

Automatic discovery of geometry theorems using minimal canonical comprehensive Gröbner systems (Extended Abstract)

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1 Introduction

The main idea in this talk is the merging of two recently developed techniques. On the one hand, we will consider a recent proposal (named *MCCGS*, standing for *minimal canonical comprehensive Gröbner systems*) [MaMo06], that is –roughly speaking– a computational tool yielding “good” bases for ideals of polynomials over a field depending on several parameters, where “good” means that the obtained bases should specialize “well”, for instance, regarding the number of solutions for the given ideal, for different values of the parameters.

The second ingredient of our contribution is about automatic theorem discovery in elementary geometry. Automatic discovery aims to obtain complementary hypotheses for a (generally false) geometric statement to become true. For instance, we can consider an arbitrary triangle and the feet on each sides of the three altitudes. These three feet give us another triangle, and now we want to conclude that such triangle is isosceles. This is generally false, but, under what extra hypotheses on the given triangle will it become true?

Finding, in an automatic way, the necessary and sufficient conditions for this statement to become a theorem, is the task of automatic discovery.

Our goal in this talk is to show how performing a *MCCGS* procedure on a certain ideal built up from the given hypotheses and thesis, depending on the free coordinates of some elements of the geometric setting, can improve the automatic discovery of geometry theorems.

Our contribution differs from [CLLW] in two senses: first, we focus on automatic discovery, and not in automatic proving. Second, the use of *MCCGS* provides not only the specialization property (which is the key for the application of partitioned parametric bases in [CLLW]) but also a case distinction, that allows a richer understanding of the underlying geometry.

2 A digest on automatic discovery

The simple idea behind the different approaches for automatic discovery is, essentially, that of adding the conjectural thesis to the collection of hypotheses, and then deriving, from this new ideal of thesis plus hypotheses, some new constraints in terms of the free parameters ruling the geometric situation. A detailed description of the procedure we will follow for automatic discovery appears in [RV99], and it has been recently revised in [BDR] and [DR], showing that, in some precise sense, the procedure is intrinsically unique.

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Let us recall here that the approach to discovery in [RV99] proceeds, roughly speaking, first identifying a set of independent variables (those ruling the construction of the hypothesis variety, defined by the zeroes over an algebraically closed field of the hypotheses ideal H). The corresponding components of this variety, where these variables are independent (and maximally independent, as well) are called *privileged*. Let us consider the *Saturation* (e.f. [KR]) of the ideal of hypotheses by the ideal T of thesis, $H : T^\infty$.

With this notation it can be shown (see [RV99], [DR]) that the elimination ideal (over the independent variables), of $H : T^\infty$ is not zero if and only if the theorem is true over all the privileged components (and then the theorem is called “generally true” [Ch88]).

When the given theorem is not generally true, it turns that the elimination ideal of the ideal generated by the hypotheses plus the thesis is not zero if and only if the thesis does not hold over any privileged component (the so called “generally false” case, the one suitable for discovery).

3 Overview on the *MCCGS* algorithm

Given a parametric polynomial system of equations, our interest focuses on discussing the type of solutions depending on the values of the parameters. Let $\mathbf{x} = (x_1, \dots, x_n)$ be the set of variables, $\mathbf{u} = (u_1, \dots, u_m)$ the set of parameters and $I \subset \mathbb{Q}[\mathbf{u}][\mathbf{x}]$ the parametric ideal we want to discuss. We want to study how the complex solutions of the equation system defined by I vary when we specialize the values of the parameters \mathbf{u} to concrete values $\mathbf{u}_0 \in \mathbb{C}$. Denote by $A = \mathbb{Q}[\mathbf{u}]$, and by $\sigma_{\mathbf{u}_0} : A[\mathbf{x}] \rightarrow \mathbb{C}[\mathbf{x}]$ the homomorphism corresponding to the specialization (substitution of \mathbf{u} by some $\mathbf{u}_0 \in \mathbb{C}$).

A Gröbner System of the ideal $I \subset A[\mathbf{x}]$ wrt (with respect to) the termorder $\succ_{\mathbf{x}}$ is a set

$$\text{GS}(I, \succ_{\mathbf{x}}) = \{(S_i, B_i) : 1 \leq i \leq s, S_i \subset \mathbb{C}^m, B_i \subset A[\mathbf{x}], \bigcup_i S_i = \mathbb{C}^m, \\ \forall \mathbf{u}_0 \in S_i, \sigma_{\mathbf{u}_0}(B_i) \text{ is a Gröbner basis of } \sigma_{\mathbf{u}_0}(I) \text{ wrt } \succ_{\mathbf{x}}\}.$$

The algorithm *MCCGS* (Minimal Canonical Comprehensive Gröbner System) [MaMo06] of the ideal $I \subset A[\mathbf{x}]$ wrt the monomial order $\succ_{\mathbf{x}}$ for the variables, builds up the unique Gröbner System having the following properties:

1. $\mathcal{S} = \{S_1, \dots, S_s\}$ is a partition of the parameter space \mathbb{C}^m .
2. The bases B_i are normalized to have content 1 wrt \mathbf{x} over $\mathbb{Q}[\mathbf{u}]$, and the leading coefficients are different from zero on every point of S_i . Moreover, the B_i specialize to the reduced Gröbner basis of $\sigma_{\mathbf{u}_0}(I)$, keeping the same lpp’s (leading power products set) for each $\mathbf{u}_0 \in S_i$. Thus a concrete set of lpp’s can be associated to a given S_i . Moreover, although a same set of lpp’s can be attached to different S_i ’s, if two segments S_i, S_j share the same lpp’s, then there is not a common basis B specializing to both B_i, B_j .
3. The partition \mathcal{S} is canonical (unique for a given I and monomial order).
4. The partition is minimal, in the sense it does not exists another partition having property 2 with less sets S_i .
5. The sets S_i (often called *segments*) are constructible and are described in a canonical form.

4 Using *MCCGS* for automatic theorem discovering

As stated in sections 2 and 3, automatic discovery can be approached considering as new hypotheses the generators of the elimination, over the free parameters of the problem, of the ideal of hypotheses and thesis $I = (H, T)$. This is, precisely, the (Zariski closure of the) projection, over the parameter space, of the zero set of this ideal I . It is clear, then, that, for automatic discovery through *MCCGS* one must perform such decomposition over I , selecting those segments S_i such that the system I has at least one solution (in

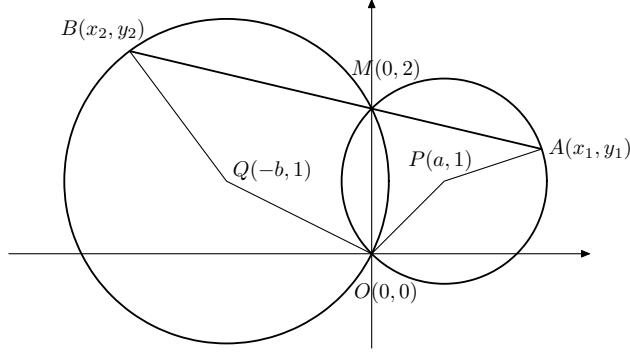


Figure 1: Skaters problem

the complex field or in whatever algebraically closed field we are working with) over S_i . In other words, discarding the S_i 's with B_i equal to 1 and keeping the remaining S_i 's. The description of these S_i 's gives, precisely, the new conditions for the thesis to hold over the hypotheses variety.

Let us see how this works in one example, where we have just detailed the discovery step in the procedure outlined above. That is, we have not included here the verification in each case that the newly found hypotheses actually lead to a true statement (the proving step, which should be performed in the standard way; in particular, it could be done using *MCCGS* to test if 1 belongs to the saturation of the ideal of new hypotheses by the thesis).

Example 1.¹ Consider two circles with centers at $P(a, 1)$ and $Q(-b, 1)$ and radius $r_1^2 = a^2 + 1$ and $r_2^2 = b^2 + 1$, as shown in Figure 1, intersecting at points $O(0, 0)$ and $M(0, 2)$. Consider two generic skaters at points $A(x_1, y_1)$ and $B(x_2, y_2)$ running on the respective circles, parametrised respectively by the oriented angles $v = \widehat{OPA}$ and $w = \widehat{OQB}$. Angle zero corresponds to the starting position $A = B = O$.

We claim that, for whatever position of the two skaters *the points A, M, B are aligned*, which is obviously false in general. But we want to determine the relation between the two oriented angles v and w making this statement to hold true. Denote c_v, s_v, c_w, s_w the cosine and sine of the angles v and w . It is easy to establish the basic hypotheses:

$$H = [(x_1 - a)^2 + (y_1 - 1)^2 - a^2 - 1, (x_2 + b)^2 + (y_2 - 1)^2 - b^2 - 1, \\ a(x_1 - a) + (y_1 - 1) + (a^2 + 1)c_v, -b(x_2 + b) + (y_2 - 1) + (1 + b^2)c_w, \\ a(y_1 - 1) - (x_1 - a) + (a^2 + 1)s_v, -b(y_2 - 1) - (x_2 + b) + (b^2 + 1)s_w]$$

The thesis is, clearly: $T = x_1 y_2 - 2x_1 - x_2 y_1 + 2x_2$. The radii of the circles are $r_1^2 = a^2 + 1$ and $r_2^2 = b^2 + 1$ and for $r_1 \neq 0$ and $r_2 \neq 0$ we have

$$c_{v_0} = \cos v_0 = \cos \widehat{OPM} = \frac{a^2 - 1}{a^2 + 1}, \quad s_{v_0} = \sin v_0 = \sin \widehat{OPM} = \frac{-2a}{a^2 + 1}, \\ c_{w_0} = \cos w_0 = \cos \widehat{OQM} = \frac{b^2 - 1}{b^2 + 1}, \quad s_{w_0} = \sin w_0 = \sin \widehat{OQM} = \frac{2b}{b^2 + 1}.$$

We want to take a, b and the angles v and w –in terms of the sines and cosines– as parameters. So we must introduce the constraints on the sine and cosine parameters. Moreover, we notice there are also some obvious degenerate situations, namely $r_1 = 0$, $r_2 = 0$ and $a + b = 0$, corresponding to null radii or coincident circles, and we want to avoid them.

¹Taken from the pastimes section of the French journal *Le Monde*, published on the printed edition of Jan. 8, 2007. This example is there attributed to E. Busser and G. Cohen.

Currently, *MCCGS* allows us to introduce all these constraints in order to discuss the parametric system. The call is

$$\text{mccgs}(H \cup T, \text{lex}(x_1, y_1, x_2, y_2), \text{lex}(a, b, s_v, c_v, s_w, c_w), \\ \text{null} = [c_v^2 + s_v^2 - 1, c_w^2 + s_w^2 - 1], \text{nonnull} = \{a^2 + 1, b^2 + 1, a + b\}).$$

including the constraints on the parameters and eluding degenerate situations as options for *MCCGS*.

The result is that *MCCGS* outputs only 2 cases. The first one has basis [1], showing that, in general, there is no solution to our query. The second one has lpp = $[y_2, x_2, y_1, x_1]$ determining in a unique form the points A and B for the given values of the parameters. The associated basis is

$$[y_2 + c_w - bs_w - 1, x_2 - bc_w - s_w + b, y_1 + c_v + as_v - 1, x_1 + ac_v - s_v - a]$$

with parameter conditions that are expressed as the union of three irreducible varieties:

$$\begin{aligned} V_1 &= \mathbb{V}(c_w^2 + s_w^2 - 1, c_v - c_w, s_v - s_w) \\ V_2 &= \mathbb{V}(c_w^2 + s_w^2 - 1, c_v^2 + s_v^2 - 1, s_w + bc_w - b, bs_w - c_w - 1) \\ V_3 &= \mathbb{V}(c_w^2 + s_w^2 - 1, c_v^2 + s_v^2 - 1, -s_v + ac_v - a, as_v + c_v + 1) \end{aligned}$$

The interpretation is easy: V_1 corresponds to the essential condition $v = w$, stating that our conjecture requires (and it is easy to show that this condition is sufficient) that both skaters keep moving with the same angular speed. V_2 and V_3 correspond to the degenerate cases where either $B = M$ and A can take any position or $A = M$ and B anywhere. We can summarize the above discussion in the following

Theorem 2. *Given two non coincident circles of non-null radii and centers P and Q , intersecting at two points O and M , let us consider points A, B on each of the circles. Then the three points A, M, B are aligned if and only if the oriented angles \widehat{OPA} and \widehat{OQB} are equal or A or B or both coincide with M .*

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