,2],[1,-1]], if and only if
      [p6, [1,0,0,-1,-1,0]]
(4) [[1,3],[1,-1]], if and only if
      [p6, [1,0,0,-1,0,0]], [p62, [1,0]]
(5) [[2,2],[1,-1]], if and only if
      [p6, [1,0,0,-1,0,0]], [p62, [1,1]]
(6) [[1,1],[2,-2]], if and only if
      [p6, [1,0,0,-1,0,0]], [p62, [1,-1]]
(7) [[1,1],[1,-1,1,-1]], if and only if
      [p6, [1,0,0,0,0,1],[1,0,0,-1,-1,1],[1,0,0,-1,0,1],
        [1,0,0,1,1,1],[1,0,0,0,1,1],[1,0,0,-1,1,1]]
(8) [[2],[1,-1,1,-1]], if and only if
      [p6, [1,0,0,1,1,0],[1,0,0,0,1,0],[1,0,0,-1,1,0]]
(9) [[],[1,-1,2,-2]], if and only if
      [p6, [1,0,0,1,0,0]]
(10) [[],[1,-1,1,-1,1,-1]], if and only if
      [p6, [1,0,0,0,1,-1],[1,0,0,-1,1,-1],[1,0,0,1,0,-1],
        [1,0,0,0,0,-1],[1,0,0,1,-1,-1],[1,0,0,1,1,-1]]
Where,
(#1) p6:=x^6+a*x^2+b*x+c,
(#2) p62:=4*a*x^2+5*b*x+6*c,
```

This is read as follows. Each numbered line describes one root classification. Thus, the first line describes the RC for a single real root of multiplicity 6. The condition for this RC is that the polynomial p_6 has the sign list $[1, 0, 0, 0, 0, 0]$. Since the discriminant sequence for p_6 is $D(p_6) = [1, 0, 0, a^3, D_5, D_6]$,

$$\begin{aligned} D_5 &= 256a^5 + 1728a^2c^2 - 5400ab^2c + 1875b^4 \ , \\ D_6 &= -1024a^6c + 256a^5b^2 - 13824a^3c^3 + 43200a^2b^2c^2 - 22500ab^4c \\ &\qquad +3125b^6 - 46656c^5 \ . \end{aligned}$$

the condition on the sign list implies $a = b = c = 0$, which is what we expect.

Now consider line (5). The RC describes 2 real roots, each of multiplicity 2, and one complex conjugate pair of multiplicity 1. The conditions for this case require the discriminant sequence $D(p_6)$ and the discriminant sequence for p_{62}

$$E = [1, E_2], \qquad E_2 = 25b^2 - 96ac.$$

Therefore for this RC, we require $a < 0$ and $D_5 = D_6 = 0$ and $E_2 > 0$.

Notice that some RCs are missing from the list. For example, the RC $[[1,1,1,1,1,1],[\,]]$, representing 6 distinct real roots, is not present. This is because this case is impossible for p_6. Also note that in line (10), the two non-realizable sign lists $[1,0,0,0,-1,-1]$ and $[1,0,0,-1,0,-1]$ are detected and deleted by the improved algorithm automatically. Finally, the expression for p_{62} has been automatically simplified by removing the content a^2, which must be nonzero.

Problem 26. Find the CRC of the polynomial $p_6 = x^6 + ax^3 + bx^2 + cx + d$.

Here is the MAPLE output.

```
(*) p6:=x^6+a*x^3+b*x^2+c*x+d
The Complete Root Classification of p6 is:
(The condition format is: [poly,its all possible sign lists])
(1) [[6],[]], if and only if
      [p6, [1,0,0,0,0,0]]
(2) [[1,1,1,1],[1,-1]], if and only if
      [p6, [1,0,-1,-1,-1,-1],[1,0,0,-1,-1,-1]]
(3) [[1,1,2],[1,-1]], if and only if
      [p6, [1,0,-1,-1,-1,0],[1,0,0,-1,-1,0]]
(4) [[1,3],[1,-1]], if and only if
      [p6, [1,0,-1,-1,0,0],[1,0,0,-1,0,0]], [p62, [1,0]]
(5) [[2,2],[1,-1]], if and only if
      [p6, [1,0,-1,-1,0,0],[1,0,0,-1,0,0]], [p62, [1,1]]
(6) [[4],[1,-1]], if and only if
      [p6, [1,0,-1,0,0,0]], [p63, [1,0,0]]
(7) [[1,1],[2,-2]], if and only if
      [p6, [1,0,-1,-1,0,0],[1,0,0,-1,0,0]], [p62, [1,-1]]
(8) [[1,1],[1,-1,1,-1]], if and only if
      [p6, [1,0,-1,0,0,1],[1,0,0,0,0,1],[1,0,-1,-1,0,1],
        [1,0,0,-1,0,1],[1,0,-1,-1,-1,1],[1,0,0,-1,-1,1],
        [1,0,-1,1,1,1],[1,0,0,1,1,1],[1,0,-1,0,1,1],
        [1,0,0,0,1,1],[1,0,-1,-1,1,1],[1,0,0,-1,1,1]]
(9) [[2],[2,-2]], if and only if
      [p6, [1,0,-1,0,0,0]], [p63, [1,1,-1],[1,0,-1],[1,-1,-1]]
(10) [[2],[1,-1,1,-1]], if and only if
      [p6, [1,0,-1,1,1,0],[1,0,0,1,1,0],[1,0,-1,0,1,0],
        [1,0,0,0,1,0],[1,0,-1,-1,1,0],[1,0,0,-1,1,0]]
(11) [[],[1,-1,2,-2]], if and only if
      [p6, [1,0,0,1,0,0],[1,0,-1,1,0,0]]
(12) [[],[1,-1,1,-1,1,-1]], if and only if
      [p6, [1,0,0,1,0,-1],[1,0,-1,0,0,-1],[1,0,0,0,0,-1],
        [1,0,-1,1,-1,-1],[1,0,0,1,-1,-1],[1,0,-1,1,1,-1],
        [1,0,0,1,1,-1],[1,0,-1,0,1,-1],[1,0,0,0,1,-1],
        [1,0,-1,-1,1,-1],[1,0,0,-1,1,-1],[1,0,-1,1,0,-1]]

Where,
(#1) p62:=-9*c*a^3-180*d*c*a+192*d*b^2+Q1*x+Q2*x^2,
(#2) p6:=x^6+a*x^3+b*x^2+c*x+d,
```

```
(#3) p63:=-3*a*x^3-4*b*x^2-5*c*x-6*d,
 and
Q1:=160*c*b^2-18*b*a^3-150*a*c^2-144*a*d*b,
Q2:=-27*a^4+108*d*a^2-240*a*b*c+128*b^3,
```

The discriminant sequence of p_6 is $D := [1, 0, -a^2, D_4, D_5, D_6]$, where

$$D_4 = -27a^4 + 108da^2 - 240abc + 128b^3$$
$$D_5 = 81a^5c - 27a^4b^2 - 1134dca^3 + 648a^2db^2 + 1620c^2a^2b$$
$$-1344acb^3 + 3240acd^2 + 256b^5 + 1728d^2b^2 - 5400bdc^2 + 1875c^4$$
$$D_6 = 108c^3a^5 + 729d^2a^6 - 8748d^3a^4 + 34992a^2d^4 - 46656d^5 - 486ca^5db$$
$$+21384ca^3d^2b - 9720c^2b^2da^2 - 77760d^3cab - 22500dc^4b + 43200c^2b^2d^2$$
$$+6912ab^4cd + 3125c^6 - 27b^2a^4c^2 + 108b^3a^4d - 8640b^3a^2d^2 - 1350dc^3a^3$$
$$+2250c^4ba^2 - 13824d^3b^3 + 27000d^2ac^3 - 1600ab^3c^3 + 256b^5c^2 - 1024b^6d.$$

From the CRC of p_6, we can obtain the conditions on a, b, c, d such that $(\forall x)[p_6 > 0]$. $(\forall x)[p_6 > 0] \Leftrightarrow$ case (11) or case (12) holds. We can directly write down the sign conditions. No mapping of conditions is necessary.

Case (11) holds $\Leftrightarrow$ the sign list of p_6 be $[1, 0, 0, 1, 0, 0]$ or $[1, 0, -1, 1, 0, 0] \Leftrightarrow [-a^2 = 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0] \vee [-a^2 < 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0] \Leftrightarrow D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0$.

Case (12) holds $\Leftrightarrow$ the sign list of p_6 be one of the following 12 lists:
$[1, 0, 0, 1, 0, -1] \Leftrightarrow [-a^2 = 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 < 0]$
$[1, 0, -1, 0, 0, -1] \Leftrightarrow [-a^2 < 0 \wedge D_4 = 0 \wedge D_5 = 0 \wedge D_6 < 0]$
$[1, 0, 0, 0, 0, -1] \Leftrightarrow [-a^2 = 0 \wedge D_4 = 0 \wedge D_5 = 0 \wedge D_6 < 0]$
$[1, 0, -1, 1, -1, -1] \Leftrightarrow [-a^2 < 0 \wedge D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0]$
$[1, 0, 0, 1, -1, -1] \Leftrightarrow [-a^2 = 0 \wedge D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0]$
$[1, 0, -1, 1, 1, -1] \Leftrightarrow [-a^2 < 0 \wedge D_4 > 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1, 0, 0, 1, 1, -1] \Leftrightarrow [-a^2 = 0 \wedge D_4 > 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1, 0, -1, 0, 1, -1] \Leftrightarrow [-a^2 < 0 \wedge D_4 = 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1, 0, 0, 0, 1, -1] \Leftrightarrow [-a^2 = 0 \wedge D_4 = 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1, 0, -1, -1, 1, -1] \Leftrightarrow [-a^2 < 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1, 0, 0, -1, 1, -1] \Leftrightarrow [-a^2 = 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1, 0, -1, 1, 0, -1] \Leftrightarrow [-a^2 < 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 < 0]$

Simplifying by hand or by QEPCAD (Brown, 2004), we conclude that case (12) holds $\Leftrightarrow D_6 < 0 \wedge [D_4 > 0 \vee D_5 > 0 \vee [D_4 = 0 \wedge D_5 = 0]]$.

Finally, by combining the conditions for case (11) and case (12), we obtain the desired result: $(\forall x)[p_6 > 0] \Leftrightarrow [D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0] \vee [D_4 = 0 \wedge D_5 = 0 \wedge D_6 < 0] \vee [D_4 > 0 \wedge D_6 < 0] \vee [D_5 > 0 \wedge D_6 < 0]$.

At the end of this paper, we give some more examples. As mentioned in the introduction section of the paper, in order to get compact results, we concentrate on sparse polynomials. Notice that, to our knowledge, no other methods have succeeded in solving these problems except Example 25.

Example 27. Find the conditions on a, b, c, d such that $(\forall x)[x^8 + ax^3 + bx^2 + cx + d > 0]$.

For this problem, it is not necessary to generate the whole CRC. Only those RCs with no real roots are needed. The solution is
$(\forall x)[x^8 + ax^3 + bx^2 + cx + d > 0] \Leftrightarrow [D_6 < 0 \wedge D_7 = 0 \wedge D_8 = 0] \vee [D_6 < 0 \wedge D_8 > 0] \vee [D_7 < 0 \wedge D_8 > 0] \vee [D_6 = 0 \wedge D_7 = 0 \wedge D_8 > 0], where$

$$D_6 = 9375a^6 + 112000a^3cd - 172800a^2b^2d - 176400a^2bc^2 + 241920ab^3c - 62208b^5,$$

$$D_7 = -9375a^7c + 3125a^6b^2 - 332500a^4c^2d - 152000a^4bd^2$$
$$+416500a^3bc^3 + 744000a^3b^2cd - 409600a^2d^4 - 718200a^2b^3c^2$$
$$-216000a^2b^4d + 2580480abcd^3 - 1756160ac^3d^2 + 334368ab^5c$$
$$+2304960bc^4d - 46656b^7 - 2709504b^2c^2d^2 - 470596c^6 + 442368b^3d^3$$

$$D_8 = -823543c^8 + 42147840abc^3d^3 + 16777216d^7 + 3931200a^2b^4c^2d$$
$$-41287680ab^2cd^4 + 7529536bc^6d + 56250a^7bcd - 1960000a^3b^2c^3d$$
$$-8524800a^3b^3cd^2 + 186624b^8d + 381024ab^5c^3 - 46656b^7c^2 + 3125a^6b^2c^2$$
$$-2880000a^5cd^3 - 20070400a^2c^2d^4 + 19660800a^2bd^5 - 84375a^8d^2$$
$$+21676032b^3c^2d^3 - 22127616b^2c^4d^2 - 1617408ab^6cd + 600250a^3bc^5$$
$$-926100a^2b^3c^4 + 1907712a^2b^5d^2 + 4224000a^4b^2d^3 - 428750a^4c^4d$$
$$-12500a^6b^3d - 8605184ac^5d^2 - 3538944d^4b^4 - 12500a^7c^3 + 5992000a^4bc^2d^2.$$

Example 28. Find the conditions on a, b, c such that $(\forall x)[x^{10} + ax^2 + bx + c > 0]$.

A solution to this problem was given in Liang & Jeffrey (2006). A new solution is obtained by using the new algorithm.
$(\forall x)[x^{10} + ax^2 + bx + c > 0] \Leftrightarrow [a > 0 \wedge D_9 = 0 \wedge D_{10} = 0] \vee [a > 0 \wedge D_{10} < 0] \vee [D_9 > 0 \wedge D_{10} < 0] \vee [a = 0 \wedge D_9 = 0 \wedge D_{10} < 0]$.

Where D_9 and D_{10} were given in Liang & Jeffrey (2006). Please notice the difference between the two solutions. The solution given here has been refined.

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The Automatic Computation of the Complete Root Classification for a Parametric Polynomial (Extended Abstract)

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Key words: Complete root classification, parametric polynomial, real quantifier elimination, root classification.

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6. Introduction

Counting and classifying the roots of a polynomial is a well-established problem area; see Basu, Pollack & Roy (2003) for references. One approach that has been successful used in the past is based on the Complete Root Classification (CRC) of a polynomial, as defined in Yang, Hou & Zeng (1996) and Yang (1999). Algorithms suitable for manual computation were given by those authors, while algorithms suitable for automatic computation were described in Liang & Zhang (1999) and Liang & Jeffrey (2006). In this paper, a new algorithm which allows significant efficiency improvements is presented and its implementation in Maple is used to solve a series of problems in quantifier elimination, specifically in positive-definiteness testing. It is shown that for a class of sparse polynomials, remarkably compact conditions are possible.

The CRC of a real parametric quartic polynomial was found by Arnon (1988), but the first method for establishing the CRC of a real parametric polynomial of any degree was given by Yang, Hou & Zeng (1996). They illustrated their method by computing the CRC of a reduced sextic polynomial. Liang & Zhang (1999) enabled the automatic generation of CRCs, and improvements were proposed by Liang & Jeffrey (2006).

This paper presents several advances on the above works. The automatic generation of CRCs is made more efficient, and new methods are used to filter extraneous cases from the results. In addition, the algorithm of Yang, Hou & Zeng (1996) defines a 'discrimination matrix' and a 'discriminant sequence' which are close to, but not identical with, similar quantities used elsewhere in real algebraic geometry. Here we connect the method of Yang, Hou & Zeng (1996) with standard definitions. For example, we relate their discriminant sequence to the signed subresultant coefficient sequence. These identifications help with the more efficient algorithm just referred to.

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7. Previous work

We review the algorithms of Yang, Hou & Zeng (1996) and Liang & Jeffrey (2006).

Definition 29. Let $p \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. The $2n \times 2n$ matrix

$$M = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_0 & & & & \\ 0 & na_n & (n-1)a_{n-1} & \dots & a_1 & & & & \\ & a_n & a_{n-1} & \dots & a_1 & a_0 & & & \\ & 0 & na_n & \dots & 2a_2 & a_1 & & & \\ & & & \vdots & \vdots & & & & \\ & & & a_n & a_{n-1} & a_{n-2} & \dots & a_0 \\ & & & 0 & na_n & (n-1)a_{n-1} & \dots & a_1 \end{pmatrix} \quad (8)$$

is called the discrimination matrix of p.

Definition 30. For $1 \le k \le 2n$, let M_k be the kth principal minor of M, and let $D_k = M_{2k}$. The n-tuple $D = [D_1, D_2, \dots, D_n]$ is called the discriminant sequence of p.

Definition 31. If $\operatorname{sgn} x$ is the signum function, with $\operatorname{sgn} 0 = 0$, then the list

$$[\operatorname{sgn} D_1, \operatorname{sgn} D_2, \dots, \operatorname{sgn} D_n]$$

is called the sign list of p.

Definition 32. The revised sign list $[e_1, e_2, \dots, e_n]$ of p is constructed from the sign list $[s_1, s_2, \dots, s_n]$ of p as follows. If $[s_i, s_{i+1}, \dots, s_{i+j}]$ is a section of the given list, where $s_i \neq 0$, $s_{i+1} = s_{i+2} = \dots = s_{i+j-1} = 0$ and $s_{i+j} \neq 0$, then we replace the subsection $[s_{i+1}, \dots, s_{i+j-1}]$ by $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \dots]$.

The revised sign list of p is denoted $\operatorname{rsl}(p)$.

Definition 33. Let $\Delta(p)$ denote $\gcd(p(x), p'(x))$. Let $\Delta^0(p) = p(x)$, $\Delta^j(p) = \Delta(\Delta^{j-1}(p))$, for $j = 1, 2, \dots$. Then $\Delta^0(p), \Delta^1(p), \Delta^2(p), \dots$ is called the Δ-sequence of p.

Definition 34. The Multiple Factor Sequence $\Delta_0(p), \Delta_1(p), \dots, \Delta_{n-1}(p)$ of p is defined using the discrimination matrix M defined above. Let M_{ki} denote the submatrix defined using Matlab matrix notation as `[1:2k;1:2k-1,2k+i]`, then

$$\Delta_k(p) = \sum_{i=0}^{k} \det M_{n-k,i} x^{k-i}$$

Lemma 35. *If* $\operatorname{rsl}(p)$ *contains* k *zeros, then* $\Delta_k(p) = \gcd(p(x), p'(x))$.

8. Basic Definitions and Theorems

We first connect the discriminant sequence to the signed subresultant coefficient sequence defined in Basu, Pollack & Roy (2003).

Lemma 36. *Let* $D = [D_1, \ldots, D_n]$ *be the discriminant sequence of* $p(x)$*. Then for* $1 \le j \le n$*,* $D_j = a_n \cdot \mathrm{sRes}_{n-j}(p, p')$*, where* sRes *is defined in Basu, Pollack & Roy (2003).*

The main theorem is the following. All terms are defined in Basu, Pollack & Roy (2003).

Theorem 37. *Let* $D = [D_1, \ldots, D_n]$ *be the discriminant sequence of a real polynomial* $p(x)$ *of degree* n*, and let* ℓ *be the maximal subscript such that* $D_\ell \neq 0$*. If* $\mathrm{PmV}(D) = r$*, then* $p(x)$ *has* $r+1$ *distinct real roots and* $\frac{1}{2}(\ell - r - 1)$ *pairs of distinct complex conjugate roots.*

The following corollary is necessary for detecting the non-realizable sign lists.

Corollary 38. *Let* $S = [s_1, \ldots, s_n]$ *and* $R = [r_1, \ldots, r_n]$ *be the sign list and the revised sign list of* $p(x)$ *respectively. Then* $\mathrm{PmV}(S) = \mathrm{PmV}(R)$*.*

9. An Improved CRC Algorithm

A Root Classification (RC) is denoted by the list L

$$L = [L_1, L_2] = [\,[n_1, n_2, \ldots], [m_1, -m_1, m_2, -m_2, \ldots]\,]\ , \tag{9}$$

where the $n_k \in \mathbb{Z}$ are the multiplicities of the real roots, and the $m_k \in \mathbb{Z}$ are the multiplicities of complex conjugate pairs of roots. In what follows, we use the functions in our Maple implementation of the algorithm.

Algorithm 5. SLReal
Input: A real polynomial $g(x)$ with parametric coefficients.
Output: A set of all possible sign lists.

Algorithm 6. GenAllSL.
Input: A real parametric polynomial $g(x)$ and a root classification L.
Output: All sign lists of $g(x)$ corresponding to L.

The following algorithm generates the conditions for $p(x)$ having L as its root classification.

Algorithm 7. Cond
Input: a real parametric polynomial $p(x)$; an RC L.
Output: a sequence of mixed lists. Each list consists of a polynomial in the Δ-sequence of $p(x)$, followed by its all possible sign lists.

The improved algorithm is now the following.

Algorithm 8. CRC
Input: A real parametric polynomial $p(x)$ and a complete list $\mathcal{L}$ of RCs of $p(x)$.
Output: the CRC of $p(x)$.

```
for L in ℒ do
    if Cond(p, L) ≠ NULL
        output L and Cond(p, L)
```

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