MINIMAL FAITHFUL UPPER-TRIANGULAR MATRIX REPRESENTATIONS FOR LOW-DIMENSIONAL SOLVABLE LIE ALGEBRAS

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Abstract. A well-known result on Lie Theory states that every finite-dimensional complex solvable Lie algebra can be represented as a matrix Lie algebra, formed by upper-triangular square matrices. However, that result does not specify which is the minimal order of the matrices involved in such representations. Hence, the main goal of this paper is to find the minimal order of these representations for each solvable Lie algebra of dimension less than five. Besides, a minimal matrix representation is given for each of these algebras.

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1 Introduction

According Ado's Theorem, given any finite-dimensional complex Lie algebra $\mathfrak{g}$, there exists a matrix algebra isomorphic to $\mathfrak{g}$ (the classical proof can be seen in [7], but a very short proof is available in [10]). In this way, every finite-dimensional complex Lie algebra can be represented as a Lie subalgebra of the complex general linear algebra $\mathfrak{gl}(n,\mathbb{C})$, formed by all the complex $n \times n$ matrices, for some $n \in \mathbb{N}$.

In this paper we consider the complex Lie algebra $\mathfrak{b}_n$, formed by all the $n \times n$ complex upper-triangular matrices. Obviously, this Lie algebra is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. Besides, the Lie algebra $\mathfrak{b}_n$ is solvable and, hence, each Lie subalgebra $\mathfrak{s} \subset \mathfrak{b}_n$ is also solvable. The Lie algebras $\mathfrak{b}_n$ and their subalgebras are interesting because any complex finite-dimensional solvable Lie algebra can be represented by a Lie subalgebra of some $\mathfrak{b}_n$ (see [5, Theorem 9.11] or [14, Theorem 3.7.3]).

Given $n \in \mathbb{N}$, we now wonder if every n-dimensional complex solvable Lie algebra can be represented as a Lie subalgebra of $\mathfrak{b}_n$. To answer this question, each n-dimensional Lie subalgebra of $\mathfrak{b}_n$ has to be studied and its isomorphism class has to be determined to rule out the algebras which do not verify this property.

In order to settle which n-dimensional solvable Lie algebras can be faithfully represented in the Lie algebra $\mathfrak{b}_n$, we are determining the minimal $m \in \mathbb{N}$ such that $\mathfrak{b}_m$ contains a Lie subalgebra isomorphic to a given solvable Lie algebra. In this way, if m is less than the dimension of the algebra our question can be answered affirmatively.

Regarding to this, Benoist [2] already proved, by using formal resolution of polynomial systems, that a 13-dimensional nilpotent Lie algebra does not admit faithful linear representation of dimension 13. Moreover, there exists a general construction by using geometrical and algebraic methods as can be seen in [12].

In this paper, we study all the complex solvable Lie algebras whose dimension is less than 5 and we obtain minimal representations of these algebras by using complex upper-triangular matrices; that is, for each complex solvable Lie algebra of such dimensions, we find the minimal $n \in \mathbb{N}$ such that this algebra can be represented by a subalgebra of $\mathfrak{b}_n$. This is the main goal of the paper.

We have used the classification given by Mubarakzyanov [1, 8] to deal with all the solvable Lie algebras (up to isomorphism classes) of dimension less than 5. Nevertheless, the classification of solvable Lie algebras is also known for dimension 5 (see [9]) and 6 (see [13]).

The structure of this paper is as follows: in Section 2, some concepts and results on solvable Lie algebras are recalled. Section 3 is devoted to explain the method used to obtain minimal representations of solvable Lie algebras by using the Lie algebras $\mathfrak{b}_n$. Next, minimal representations are computed for every solvable Lie algebra of dimension less than 5. So, Section 4 shows a minimal representation for each solvable Lie algebra of dimension less than 4. Finally, the four-dimensional solvable Lie algebras are represented in Section 5. It is convenient to note that all the computations in this paper have been got by using MAPLE 9.5 and MATHEMATICA 5.2 to benefit from each package. Let us note that an open problem could be to consider what is the largest dimension in which this method still works.

2 Preliminaries

For a general overview on Lie algebras and Representation Theory, the reader can consult [14], for instance.

Given a Lie algebra $\mathfrak{g}$, the *commutator central series* or *derived series* is defined as:

$$\mathcal{C}_1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}_2(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}],\ \ldots,\ \mathcal{C}_k(\mathfrak{g}) = [\mathcal{C}_{k-1}(\mathfrak{g}),\mathcal{C}_{k-1}(\mathfrak{g})],\ \ldots$$

So, $\mathfrak{g}$ is called *solvable* if there exists a natural integer m such that $\mathcal{C}_m(\mathfrak{g}) \equiv \{0\}$. The minimal integer m verifying such a property is called the *solvability index* of $\mathfrak{g}$.

Proposition 2.1. *Given a Lie algebra $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, their respective commutator central series are related as follows: $\mathcal{C}_k(\mathfrak{h}) \subseteq \mathcal{C}_k(\mathfrak{g})$, $\forall k \in \mathbb{N}$.*

A *linear or matrix algebra* is that whose vectors are matrices and the Lie bracket is defined by the commutator $[a, b] = a \cdot b - b \cdot a$. So, they are subalgebras of some general linear algebra $\mathfrak{gl}(\mathbb{C}, n)$, formed by $n \times n$ complex regular matrices.

We are dealing with a particular family of linear algebras in this paper: the Lie algebras $\mathfrak{b}_n$. Given $n \in \mathbb{N}$, the Lie algebra $\mathfrak{b}_n$ is the linear algebra formed by all the $n \times n$ upper-triangular complex matrices. The Lie algebra $\mathfrak{b}_n$ has dimension $\binom{n+1}{2}$ and a basis $\mathcal{B} = \{X_{ij} \in \mathfrak{b}_n \mid 1 \le i \le j \le n\}$ of this algebra is given by the following upper-triangular matrices:

$$X_{ij} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & x_{2,2} & \cdots & x_{2,n-1} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n-1,n-1} & x_{n-1,n} \\ 0 & 0 & \cdots & 0 & x_{n,n} \end{pmatrix}, \qquad \text{with } x_{r,s} = \begin{cases} 0, & \text{if } (r,s) \neq (i,j) \\ 1, & \text{if } (r,s) = (i,j) \end{cases}$$

By considering this basis $\mathcal{B}$, the following expression for the law of this algebra is obtained:

Theorem 2.2. *Given $n \in \mathbb{N}$, the law of the Lie algebra $\mathfrak{b}_n$ with respect to the basis $\mathcal{B}$ is given by the following nonzero brackets:*

$$[X_{i,h}, X_{h,k}] = X_{i,k}, \quad 1 \le i \le h \le k \le n \ \wedge \ i \neq k. \qquad \square$$

Given a Lie algebra $\mathfrak{g}$, a *representation* of $\mathfrak{g}$ in $\mathbb{C}^n$ is a homomorphism of Lie algebra $\phi : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(\mathbb{C}, n)$. The natural integer n is called the *dimension* of this representation. In this paper, we consider faithful representations because such representations allow us to identify a given Lie algebra with its image under the representation, which is a Lie subalgebra of $\mathfrak{gl}(\mathbb{C}, n)$.

Representations can be also defined by using arbitrary n-dimensional vector spaces V (see [5]). In such a case, a representation would be a homomorphism of Lie algebra from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of the vector space V, which is called $\mathfrak{g}$-module. However, it is sufficient to consider representations on $\mathbb{C}^n$ because there always exists a unique $n \in \mathbb{N}$ such that V is isomorphic to $\mathbb{C}^n$.

With respect to minimal representations of Lie algebras, Burde [3] already dealt with the following classical invariant for an arbitrary Lie algebra $\mathfrak{g}$:

$$\mu(\mathfrak{g}) = \min\{\dim(M) \mid M \text{ is a faithful } \mathfrak{g}\text{-module}\}.$$

This value is also equal to the minimal value n such that $\mathfrak{gl}(\mathbb{C}^n)$ contains a subalgebra isomorphic to $\mathfrak{g}$. However, the algebras considered in [3] are nilpotent and the non-nilpotent solvable Lie algebras were not studied.

In this paper, minimal representations of solvable Lie algebras are found. Besides, we are interested in minimal representations of these algebras with a particular restriction: the representations have to be given by upper-triangular matrices. In this way, given a solvable Lie algebra $\mathfrak{g}$, we are going to compute the minimal value n such that $\mathfrak{b}_n$ contains $\mathfrak{g}$. This value is an invariant of $\mathfrak{g}$ and its expression is given by:

$$\bar{\mu}(\mathfrak{g}) = \min\{n \in \mathbb{N} \mid \exists \text{ subalgebra of } \mathfrak{b}_n \text{ isomorphic to } \mathfrak{g}\}.$$

The invariant $\bar{\mu}(\mathfrak{g})$ exists for any solvable Lie algebra $\mathfrak{g}$ because every solvable complex Lie algebra can be represented by a Lie subalgebra of the Lie algebra $\mathfrak{b}_n$, in virtue of Theorem 3.7.3 of [14].

3 Method to determine a minimal representation by using Lie algebras $\mathfrak{b}_n$

In this section, we explain briefly the method which we are applying to obtain a minimal representation of a given complex solvable Lie algebra $\mathfrak{g}$ by using the matrix algebras $\mathfrak{b}_n$. The steps followed to obtain such a representation are shown next:

1. By using Proposition 2.1, the commutator central series of the Lie algebra $\mathfrak{g}$ is compared with the one of the Lie algebra $\mathfrak{b}_n$, beginning with $n = 1$. We pass to the following step when we have found the minimal n verifying Proposition 2.1.

2. Given a basis of the Lie algebra $\mathfrak{g}$, we express each vector of this basis as a linear combination of the vectors X_{ij} in the basis $\mathcal{B}$ of $\mathfrak{b}_n$. To do it, for each basic vector of $\mathfrak{g}$, we consider the greatest $k \in \mathbb{N}$ such that the ideal $\mathcal{C}_k(\mathfrak{b}_n)$ contains to this basic vector. In this way, this vector is expressed as a linear combination of the vectors X_{ij} in the chosen $\mathcal{C}_k(\mathfrak{b}_n)$.

3. All the brackets between two basic vectors of $\mathfrak{g}$ are computed. By imposing the law of the Lie algebra $\mathfrak{g}$, a system of non-linear equations are obtained by comparing coordinate to coordinate with respect to the basis $\mathcal{B}$. The unknowns of this system are the coefficients of the vectors in the basis of the solvable Lie algebra $\mathfrak{g}$ studied.

4. We solve the system of equations (by using some symbolic computation package) and a solution of the system will provide us a representation of the Lie algebra $\mathfrak{g}$ in $\mathfrak{b}_n$ if such a solution satisfies that the vectors resulting are linearly independent. When no solution is obtained from the system, the Lie algebra $\mathfrak{g}$ cannot be represented by a subalgebra of $\mathfrak{b}_n$. In this case, we go back to step 2 and repeat all the steps with the Lie algebra $\mathfrak{b}_{n+1}$.

Let us note that the representation obtained for the Lie algebra $\mathfrak{g}$ is minimal because we start with $n = 1$ and n is increased when no representations can be obtained in $\mathfrak{b}_n$. This method is going to be used in the next sections to get minimal upper-triangular matrix representations.

4 Solvable Lie algebras of dimension less than 4

This section is devoted to obtain a minimal upper-triangular representation for each solvable Lie algebra of dimension less than 4. Let us note that we consider solvable Lie algebra up to isomorphism. In this way, we only need to represent a representative of each class to get our goal, because the representations of two isomorphic Lie algebras are isomorphic between themselves. So, we have used the classification of Lie algebras given in [1, 8] and $\{Z_i\}_{i=1}^n$ denotes a basis of each n-dimensional solvable Lie algebra. For each non-abelian Lie algebra, we only have written its nonzero brackets.

Dimension 1:	$\mathfrak{s}_1^1$:	$[Z_1, Z_1]=0$	(abelian).
Dimension 2:	$\mathfrak{s}_2^1$:	$[Z_i, Z_j]=0, \quad i,j \in \{1,2\}$	(abelian).
	$\mathfrak{s}_2^2$:	$[Z_1, Z_2]=Z_1$.	
Dimension 3:	$\mathfrak{s}_3^1$:	$[Z_i, Z_j]=0, \quad i,j \in \{1,2,3\}$	(abelian).
	$\mathfrak{s}_3^2$:	$[Z_1, Z_3]=Z_2$	(Heisenberg algebra).
	$\mathfrak{s}_3^3$:	$[Z_1, Z_3]=Z_1, \ [Z_2, Z_3]=Z_2$.	
	$\mathfrak{s}_3^4$:	$[Z_1, Z_3]=Z_2, \ [Z_2, Z_3]=-Z_1$.	
	$\mathfrak{s}_3^5$:	$[Z_1, Z_3]=-Z_1, \ [Z_2, Z_3]=-Z_1-Z_2$.	
	$\mathfrak{s}_3^6$:	$[Z_1, Z_3]=-Z_1$,	$(\mathfrak{s}_1^1 \oplus \mathfrak{s}_2^2)$.
	$\mathfrak{s}_3^7$:	$[Z_1, Z_2]=Z_3, \ [Z_1, Z_3] = \alpha Z_2 + Z_3$.	(With $\alpha \neq 0$ and $1-4\alpha \neq 0$).

Let us note that another one-parametric family of solvable Lie algebra is given in the classifications used in this paper. We are referring to the family L_α^4 (in [6]) or $L_3, 5$ (in [11]), whose law is given by: $[Z_1, Z_2]= Z_3$, $[Z_1, Z_3] = \alpha Z_2$. Two algebras L_α^4 and L_β^4 of this family are isomorphic if and only if there exists $c \in \mathbb{C}$ such that $\alpha = c^2 \cdot \beta$. Obviously, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, the complex number c can be get as $c = \sqrt{\alpha\beta^{-1}}$. In this way, this family provides us two non-isomorphic complex solvable Lie algebras: $\mathfrak{s}_3^2$ (for $\alpha = 0$) and $\mathfrak{s}_3^4$ (for $\alpha \neq 0$).

To represent the algebras shown in the previous table, we only need the Lie algebras $\mathfrak{b}_1$, $\mathfrak{b}_2$ and $\mathfrak{b}_3$. This is proved in the following result:

Theorem 4.1. *A minimal representation by upper-triangular matrices for each solvable Lie algebra of dimension less than 4 is given in the next table with the dimension of such a minimal representation:*

Lie algebra	Minimal representation	Dim. ($\bar{\mu}$)
$\mathfrak{s}_1^1$:	$\langle Z_1 = X_{11} \rangle \subseteq \mathfrak{b}_1$	1
$\mathfrak{s}_2^1$:	$\langle Z_1 = X_{11}, Z_2 = X_{22} \rangle \subseteq \mathfrak{b}_2$	2
$\mathfrak{s}_2^2$:	$\langle Z_1 = X_{11}, Z_2 = X_{12} \rangle \subseteq \mathfrak{b}_2$	2
$\mathfrak{s}_3^1$:	$\langle Z_1 = X_{11}, Z_2 = X_{22}, Z_3 = X_{33} \rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_3^2$:	$\langle Z_1 = X_{12}, Z_2 = X_{13}, Z_3 = X_{23} \rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_3^3$:	$\langle Z_1 = X_{13}, Z_2 = X_{12}, Z_3 = -X_{11} \rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_3^4$:	$\langle Z_1 = X_{12}, Z_2 = iX_{12} + X_{13}, Z_3 = iX_{22} + X_{23} - iX_{33} \rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_3^5$:	$\langle Z_1 = X_{13}, Z_2 = X_{23}, Z_3 = X_{12} - X_{33} \rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_3^6$:	$\langle Z_1 = X_{12}, Z_2 = X_{11} + X_{22}, Z_3 = X_{22} \rangle \subseteq \mathfrak{b}_2$	2
$\mathfrak{s}_3^7$:	$\langle Z_1 = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1+4\alpha}\right) X_{11} + \sqrt{1+4\alpha} X_{22} + X_{23},$ $Z_2 = \left(-1 - \sqrt{1+4\alpha}\right) X_{12} - 2X_{13}, Z_3 = 2\alpha X_{12} \rangle \subseteq \mathfrak{b}_3$	3

Proof. Due to reasons of length, we will only show the complete proof for the 3-dimensional Lie algebra $\mathfrak{s}_3^4$. To obtain its minimal representation, we use the method explained in Section 3. The minimal representations of the remaining algebras have been computed with the same method and its application is analogous to the one shown for the Lie algebra $\mathfrak{s}_3^4$.

Next, each step of our method is applied to the solvable Lie algebra $\mathfrak{s}_3^4$:

1. **Comparing the commutator central series:** In the following table, the ideals of this series are shown for Lie algebras $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ and $\mathfrak{s}_3^4$:

Lie algebra	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathfrak{b}_1$	$\langle X_{11} \rangle$	$\{0\}$	$\{0\}$
$\mathfrak{b}_2$	$\langle X_{11}, X_{12}, X_{22} \rangle$	$\langle X_{12} \rangle$	$\{0\}$
$\mathfrak{b}_3$	$\langle X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33} \rangle$	$\langle X_{12}, X_{13}, X_{23} \rangle$	$\langle X_{13} \rangle$
$\mathfrak{s}_3^4$	$\langle Z_1, Z_2, Z_3 \rangle$	$\langle Z_1, Z_2 \rangle$	$\{0\}$

In virtue of Proposition 2.1, the solvable Lie algebra $\mathfrak{s}_3^4$ cannot be represented by using the Lie algebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Due to it, we consider the Lie algebra $\mathfrak{b}_3$ which is not in contradiction with Proposition 2.1.

2. **Writing the basis of $\mathfrak{s}_3^4$:** The vectors Z_1 and Z_2 belong to $\mathcal{C}_2(\mathfrak{s}_3^4)$ and Z_3 belongs to $\mathcal{C}_1(\mathfrak{s}_3^4) \setminus \mathcal{C}_2(\mathfrak{s}_3^4)$. In this way, we can express these three vectors as the following linear combinations of the basis $\{X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}\}$ of the Lie algebra $\mathfrak{b}_3$:

$$\begin{aligned}
Z_1 &= a_{12} \cdot X_{12} + a_{13} \cdot X_{13} + a_{23} \cdot X_{23};\\
Z_2 &= b_{12} \cdot X_{12} + b_{13} \cdot X_{13} + b_{23} \cdot X_{23};\\
Z_3 &= c_{11} \cdot X_{11} + c_{12} \cdot X_{12} + c_{13} \cdot X_{13} + c_{22} \cdot X_{22} + c_{23} \cdot X_{23} + c_{33} \cdot X_{33}.
\end{aligned}$$

3. **Computing the brackets and comparing them with the law of $\mathfrak{s}_3^4$:** All the brackets can be computed with the help of any symbolic computation package as MAPLE 9.5. After inputting the vectors Z_i $(i = 1, 2, 3)$ defined in the previous step, we define the commutator operator and, then, we compute all the brackets between two different vectors Z_i. A way to implement this is shown in the following routine:

```
> z1:=matrix(3,3,[0,a12,a13,0,0,a23,0,0,0]);
> z2:=matrix(3,3,[0,b12,b13,0,0,b23,0,0,0]);
> z3:=matrix(3,3,[c11,c12,c13,0,c22,a23,0,0,c33]);
> cor:= proc(i,j)
  evalm(z||i&*z||j-z||j&*z||i):
  end:
> ec:=[]:
> for i from 1 to 3 do
  for j from i+1 to 3 do
  ec:=[op(ec),cor(i,j)];
  od;
  od;
```

The variable `ec` saves all these brackets. Now, these brackets are equalled to those which define the Lie algebra $\mathfrak{s}_3^4$. In this case, its nonzero brackets are $[Z1, Z3] = Z_2$ and $[Z_2, Z_3] = -Z_1$ and the comparison leads to the following system of non-linear equations, which is obtained by identifying coordinate by coordinate in each bracket:

$$\begin{aligned}
a_{12} \cdot b_{23} - a_{23} \cdot a_{12} &= 0\\
a_{12} \cdot (-c_{11} + c_{22}) &= b_{12}\\
a_{12} \cdot c_{23} - a_{13} \cdot c_{11} + a_{13} \cdot c_{33} - a_{23} \cdot c_{12} &= b_{13}\\
a_{23} \cdot (-c_{22} + c_{33}) &= b_{23}\\
b_{12} \cdot (-c_{11} + c_{22}) &= -a_{12} \qquad (1)\\
b_{12} \cdot c_{23} - b_{13} \cdot c_{11} + b_{13} \cdot c_{33} - b_{23} \cdot c_{12} &= -a_{13}\\
a_{2,3} &= -a_{2,3}(c_{3,3} - c_{2,2})^2 \qquad (2)
\end{aligned}$$

We can obtain a system equivalent to the previous one by replacing the equations

(1) and (2) with the following two equations:

$$a_{12}(c_{22}-c_{11})^2 = -a_{12} \tag{3}$$

$$a_{23}\cdot(-c_{22}+c_{33})^2 = -a_{23} \tag{4}$$

4. **Solving the system of non-linear equations:** Before using the command `Solve` of MATHEMATICA 5.2 to obtain the solutions of the system obtained in the previous step, let us note that a solution of this system has to be given by complex numbers if we impose that the set $\{Z_1, Z_2, Z_3\}$ is linearly independent because of equations (3) and (4). Effectively, if $a_{12} = a_{23} = 0$ then $b_{12} = b_{23} = 0$ and the vectors Z_1 and Z_2 are proportional. So, it is necessary that $a_{12} \neq 0$ or $a_{23} \neq 0$ and, hence, the solution is given by complex numbers, at least, for one coefficient.

   In this way, a family of solutions of the system obtained with MATHEMATICA 5.2 is the following:

   $$b_{12} = ia_{12},\ c_{23} = \frac{ia_{13}+b_{13}}{a_{12}},\ c_{33} = -i + c_{11},\ c_{22} = i + c_{11},\ a_{23} = b_{23} = 0.$$

   A particular solution of this family is given by:

   $$a_{12}= b_{13}= c_{23}= 1,\ a_{13}= a_{23}= b_{23}= c_{11}= c_{12}= c_{13}= 0,\ b_{12}= c_{22}= i,\ c_{33}= -i.$$

   Obviously, the set $\{Z_1, Z_2, Z_3\}$ given by this particular solution is linearly independent and we have got a minimal representation by upper-triangular matrices for the Lie algebra $\mathfrak{s}_3^4$.

   Let us note that we have made use of MATHEMATICA 5.2 to solve the system because this package is more effective than MAPLE 9.5 to solve non-linear equations.

   □

5 Solvable Lie algebras of dimension 4

The following stage in this paper consists in dealing with 4-dimensional solvable Lie algebras. In this dimension, the systems of non-linear equations which have to be solved are computationally quite a lot more difficult to tackle. Besides, there are a bigger number of isomorphism class of 4-dimensional solvable Lie algebras: 19, which are arranged in 12 non-decomposable Lie algebras and 7 decomposable ones. Let us remember that a Lie algebra is decomposable when it is isomorphic to the direct sum of other two Lie algebras.

In this section, we only study the non-decomposable solvable Lie algebras of dimension 4. So, we show the list of isomorphism class of these algebras according to [1, 8]. In an analogous way in which the isomorphism class was given for dimension 3 in Section 4, $\{Z_i\}_{i=1}^4$ denotes a basis of each 4-dimensional solvable Lie algebra and only the nonzero brackets are written for each non-abelian Lie algebra.

$\mathfrak{s}_4^1$: $[Z_i, Z_j]=0,\ i, j \in \{1, 2, 3, 4\}$ (abelian).

$\mathfrak{s}_4^2$: $[Z_1, Z_3]=Z_2,\ \ [Z_1, Z_4]=Z_3.$

$\mathfrak{s}_4^3$: $[Z_1, Z_3]=Z_3,\ \ [Z_1, Z_4]=Z_4,\ \ [Z_2, Z_3]=Z_4.$

$\mathfrak{s}_4^4$: $[Z_1, Z_3]=Z_3,\ \ [Z_1, Z_4]=Z_4,\ \ [Z_2, Z_3]=-Z_4,\ \ [Z_2, Z_4]=Z_3.$

$\mathfrak{s}_4^5$: $[Z_1, Z_3]=Z_3,\ \ [Z_1, Z_2]=Z_4.$

$\mathfrak{s}_4^6$: $[Z_4, Z_1]=Z_1,\ \ [Z_4, Z_2]=\alpha Z_2,\ \ [Z_4, Z_3]=\beta Z_3.$

$\mathfrak{s}_4^7$: $[Z_3, Z_1]=\alpha Z_1,\ \ [Z_3, Z_2]=Z_2,\ \ [Z_3, Z_4]=Z_2+Z_4.$

$\mathfrak{s}_4^8$: $[Z_1, Z_2]=Z_2+Z_3,\ \ [Z_1, Z_3]=Z_3+Z_4,\ \ [Z_1, Z_4]=Z_4.$

$\mathfrak{s}_4^9$: $[Z_1, Z_4]=\alpha Z_4,\ \ [Z_1, Z_2]=\beta Z_2-Z_3,\ \ [Z_1, Z_3]=Z_2+\beta Z_3.$

$\mathfrak{s}_4^{10}$: $[Z_2, Z_3]=Z_4,\ \ [Z_1, Z_2]=(\alpha-1)Z_2,\ \ [Z_1, Z_3]=Z_3,\ \ [Z_1, Z_4]=\alpha Z_4.$

$\mathfrak{s}_4^{11}$: $[Z_2, Z_3]=Z_4,\ \ [Z_1, Z_2]=Z_2+Z_3,\ \ [Z_1, Z_3]=Z_3,\ \ [Z_1, Z_4]=2Z_4$

$\mathfrak{s}_4^{12}$: $[Z_2, Z_3]=Z_4,\ \ [Z_1, Z_2]=\alpha Z_2-Z_3,\ \ [Z_1, Z_3]=Z_2+\alpha Z_3,\ \ [Z_1, Z_4]=2\alpha Z_4.$

Every 4-dimensional solvable Lie algebra shown in the previous table can be represented by subalgebras of $\mathfrak{b}_4$. However, some of them have their minimal representation by a subalgebra of $\mathfrak{b}_3$ instead of a subalgebra of $\mathfrak{b}_4$. In the following theorem, we have computed the dimension of a minimal representation for each of these algebras, besides of giving a minimal representation:

Theorem 5.1. *A minimal representation by upper-triangular matrices for each non-decomposable solvable Lie algebra of dimension 4 is given in the next table with the dimension of such a minimal representation:*

L.A.	Minimal representation	Dim. ($\bar{\mu}$)
$\mathfrak{s}_4^1$:	$\langle Z_1=X_{11}, Z_2=X_{22}, Z_3=X_{33}, Z_4=X_{44}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^2$:	$\langle Z_1=-(X_{23}+X_{34}), Z_2=X_{14}, Z_3=X_{13}, Z_4=X_{12}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^3$:	$\langle Z_1=X_{11}, Z_2=-X_{23}, Z_3=X_{12}, Z_4=X_{13}\rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_4^4$:	$\langle Z_1=X_{11}, Z_2=-iX_{22}+X_{23}+iX_{33}, Z_3=X_{12}, Z_4=-iX_{12}+X_{13}\rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_4^5$:	$\langle Z_1=X_{12}+X_{33}, Z_2=X_{24}, Z_3=X_{34}, Z_4=X_{14}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^6$:	$\langle Z_1=X_{14}, Z_2=X_{24}, Z_3=X_{34}, Z_4=(\alpha-1)X_{22}+(\beta-1)X_{33}-X_{44}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^7$:	$\langle Z_1=X_{14}, Z_2=-X_{13}, Z_3=X_{11}+X_{23}+(1-\alpha)X_{44}, Z_4=X_{12}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^8$:	$\langle Z_1=X_{12}-X_{23}-X_{44}, Z_2=X_{34}, Z_3=-X_{24}, Z_4=-X_{14}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^9$:	$\langle Z_1=\alpha X_{11}+(\alpha-\beta-i)X_{22}+(\beta-i)X_{33}, Z_2=i(X_{12}-X_{34}), Z_3=X_{12}+X_{34}, Z_4=X_{14}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^{10}$:	$\langle Z_1=\alpha X_{11}+X_{22}, Z_2=X_{12}, Z_3=X_{23}, Z_4=X_{13}\rangle \subseteq \mathfrak{b}_3$	3
$\mathfrak{s}_4^{11}$:	$\langle Z_1=X_{12}+X_{11}+X_{22}-X_{44}, Z_2=X_{23}-X_{34}, Z_3=X_{13}, Z_4=X_{14}\rangle \subseteq \mathfrak{b}_4$	4
$\mathfrak{s}_4^{12}$:	$\langle Z_1=2\alpha X_{11}+(\alpha-i)X_{22}, Z_2=X_{23}+X_{12}, Z_3=iX_{23}-iX_{12}, Z_4=2iX_{13}\rangle \subseteq \mathfrak{b}_3$	3

Proof. We are going to proceed by applying the method shown in Section 3 to obtain a minimal representation by upper-triangular matrices for the Lie algebra $\mathfrak{s}_4^9$. A minimal representation can be got for any other 4-dimensional solvable Lie algebra by using this same method in an analogous way.

We have considered this algebra because of two reasons. First, it corresponds to a bi-parameter family of Lie algebras. Second, computing a minimal representation of this algebra involves to prove that no subalgebra of the Lie algebra b_3 is isomorphic to $\mathfrak{s}_4^9$. This fact will be proved when searching the minimal representation:

1. **Comparing the commutator central series:** In the following table, the ideals of this series are shown for Lie algebras $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$, $\mathfrak{b}_4$ and $\mathfrak{s}_3^4$:

L.A.	$\mathcal{C}_1$	$\mathcal{C}_2$
$\mathfrak{b}_1$	$\langle X_{11}\rangle$	$\{0\}$
$\mathfrak{b}_2$	$\langle X_{11}, X_{12}, X_{22}\rangle$	$\langle X_{12}\rangle$
$\mathfrak{b}_3$	$\langle X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}\rangle$	$\langle X_{12}, X_{13}, X_{23}\rangle$
$\mathfrak{b}_4$	$\langle X_{11}, X_{12}, X_{13}, X_{14}, X_{22}, X_{23}, X_{24}, X_{33}, X_{34}, X_{44}\rangle$	$\langle X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}\rangle$
$\mathfrak{s}_4^9$	$\langle Z_1, Z_2, Z_3, Z_4\rangle$	$\langle Z_2, Z_3, Z_4\rangle$

   In virtue of Proposition 2.1, the solvable Lie algebra $\mathfrak{s}_4^9$ cannot be represented by using the Lie algebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Due to it, we consider the Lie algebra $\mathfrak{b}_3$ which is not in contradiction with Proposition 2.1 because the ideal $\mathcal{C}_3(\mathfrak{s}_4^9)$ is $\{0\}$.

2. **Writing the basis of $\mathfrak{s}_4^9$ with respect to $\mathfrak{b}_3$:** The vectors Z_2, Z_3 and Z_4 belong to $\mathcal{C}_2(\mathfrak{s}_4^9)$ and Z_1 belongs to $\mathcal{C}_1(\mathfrak{s}_4^9)\backslash\mathcal{C}_2(\mathfrak{s}_4^9)$. In this way, we can express these four vectors as the following linear combinations of the basis $\{X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}\}$ of the Lie algebra $\mathfrak{b}_3$:

$$\begin{aligned} Z_1 &= \sum_{\substack{j=i\\ i=1}}^{\substack{i=3\\ j=3}} a_{ij}X_{ij},\\ Z_2 &= b_{12}X_{12} + b_{13}X_{13} + b_{23}X_{23},\\ Z_3 &= c_{12}X_{12} + c_{13}X_{13} + c_{23}X_{23},\\ Z_4 &= d_{12}X_{12} + d_{13}X_{13} + d_{23}X_{23}. \end{aligned}$$

3. **Computing the brackets and comparing them with the law of $\mathfrak{s}_4^9$:** The brackets of this algebra are computed by using MAPLE 9.5. In fact, we can use the routine shown in Theorem 4.1 if the dimension of the algebra and the new expression of the fields Z_i are adjusted in it.

   By equalling the obtained brackets to those which define the Lie algebra $\mathfrak{s}_4^9$, the following system of non-linear equations is obtained:

$$\begin{cases}
\beta b_{23} - c_{23} = a_{22} b_{23} - a_{33} b_{23}, \\
\beta b_{13} - c_{13} = -a_{23} b_{12} + a_{12} b_{23} + a_{11} b_{13} - a_{33} b_{13}, \\
\beta b_{12} - c_{12} = a_{11} b_{12} - a_{22} b_{12}, \\
b_{23} + \beta c_{23} = a_{22} c_{23} - a_{33} c_{23}, \\
b_{13} + \beta c_{13} = -a_{23} c_{12} + a_{12} c_{23} + a_{11} c_{13} - a_{33} c_{13}, \\
b_{12} + \beta c_{12} = a_{11} c_{12} - a_{22} c_{12}, \\
\alpha d_{23} = a_{22} d_{23} - a_{33} d_{23}, \\
\alpha d_{13} = -a_{23} d_{12} + a_{12} d_{23} + a_{11} d_{13} - a_{33} d_{13}, \\
\alpha d_{12} = a_{11} d_{12} - a_{22} d_{12}, \\
0 = b_{23} c_{12} - b_{12} c_{23}, \\
0 = b_{23} d_{12} - b_{12} d_{23}, \\
0 = c_{23} d_{12} - c_{12} d_{23}.
\end{cases}$$

4. **Solving the system of non-linear equations:** By solving the previous system with the command `Solve` of MATHEMATICA 5.2, we obtain that every solution implies that the set $\{Z_2, Z_3, Z_4\}$ is linearly dependent and, hence, we cannot get that $\{Z_1, Z_2, Z_3, Z_4\}$ is a basis. In consequence, no subalgebra of the Lie algebra $\mathfrak{b}_3$ is isomorphic to the Lie algebra $\mathfrak{s}_4^9$ and, hence, $\mathfrak{s}_4^9$ cannot be represented by 3×3 upper-triangular matrices.

   Now, we have to go back to step 2 to search the representation in the Lie algebra $\mathfrak{b}_4$ of 4×4 upper-triangular matrices.

5. **Writing the basis of $\mathfrak{s}_4^9$ with respect to $\mathfrak{b}_4$:** With an analogous reasoning to which was used in step 2 of this proof, the vectors Z_1, Z_2, Z_3 and Z_4 are now expressed as the following linear combinations of the basis $\{X_{11},\ X_{12},\ X_{13},\ X_{14},\ X_{22},\ X_{23},\ X_{24},\ X_{33},\ X_{34},\ X_{44}\}$ of the Lie algebra $\mathfrak{b}_4$:

$$Z_1 = \sum_{\substack{j=i \\ i=1}}^{\substack{i=4 \\ j=4}} a_{ij} X_{ij}; \quad Z_2 = \sum_{\substack{j=i+1 \\ i=1}}^{\substack{i=4 \\ j=4}} b_{ij} X_{ij}; \quad Z_3 = \sum_{\substack{j=i+1 \\ i=1}}^{\substack{i=4 \\ j=4}} c_{ij} X_{ij}; \quad Z_4 = \sum_{\substack{j=i+1 \\ i=1}}^{\substack{i=4 \\ j=4}} d_{ij} X_{ij}.$$

6. **Computing the brackets and comparing them with the law of $\mathfrak{s}_4^9$:** We compute the brackets again with MAPLE 9.5 and we obtain now the following system:

$$\begin{cases} \beta b_{34} - c_{34} = a_{33}b_{34} - b_{34}a_{44}, \\ \beta b_{24} - c_{24} = a_{22}b_{24} + a_{23}b_{34} - b_{23}a_{34} - b_{24}a_{44}, \\ \beta b_{23} - c_{23} = a_{22}b_{23} - a_{33}b_{23}, \\ \beta b_{14} - c_{14} = a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} - b_{12}a_{24} - b_{13}a_{34} - b_{14}a_{44}, \\ \beta b_{13} - c_{13} = a_{11}b_{13} + a_{12}b_{23} - a_{23}b_{12} - a_{33}b_{13}, \\ \beta b_{12} - c_{12} = a_{11}b_{12} - a_{22}b_{12}, \\ b_{34} + \beta c_{34} = a_{33}c_{34} - c_{34}a_{44}, \\ b_{24} + \beta c_{24} = a_{22}c_{24} + a_{23}c_{34} - c_{23}a_{34} - c_{24}a_{44}, \\ b_{23} + \beta c_{23} = a_{22}c_{23} - a_{33}c_{23}, \\ b_{14} + \beta c_{14} = a_{11}c_{14} + a_{12}c_{24} + a_{13}c_{34} - c_{12}a_{24} - c_{13}a_{34} - c_{14}a_{44}, \\ b_{13} + \beta c_{13} = a_{11}c_{13} + a_{12}c_{23} - a_{23}c_{12} - a_{33}c_{13}, \\ b_{12} + \beta c_{12} = a_{11}c_{12} - a_{22}c_{12}, \\ \alpha d_{34} = a_{33}d_{34} - d_{34}a_{44}, \\ \alpha d_{24} = a_{22}d_{24} + a_{23}d_{34} - d_{23}a_{34} - d_{24}a_{44}, \\ \alpha d_{23} = a_{22}d_{23} - a_{33}d_{23}, \\ \alpha d_{14} = a_{11}d_{14} + a_{12}d_{24} + a_{13}d_{34} - d_{12}a_{24} - d_{13}a_{34} - d_{14}a_{44}, \\ \alpha d_{13} = a_{11}d_{13} + a_{12}d_{23} - a_{23}d_{12} - a_{33}d_{13}, \\ \alpha d_{12} = a_{11}d_{12} - a_{22}d_{12}, \\ 0 = b_{23}c_{34} - c_{23}b_{34}, \\ 0 = b_{12}c_{24} + b_{13}c_{34} - c_{12}b_{24} - c_{13}b_{34}, \\ 0 = b_{12}c_{23} - b_{23}c_{12}, \\ 0 = b_{23}d_{34} - d_{23}b_{34}, \\ 0 = b_{12}d_{24} + b_{13}d_{34} - d_{12}b_{24} - d_{13}b_{34}, \\ 0 = b_{12}d_{23} - b_{23}d_{12}, \\ 0 = c_{23}d_{34} - d_{23}c_{34}, \\ 0 = c_{12}d_{24} + c_{13}d_{34} - d_{12}c_{24} - d_{13}c_{34}, \\ 0 = c_{12}d_{23} - c_{23}d_{12}. \end{cases}$$

7. **Solving the system of non-linear equations:** By solving the previous system with the help of MATHEMATICA 5.2, we have obtained the following particular solution in which all the unknowns are equal to 0, with the exception of:

$$a_{11} = \alpha, \quad a_{22} = \alpha - \beta - i, \quad a_{33} = \beta - i, \quad b_{14} = 1, \quad c_{12} = 1, \quad d_{34} = 1.$$

   Obviously, the set $\{Z_1, Z_2, Z_3\}$ given by this particular solution is linearly independent and we have got a minimal representation by upper-triangular matrices for the Lie algebra $\mathfrak{s}_4^9$. □

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