BINOMIAL D-MODULES

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ABSTRACT. We study quotients of the Weyl algebra by left ideals whose generators consist of an arbitrary $\mathbb{Z}^d$-graded binomial ideal I in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ along with Euler operators defined by the grading and a parameter $\beta \in \mathbb{C}^d$. We determine the parameters β for which these D-modules (i) are holonomic (equivalently, regular holonomic, when I is standard-graded); (ii) decompose as direct sums indexed by the primary components of I; and (iii) have holonomic rank greater than the rank for generic β. In each of these three cases, the parameters in question are precisely those outside of a certain explicitly described affine subspace arrangement in $\mathbb{C}^d$. In the special case of Horn hypergeometric D-modules, when I is a lattice basis ideal, we furthermore compute the generic holonomic rank combinatorially and write down a basis of solutions in terms of associated A-hypergeometric functions. Fundamental in this study is an explicit lattice point description of the primary components of an arbitrary binomial ideal in characteristic zero, which we derive from a characteristic-free combinatorial result on binomial ideals in affine semigroup rings. Effective methods can be derived for the computation of primary components of arbitrary binomial ideals and series solutions to classical Horn systems.

1. EXTENDED ABSTRACT

1.1. **Hypergeometric series.** A univariate power series is *hypergeometric* if the successive ratios of its coefficients are given by a fixed rational function. These functions, and the elegant differential equations they satisfy, have proven ubiquitous in mathematics. As a small example of this phenomenon, consider the Hermite polynomials. These hypergeometric functions naturally occur, for instance, in physics (energy levels of the harmonic oscillator) [CDL77], numerical analysis (Gaussian quadrature) [SB02], combinatorics (matching polynomials of complete graphs) [God81], and probability (iterated Itô integrals of standard Wiener processes) [Itô51].

Perhaps the most natural definition of hypergeometric power series in several variables is the following, whose bivariate specialization was studied by Jakob Horn as early as 1889 [Hor1889]. More references include [Hor31], the first of six articles, all in Mathematische Annalen between 1931 and 1940, and all containing "Hypergeometrische Funktionen zweier Veränderlichen" (hypergeometric functions in two variables) in their titles.

Definition 1.1. A formal series $F(z) = \sum_{\alpha \in \mathbb{N}^m} a_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ in m variables with complex coefficients is *hypergeometric in the sense of Horn* if there exist rational functions

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$r_1, r_2, \ldots, r_m$ in m variables such that

$$\frac{a_{\alpha+e_k}}{a_\alpha} = r_k(\alpha) \quad \text{for all } \alpha \in \mathbb{N}^m \text{ and } k = 1, \ldots, m. \tag{1.1}$$

Here we denote by $e_1, \ldots, e_m$ the standard basis vectors of $\mathbb{N}^m$.

Write the rational functions of the previous definition as

$$r_k(\alpha) = p_k(\alpha)/q_k(\alpha + e_k) \quad k = 1, \ldots m,$$

where p_k and q_k are relatively prime polynomials and z_k divides q_k.

Since $g(z_1\partial_{z_1}, \ldots, z_m\partial_{z_m})z^\alpha = g(\alpha_1, \ldots, \alpha_m)z^\alpha$ for all monomials z^α, the series F satisfies the following *Horn hypergeometric system of differential equations*:

$$q_k(z_1\partial_{z_1}, \ldots, z_m\partial_{z_m})F(z) = z_k p_k(z_1\partial_{z_1}, \ldots, z_m\partial_{z_m})F(z) \quad k = 1, \ldots, m. \tag{1.2}$$

Of particular interest are the series where the numerators and denominators of the rational functions r_k factor into products of linear factors. (Contrast with the notion of "proper hypergeometric term" in [PWZ96].) Notice that by the fundamental theorem of algebra, this is not restrictive when the number of variables is $m = 1$.

1.2. **Binomial ideals and binomial D-modules.** The central objects of study in this article are the *binomial D-modules*, to be introduced in Definition 1.3, which reformulate and generalize the classical Horn hypergeometric systems. Our definition is based on the point of view developed by Gelfand, Graev, Kapranov, and Zelevinsky [GGZ87, GKZ89], and contains their hypergeometric systems as special cases.

To construct a binomial D-module, the starting point is an integer matrix A, about which we wish to be consistent throughout.

Convention 1.2. $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ denotes an integer $d \times n$ matrix of rank d whose columns $a_1, \ldots, a_n$ all lie in a single open linear half-space of $\mathbb{R}^d$; equivalently, the cone generated by the columns of A is pointed (contains no lines), and all of the a_i are nonzero. We also assume that $\mathbb{Z}A = \mathbb{Z}^d$; that is, the columns of A span $\mathbb{Z}^d$ as a lattice.

The reformulation of Horn systems proceeds by a change of variables, so we will use $x = x_1, \ldots, x_n$ and $\partial = \partial_1, \ldots, \partial_n$ (where $\partial_i = \partial_{x_i}$), instead of $z_1, \ldots, z_m$ and $\partial_{z_1}, \ldots, \partial_{z_m}$, whenever we work in the binomial setting. The matrix A induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_n] = \mathbb{C}[\partial]$, which we call the *$A$-grading*, by setting $\deg(\partial_i) = -a_i$. An ideal of $\mathbb{C}[\partial]$ is *A-graded* if it is generated by elements that are homogeneous for the A-grading. For example, a *binomial ideal* is generated by *binomials* $\partial^u - \lambda\partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$; such an ideal is A-graded precisely when it is generated by binomials $\partial^u - \lambda\partial^v$ each of which satisfies either $Au = Av$ or $\lambda = 0$ (in particular, monomials are allowed as generators of binomial ideals). The hypotheses on A mean that the A-grading is a *positive $\mathbb{Z}^d$-grading* [MS05, Chapter 8].

The Weyl algebra $D = D_n$ of linear partial differential operators, written with the variables x and ∂, is also naturally A-graded by additionally setting $\deg(x_i) = a_i$. Consequently, the *Euler operators* in our next definition are A-homogeneous of degree 0.

Definition 1.3. For each $i \in \{1, \dots, d\}$, the i^{th} *Euler operator* is

$$E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n.$$

Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \dots, E_d - \beta_d$. (The dependence of the Euler operators E_i on the matrix A is suppressed from the notation.)

For an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + \langle E - \beta \rangle$ in the Weyl algebra D. The *binomial D-module* associated to I is $D/H_A(I, \beta)$.

We explain how Horn systems correspond to the binomial D-modules arising from a very special class of binomial ideals called *lattice basis ideals*.

Our goal is to demonstrate not merely that the definition of binomial D-modules can be made in this generality—and that it leads to meaningful theorems—but that it *must* be made, even if one is interested only in classical questions concerning Horn hypergeometric systems, which arise from lattice basis ideals. Furthermore, once the definition has been made, most of what we wish to prove about Horn hypergeometric systems generalizes to all binomial D-modules.

Before going on, we recall some general background and basic references on D-modules. A left D-ideal $\mathcal{I}$ is *holonomic* if its characteristic variety has dimension n. Holonomicity has strong homological implications, making the class of holonomic D-modules a natural one to study. If $\mathcal{I}$ is holonomic, its *holonomic rank*, i.e. the dimension of the space of solutions of the D-ideal $\mathcal{I}$ that are holomorphic in a sufficiently small neighborhood of a point outside the singular locus, is finite (the converse of this result is not true). We refer to the texts [Bor87, Cou95, SST00] for introductory overviews of the theory of D-modules; we point out that the exposition in [SST00] is geared toward algorithms and computations. A treatment of D-modules with *regular singularities* can be found in [Bjö79, Bjö93].

1.3. **Toric ideals and A-hypergeometric systems.** The fundamental examples of binomial D-modules, and the ones which our definition most directly generalizes, are the *A-hypergeometric systems* (or *GKZ hypergeometric systems*) of Gelfand, Kapranov, and Zelevinsky [GKZ89]. Given A as in Convention 1.2, these are the left D-ideals $H_A(I_A, \beta)$, also denoted by $H_A(\beta)$, where

$$I_A = \langle \partial^u - \partial^v : Au = Av \rangle \subseteq \mathbb{C}[\partial_1, \dots, \partial_n] \tag{1.3}$$

is the *toric ideal* for the matrix A. The systems $H_A(\beta)$ have many applications; for example, they arise naturally in the moduli theory of Calabi-Yau complete intersections in toric varieties, and (therefore) they play an important role in applications of mirror symmetry in mathematical physics [BvS95, Ho99, Hos04, HLY96].

The ideal I_A is a prime A-graded binomial ideal, and the quotient ring $\mathbb{C}[\partial]/I_A$ is the semigroup ring for the affine semigroup $\mathbb{N}A$ generated by the columns of A. There is a rich theory of toric ideals, toric varieties, and affine semigroup rings, whose core philosophy is to exploit the connection between the algebra of the semigroup ring $\mathbb{C}[\partial]/I_A = \mathbb{C}[\mathbb{N}A]$ and the combinatorics of the semigroup $\mathbb{N}A$. In this way, algebro-geometric results on toric varieties can be obtained by combinatorial means, and purely combinatorial facts about polyhedral geometry can be proved using algebraic techniques. We direct the reader to the texts [Ful93, GKZ94, MS05] for more information.

Much is known about A-hypergeometric D-modules. They are holonomic for all parameters [GKZ89, Ado94], and they are regular holonomic exactly when I_A is $\mathbb{Z}$-graded in the usual sense [Hot91, SW06]. In this case, (Gamma-)series expansions for the solutions of $H_A(\beta)$ centered at the origin and convergent in certain domains can be explicitly computed [GKZ89, SST00]. The generic (minimal) holonomic rank is known to be $\mathrm{vol}(A)$, the normalized volume of the convex hull of the columns of A and the origin [GKZ89, Ado94], and holonomic rank is independent of the parameter β if and only if the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay [GKZ89, Ado94, MMW05]. We will extend all of these results, suitably modified, to the general setting of binomial D-modules. The important caveat is that a general binomial D-module can exhibit behavior that is forbidden to GKZ systems (see Example 1.8, for instance), so it is impossible for the extension to be entirely straightforward.

1.4. **Binomial Horn systems.** Classical Horn systems, which we are about to define precisely, were first studied by Appell [App1880], Mellin [Mel21], and Horn [Hor1889]. They directly generalize the univariate hypergeometric equations for the functions ${}_pF_q$; see [SK85, Sla66] and the references therein. As we mentioned earlier, our motivation to consider binomial D-modules is that they contain as special cases these classical Horn systems. The definition of these systems involves a matrix B about which, like the matrix A from Convention 1.2, we wish to be consistent throughout.

Convention 1.4. Let $B = (b_{jk}) \in \mathbb{Z}^{n\times m}$ be an integer matrix of full rank $m \leq n$. Assume that every nonzero element of the column-span of B over the integers $\mathbb{Z}$ is *mixed*, meaning that it has at least one positive and one negative entry; in particular, the columns of B are mixed. We write $b_1, \ldots, b_n$ for the rows of B. Having chosen B, we set $d = n - m$ and pick a matrix $A \in \mathbb{Z}^{d\times n}$ whose columns span $\mathbb{Z}^d$ as a lattice, such that $AB = 0$. In the case that $d \neq 0$, the mixedness hypothesis on B is equivalent to the pointedness assumption for A that appears in Convention 1.2. We do allow $d = 0$, in which case A is the empty matrix.

Definition 1.5. For a matrix $B \in \mathbb{Z}^{n\times m}$ as in Convention 1.4 and a vector $c = (c_1, \ldots, c_n)$ in $\mathbb{C}^n$, the *classical Horn system with parameter* c is the left ideal $\mathrm{Horn}(B, c)$ in the Weyl algebra D_m generated by the m differential operators

$$q_k(\theta_z) - z_k p_k(\theta_z), \quad k = 1, \ldots, m,$$

where $\theta_z = (\theta_{z_1}, \ldots, \theta_{z_m})$, $\theta_{z_k} = z_k \partial_{z_k}$ $(1 \leq k \leq m)$, and

$$q_k(\theta_z) = \prod_{b_{jk}>0} \prod_{\ell=0}^{b_{jk}-1} (b_j \cdot \theta_z + c_j - \ell) \quad \text{and} \quad p_k(\theta_z) = \prod_{b_{jk}<0} \prod_{\ell=0}^{|b_{jk}|-1} (b_j \cdot \theta_z + c_j - \ell).$$

Using ideas of Gelfand, Kapranov, and Zelevinsky, the classical Horn systems can be reinterpreted as the following binomial D-modules, with $\beta = Ac$.

Definition 1.6. Fix integer matrices B and A as in Convention 1.4, and let $I(B)$ be the *lattice basis ideal* corresponding to this matrix, that is, the ideal in $\mathbb{C}[\partial]$ generated by the binomials

$$\prod_{b_{jk}>0} \partial_{x_j}^{b_{jk}} - \prod_{b_{jk}<0} \partial_{x_j}^{-b_{jk}} \quad \text{for} \quad 1 \leq k \leq m.$$

The *binomial Horn system with parameter* β is the left ideal $H(B, \beta) = H_A(I(B), \beta)$ in the Weyl algebra $D = D_n$.

The classical-to-binomial transformation proceeds via the surjection

$$\begin{array}{rcl} (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^m \\ (x_1, \ldots, x_n) & \mapsto & x^B = (\prod_{j=1}^n x_j^{b_{j1}}, \ldots, \prod_{j=1}^n x_j^{b_{jm}}), \end{array} \tag{1.4}$$

where $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ is the group of nonzero complex numbers. A solution $f(z_1, \ldots, z_m)$ of the classical Horn system $\mathrm{Horn}(B, c)$ gives rise to a solution $x^c f(x^B)$ of the binomial Horn system $H(B, Ac)$. That this indeed defines a vector space isomorphism between the (local) solution spaces was proved in [DMS05, Section 5] for $n > m$ in the homogeneous case, where the column sums of B are zero, but the proofs (which are elementary calculations taking only a page) go through verbatim for $n \geq m$ in the inhomogeneous case.

The transformation $f(z) \mapsto x^c f(x^B)$ takes classical series solutions supported on $\mathbb{N}^m$ to Puiseux series solutions supported on the translate $c + \ker(A) \subseteq \mathbb{C}^n$ of the kernel of A in $\mathbb{Z}^n$. (Note that $\ker(A)$ contains the lattice $\mathbb{Z}B$ spanned by the columns of B as a finite index subgroup.) More precisely, the differential equations $E - \beta$, which geometrically impose torus-equivariance infinitesimally under the action of (the Lie algebra of) $\ker((\mathbb{C}^*)^n \to (\mathbb{C}^*)^m)$, result in series supported on $c + \ker(A)$, while the binomials in the lattice basis ideal $I(B) \subseteq H(B, Ac)$ impose hypergeometric constraints on the coefficients.

Although the isomorphism $f(z) \mapsto x^c f(x^B)$ is only at the level of local holomorphic solutions, not D-modules, it preserves many of the pertinent features, including the dimensions of the spaces of local holomorphic solutions and the structure of their series expansions. Therefore, although the classical Horn systems are our motivation, we take the binomial formulation as our starting point: no result in this article depends logically on the classical-to-binomial equivalence.

1.5. **Holomorphic solutions to Horn systems.** The binomial rephrasing of Horn systems led to formulas in [GGR92] for Gamma-series solutions via A-hypergeometric theory. However, Gamma-series need not span the space of local holomorphic solutions of $H(B, \beta)$ at a point of $\mathbb{C}^n$ that is nonsingular for $H(B, \beta)$, even in the simplest cases. The reason is that Gamma-series are *fully supported*: there is a cone of dimension m (the maximum possible) whose lattice points correspond to monomials with nonzero coefficients. Generally speaking, Horn systems in dimension $m \geq 2$ tend to have many series solutions without full support.

Example 1.7. In the course of studying one of Appell's systems of two hypergeometric equations in $m = 2$ variables, Arthur Erdélyi [Erd50] mentions a modified form of the following example. Given any $\beta \in \mathbb{C}^2$ and the two matrices

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

satisfying Convention 1.4, the Puiseux monomial $x_1^{\beta_1/3} x_4^{\beta_2/3}$ is a solution of $H(B, \beta)$.

A key feature of the above example is that the solutions without full support persist for arbitrary choices of the parameter vector β. The fact that this phenomenon occurs in much more generality—for arbitrary dimension $m \geq 2$, in particular—was realized only recently [DMS05]. And it is not the sole peculiarity that arises in dimension $m \geq 2$: in view of the transformation to binomial Horn systems, the following demonstrates that classical Horn systems can exhibit poor behavior for badly chosen parameters.

Example 1.8. Consider

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that

$$H(B, \beta) = \langle \partial_1\partial_3 - \partial_2, \partial_1\partial_4 - \partial_2 \rangle + \langle x_1\partial_1 - x_2\partial_2 - \beta_1, x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - \beta_2 \rangle.$$

If $\beta_1 = 0$, then any (local holomorphic) bivariate function $f(x_3, x_4)$ annihilated by the operator $x_3\partial_3 + x_4\partial_4 - \beta_2$ is a solution of $H(B, \beta)$. The space of such functions is infinite-dimensional; in fact, it has uncountable dimension, as it contains all monomials $x_3^{w_3} x_4^{w_4}$ with $w_3, w_4 \in \mathbb{C}$ and $w_3 + w_4 = \beta_2$.

Erdélyi's goal for his study of the Appell system was to give bases of solutions that converged in different regions of $\mathbb{C}^2$, eventually covering the whole space, just as Kummer had done for the Gauss hypergeometric equation more than a century before [Kum1836]. There has been extensive work since then (see [SK85] and its references) on convergence of more general hypergeometric functions in two and three variables. But already for the classical case of Horn systems, where the phenomena in Examples 1.7 and 1.8 are commonplace, Erdélyi's work raises a number of fundamental questions that remain largely open (partial answers in dimension $m = 2$ being known [DMS05]; see Remark 1.16). The purpose of this article is to answer the following completely and precisely.

Questions 1.9. *Fix B as in Convention 1.4, and consider the Horn systems determined by B.*

1. *For which parameters does the space of local holomorphic solutions around a nonsingular point have finite dimension as a complex vector space?*
2. *What is a combinatorial formula for the minimum such dimension, over all possible choices of parameters?*
3. *Which parameters are generic, in the sense that the minimum dimension is attained?*
4. *How do (the supports of) series solutions centered at the origin look, combinatorially?*

These questions make sense simultaneously for classical Horn systems and binomial Horn systems, since the answers are invariant under the classical-to-binomial transformation. That the questions also make sense for binomial D-modules is our point of departure, for they can be addressed in this generality using answers to the following.

Questions 1.9 (continued). *Consider the binomial D-modules $H_A(I, \beta)$ for varying $\beta \in \mathbb{C}^d$.*

5. *When is $D/H_A(I, \beta)$ a holonomic D-module?*
6. *When is $D/H_A(I, \beta)$ a* regular *holonomic D-module?*

The phenomena underlying all of the answers to Questions 1.9 can be described in terms of lattice point geometry, as one might hope, owing to the nature of hypergeometric recursions as relations between coefficients on monomials. The lattice point geometry is elementary, in the sense that it only requires constructions involving cosets and equivalence relations in lattices. However, modern techniques are required to make the descriptions quantitatively accurate and to prove them. In particular, our progress applies two distinct and substantial steps: precise advances in the combinatorial commutative algebra of binomial ideals in semigroup rings, and a functorial translation of those advances into D-module theory.

1.6. **Combinatorial answers to hypergeometric questions.** The supports of the various series solutions to $H(B, \beta)$ centered at the origin are controlled by how effectively the columns of B join the lattice points in the positive orthant $\mathbb{N}^n$. In essence, this is because the coefficients on a pair of Puiseux monomials are related by the binomial equations in $I(B)$ when their exponent vectors in $c+\ker(A)$ differ by a column of B. This observation prompts us to construct an undirected graph on the nodes $\mathbb{N}^n$ with an edge between pairs of points differing by a column of B. Each connected component, or *B-subgraph of* $\mathbb{N}^n$, is contained in a single fiber $(a + \mathbb{Z}B) \cap \mathbb{N}^n$ of the projection $\mathbb{N}^n \to \mathbb{Z}^n/\mathbb{Z}B$.

The geometry of B-subgraphs generalizes to an arbitrary binomial ideal I, which determines a *congruence* as follows: $u \sim v$ if $\partial^u - \lambda\partial^v \in I$ for some $\lambda \neq 0$. This generalization is key, as it allows us the flexibility to work with the congruences determined by various ideals related to $I(B)$, which might not themselves be lattice basis ideals. For example, when the binomial ideal I is the toric ideal

$$I_A = \langle \partial^u - \partial^v : u, v \in \mathbb{N}^n \text{ and } Au = Av \rangle \subseteq \mathbb{C}[\partial]$$

for the matrix A, the congruence class of $\alpha = Aa \in \mathbb{Z}A = \mathbb{Z}^d$ for $a \in \mathbb{N}^n$ consists of the lattice points in the polyhedron

$$P_\alpha = \{u \in \mathbb{R}^n : Au = \alpha \text{ and } u \geq 0\} = (a + \ker(A)) \cap \mathbb{N}^n.$$

With this picture in mind, the B-subgraphs in P_α, or the congruence classes for any binomial ideal I (homogeneous for A as in Definition 1.3), typically consist of one big continent in the interior of P_α plus a number of surrounding islands.

The extent to which $I(B)$ differs from I_A is measured by which bridges must be built—and in which directions—to join various islands to the continent. To this end, let $J \subseteq \{1, \ldots, n\}$, and write $\mathbb{Z}^J = \{v \in \mathbb{Z}^n : v_i = 0 \text{ for all } i \notin J\}$. For $\bar{J} = \{1, \ldots, n\} \smallsetminus J$, $\mathbb{N}^{\bar{J}}$ is defined in a similar manner. Suppose that $L \subseteq \mathbb{Z}^J$ is a saturated sublattice, so $\mathbb{Z}^J/L$ is torsion-free.

Just as $I(B)$ determines a congruence on $\mathbb{N}^n$, it determines one on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$. The sublattice L determines a coarsening of this congruence, by allowing bridges from u to v if $u - v \in L$. Certain choices of $L \subseteq \mathbb{Z}^J$ satisfying $L \subseteq \ker(A)$ are *associated* to $I(B)$, and for these, there exist coarsened classes in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ whose images in $(\mathbb{Z}^J/L) \times \mathbb{N}^{\bar{J}}$ are finite; let us call these classes *L-bounded*. Each L-bounded class in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ lies in a single coset of $\ker(A)$, since $L \subseteq \ker(A)$, so its image in $\mathbb{Z}^d = \mathbb{Z}^n/\ker(A)$ is a well-defined point.

The D-module theoretic consequences of L-bounded classes depend on a crucial distinction.

Definition 1.10. An associated saturated sublattice $L \subseteq \mathbb{Z}^J \cap \ker(A)$ is called *toral* if $L = \mathbb{Z}^J \cap \ker(A)$; otherwise, $L \subsetneq \mathbb{Z}^J \cap \ker(A)$ is called *Andean*.

Example 1.11. [Example 1.8 continued] With A and B as in Example 1.8, there are two associated lattices, one with $J = \{1, 2, 3, 4\}$, the other with $J = \{3, 4\}$. The first one is toral, while the second is Andean.

In what follows, A_J denotes the submatrix of A whose columns are indexed by J. We write $\mathbb{Z}A_J \subseteq \mathbb{Z}^d = \mathbb{Z}A$ for the group generated by these columns, and $\mathbb{C}A_J \subseteq \mathbb{C}^d$ for the vector subspace they generate.

Observation 1.12. The images of the L-bounded classes for all of the Andean associated sublattices $L \subseteq \mathbb{Z}^J$ comprise a finite union of cosets of $\mathbb{Z}A_J$. The union over all J of the corresponding cosets of $\mathbb{C}A_J$ is an affine subspace arrangement in $\mathbb{C}^d$ called the *Andean arrangement*.

Example 1.13. [Example 1.11 continued] The Andean arrangement in this case is

$$\mathbb{C}\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right] = \left\{ \left[\begin{smallmatrix} 0 \\ \beta_2 \end{smallmatrix}\right] : \beta_2 \in \mathbb{C} \right\}.$$

As we have already checked, the Horn system in Example 1.8 fails to be holonomic for this set of parameters.

Observation 1.14. A class in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ determined by a toral associated sublattice $L \subseteq \mathbb{Z}^J$ is L-bounded if and only if its image in $\mathbb{N}^{\bar{J}}$ is bounded. If $\mathbb{C}A_J = \mathbb{C}^d$, then the number $\mu(L, J)$ of such bounded images in $\mathbb{N}^{\bar{J}}$ is finite.

Answers 1.15. *The answers to Questions 1.9, phrased in the language of binomial Horn systems $H(B, \beta)$, are as follows.*

1. *The dimension is finite exactly for $-\beta$ not in the Andean arrangement.*
2. *The generic (minimum) rank is $\sum \mu(L, J) \cdot \mathrm{vol}(A_J)$, the sum being over all toral associated sublattices with $\mathbb{C}A_J = \mathbb{C}^d$, where $\mathrm{vol}(A_J)$ is the volume of the convex hull of A_J and the origin, normalized so a lattice simplex in $\mathbb{Z}A_J$ has volume* 1.
3. *The minimum rank is attained precisely when $-\beta$ lies outside of an affine subspace arrangement determined by certain local cohomology modules, with the same flavor as (and containing) the Andean arrangement.*
4. *When the Horn system is regular holonomic and β is general, there are $\mu(L, J) \cdot \mathrm{vol}(A_J)$ linearly independent solutions supported on (translates of) the L-bounded classes, with hypergeometric recursions determining the coefficients. Only* $g \cdot \mathrm{vol}(A)$

many Gamma series solutions have full support, where $g = |\ker(A)/\mathbb{Z}B|$ is the index of $\mathbb{Z}B$ in its saturation $\ker(A)$.

5. *Holonomicity is equivalent to the finite dimension in Answer 1.15.1.*
6. *Holonomicity is equivalent to regular holonomicity when I is standard $\mathbb{Z}$-graded—i.e., the row-span of A contains the vector* $[1 \cdots 1]$. *Conversely, if there exists a parameter β for which $D/H_A(I, \beta)$ is regular holonomic, then I is $\mathbb{Z}$-graded.*

In Answer 1.15.4, the solutions for toral sublattices $L = \ker(A) \cap \mathbb{Z}^J$ in which J is a proper subset of $\{1, \ldots, n\}$ give rise to solutions that are bounded in the $\mathbb{N}^{\bar{J}}$ directions, and hence supported on sets of dimension $\operatorname{rank}(L) = |J| - d < n - d = m$. Answer 1.15.6 is, given the other results in this paper, an (easy) consequence of the (hard) holonomic regularity results of Hotta [Hot91] and Schulze–Walther [SW06]. Finally, let us note again that most of the results quoted in Answers 1.15 are stated and proved in the context of arbitrary binomial D-modules, not just Horn systems.

Remark 1.16. The systematic study of binomial Horn systems was started in [DMS05] under the hypothesis that m (the number of columns of B) is equal to 2. See also [Sad02]. Our results here are more general than those found in [DMS05] (as we treat all binomial D-modules, not just those arising from lattice basis ideals of codimension 2), more refined (we have completely explicit control over the parameters) and stronger (for instance, our direct sum results hold at the level of D-modules and not just local solution spaces). On the other hand, the generic holonomicity of classical Horn D-modules for $m > 2$ remains unproven, the bivariate case having been treated in [DMS05].

Example 1.17. [Example 1.7, continued] There are two associated sublattices $L \subseteq \mathbb{Z}^J$ here, both toral, and both satisfying $\mathbb{C}A_J = \mathbb{C}^2$: the sublattice $\ker(A) \subseteq \mathbb{Z}^4$, where $J = \{1, 2, 3, 4\}$, and the sublattice $\mathbf{0} \subseteq \mathbb{Z}^J$ for $J = \{1, 4\}$. Both of the multiplicities $\mu\big(\ker(A), \{1, 2, 3, 4\}\big)$ and $\mu\big(\mathbf{0}, \{1, 4\}\big)$ equal 1, while $\operatorname{vol}(A) = 3$ and $\operatorname{vol}(A_{\{1,4\}}) = 1$, the latter because $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$ form a basis for the lattice they generate. Hence there are four solutions in total, three of them with full support and one—namely the Puiseux monomial in Example 1.7—with support of dimension zero. See Example 1.18 for an (easy!) computation of these associated lattices and their multiplicities.

Throughout this article we will make repeated use of two quite different tools. The first is a description of the aforementioned associated lattices. The second comes from A-hypergeometric theory [MMW05], and is homological in nature. We close this extended abstract with a discussion of these tools.

1.7. **Binomial primary decomposition.** The geometry and combinatorics of lattice point congruences control the primary decomposition of arbitrary binomial ideals in characteristic zero. These results are developed for all binomial ideals, instead of only for lattice basis ideals, because the binomial ideals arising naturally in the process of carrying out primary decompositions are sufficiently arbitrary that the general case contributes conceptual clarity without presenting additional obstacles. The developments here can be seen as a combinatorial refinement of the binomial primary decomposition theorem of Eisenbud and Sturmfels [ES96]. New effective computations can be derived from our explicit description.

Our combinatorial study of binomial primary decomposition results in a natural language for quantifying which sublattices are associated, which cosets appear in Observation 1.12, and which bounded images appear in Observation 1.14. To be precise, a binomial prime ideal $I_{\rho,J}$ in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ is determined by a subset $J \subseteq \{1, \ldots, n\}$ and a character $\rho : L \to \mathbb{C}^*$ for some sublattice $L \subseteq \mathbb{Z}^J$. The sublattice $L \subseteq \mathbb{Z}^J$ is associated when $I_{\rho,J}$ is associated to I in the usual commutative algebra sense, and the multiplicity $\mu(L, J)$ in Observation 1.14 is the commutative algebra multiplicity of $I_{\rho,J}$ in I.

Example 1.18. [Example 1.17, continued] The binomial Horn system is

$$H(B, \beta) = I(B) + \langle 3x_1\partial_1 + 2x_2\partial_2 + x_3\partial_3 - \beta_1, x_2\partial_2 + 2x_3\partial_3 + 3x_4\partial_4 - \beta_2\rangle \subseteq D_4.$$

The primary decomposition of the lattice basis ideal $I(B)$ in $\mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_4]$ is

$$I(B) = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2\rangle = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3\rangle \cap \langle \partial_2, \partial_3\rangle.$$

The first of these components is the toric ideal $I_A = I_{\rho,J}$ of the twisted cubic curve, where $\rho : \ker(A) = \mathbb{Z}B \to \mathbb{C}^*$ is the trivial character and $J = \{1, 2, 3, 4\}$. The ideal $\langle \partial_2, \partial_3\rangle$ is the binomial prime ideal $I_{\rho,J}$ for the (automatically) trivial character $\rho : \mathbf{0} \to \mathbb{C}^*$ and the subset $J = \{1, 4\}$. Both of these ideals have multiplicity 1 in $I(B)$, which is a radical ideal. This explains the associated lattices and multiplicities in Example 1.17.

The explicit lattice-point descriptions of binomial primary decomposition we give have potential applications beyond hypergeometric systems. Consider the special case of monomial ideals: certain constructions at the interface between commutative algebra and algebraic geometry, such as integral closure and multiplier ideals, admit concrete convex polyhedral descriptions. The path is now open to attempt analogous constructions for binomial ideals, including effective polyhedral computations.

For a note on motivation, this project began with the conjectural statement of Answer 1.15.4, which we concluded must hold because of evidence derived from our knowledge of series solutions. Its proof directed all of the developments in the rest of the paper. Our consequent use of B-subgraphs, and more generally the application of congruences toward the primary decomposition of binomial ideals, serves as an advertisement for hypergeometric intuition as inspiration for developments of independent interest in combinatorics and commutative algebra.

1.8. **Euler-Koszul homology.** Binomial primary decomposition is not only the natural language for lattice point geometry, it is the reason why lattice point geometry governs the D-module theoretic properties of binomial D-modules. This we demonstrate by functorially translating the commutative algebra of A-graded primary decomposition directly into the D-module setting. The functor we employ is Euler-Koszul homology, which allows us to pull apart the primary components of binomial ideals, thereby isolating the contribution of each to the solutions of the corresponding binomial D-module. Here we see again the need to work with general binomial D-modules: primary components of lattice basis ideals, and intersections of various collections of them, are more or less arbitrary A-homogeneous binomial ideals.

We stress at this point that the combinatorial geometric lattice-point description of binomial primary decomposition is a crucial prerequisite for the effective translation into the realm of D-modules. Indeed, semigroup gradings pervade the arguments demonstrating the fundamentally holonomic behavior of Euler-Koszul homology for toral modules and its resolutely non-holonomic behavior for Andean modules. This is borne out in a couple of key steps, which say that quotients by primary ideals are either toral or Andean as $\mathbb{C}[\partial]$-modules, thus constituting the bridge from the commutative binomial theory to the binomial D-module theory. Taming the homological (holonomic) and structural properties of binomial D-modules also rests squarely on having tight control over the interactions of primary decomposition with various semigroup gradings of the polynomial ring. The underlying phenomenon is thus:

Central principle. Just as toric ideals are the building blocks of binomial ideals, A-hypergeometric systems are the building blocks of binomial D-modules.

As a final indication of how structural results for binomial D-modules have concrete combinatorial implications for Horn hypergeometric systems, let us see how the primary decomposition in Example 1.18 results in the combinatorial multiplicity formula (Answer 1.15.2) for the holonomic rank at generic parameters β. The general result to which we appeal is: for generic parameters β, the binomial D-module $D/H_A(I,\beta)$ decomposes as a direct sum over the toral primary components of I.

Example 1.19. [Example 1.18, continued] The intersection in $\mathbb{C}^4 = \operatorname{Spec}(\mathbb{C}[\partial_1, \ldots, \partial_4])$ of the two irreducible varieties in the zero set of $I(B)$ is the zero set of

$$\langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3\rangle + \langle \partial_2, \partial_3\rangle = \langle \partial_1\partial_4, \partial_2, \partial_3\rangle.$$

The primary arrangement is, in this case, the line in $\mathbb{C}^2$ spanned by $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ union the line in $\mathbb{C}^2$ spanned by $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$. When β lies off the union of these two lines, our results yield an isomorphism of D_4-modules:

$$\frac{D_4}{H(B,\beta)} \cong \frac{D_4}{\langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3\rangle + \langle E - \beta\rangle} \oplus \frac{D_4}{\langle \partial_2, \partial_3\rangle + \langle E - \beta\rangle}.$$

The summands on the right-hand side are GKZ hypergeometric systems (up to extraneous vanishing variables in the $\langle \partial_2, \partial_3\rangle$ case) with holonomic ranks 3 and 1, respectively.

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