Checking if the toric ideal of an affine monomial curve is a complete intersection

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Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Denote by x^a the monomial $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. A *binomial* f in R is a difference of two monomials, i.e. , $f = x^a - x^b$ for some $a, b \in \mathbb{N}^n$. An ideal of R generated by binomials is called a *binomial ideal.*

Let $\underline{d} = \{d_1, \ldots, d_n\}$ be a set of all-different positive integers and consider the monomial curve

$$\Gamma = \{(t^{d_1}, \ldots, t^{d_n}) \in \mathbb{A}_k^n \,|\, t \in k\}\,.$$

The image of the homomorphism of k-algebras $\phi\colon R \to k[t];\ x_i \longmapsto t^{d_i}$ will be denoted by $k[\Gamma]$ and its kernel will be denoted by $I(d_1, \ldots, d_n)$. The ideal $I(d_1, \ldots, d_n)$ is called the *toric ideal* of Γ. It is a *quasi-homogeneous* ideal, i.e., a homogeneous ideal when one gives degree d_i to variable x_i for all $i \in \{1, \ldots, n\}$. Since $k[t]$ is integral over $k[\Gamma]$, the height of $I(d_1, \ldots, d_n)$ is equal to $n - 1$. By [10, Proposition 7.1.2], the toric ideal $I(d_1, \ldots, d_n)$ is generated by quasi-homogeneous binomials. According to [5, Lemma 3.4 and Remark 3.5], if either $\gcd(\underline{d}) = 1$ or k is algebraically closed, we get $\Gamma = V(I(d_1, \ldots, d_n))$, i.e., Γ is a toric variety. If k is an infinite field, by Corollary 7.1.12 in Villarreal [10] , the ideal $I(\Gamma)$ of polynomials vanishing on Γ is equal to $I(d_1, \ldots, d_n)$.

The prime ideal $I(d_1, \ldots, d_n)$ is called a *complete intersection* if there exists a system of quasi-homogeneous binomials $g_1, \ldots, g_{n-1}$ such that $I(d_1, \ldots, d_n) = (g_1, \ldots, g_{n-1})$.

The aim of this work is to obtain and implement an algorithm for determining whether $I(d_1, \ldots, d_n)$ is a complete intersection or not without computing a minimal set of quasi-homogeneous generators of the toric ideal.

If $n = 3$, setting $c_i := \min\left(\mathbb{Z}^+ d_i \cap \sum_{\substack{j \in \{1,2,3\} \\ j \neq i}} \mathbb{N} d_j\right)$ for all $i \in \{1, 2, 3\}$, Herzog obtained in [8] the following result:

$$I(d_1, d_2, d_3) \text{ is a complete intersection} \Longleftrightarrow \exists\, i, j : 1 \leq i < j \leq 3 \text{ such that } c_i = c_j.$$

In order to obtain a good implementation of the algorithm, it is necessary to provide methods for computing the c_i efficiently.

It is well-known that Herzog's characterization does not hold for $n > 3$. In this work, we use the combinatorial-arithmetical structure of complete intersections given by the existence of certain binary trees labeled by $\{d_1, \ldots, d_n\}$ in [1, Theorem 4.3] in order to obtain an algorithm based on the computation of the smallest positive multiple of an integer that belongs to a semigroup. Therefore, the contribution of this work can be seen as a generalization of Herzog's result.

It is worthwhile to mention here that Delorme in [3, Section 14] obtained another algorithm for checking whether the toric ideal $I(d_1, \ldots, d_n)$ is a complete intersection or not. The algorithm uses his characterization of complete intersections given by the existence of certain *suites distinguées* (see [3, Lemma 8]). Besides the fact that our approach is different in nature to that of Delorme, the most important difference between the two algorithms is that while our algorithm needs the computation of the smallest positive multiple of an integer in a semigroup, Delorme's algorithm is based on the computation of the smallest positive integer in the intersection of two semigroups, being the cost of this computation higher than the previous one.

To present our algorithm, let us start with the following result, where c_i denotes $\min\left(\mathbb{Z}^+ d_i \cap \sum_{\substack{j\in\{1,\ldots,n\}\\ j\neq i}} \mathbb{N} d_j\right)$ for all $i \in \{1, \ldots, n\}$:

Theorem 1. *Let k be an arbitrary field. If $I(d_1, \ldots, d_n) \subset k[x_1, \ldots, x_n]$ is a complete intersection, then there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that $c_i = c_j$. Moreover, if $c_i = c_j$ for $i, j : 1 \leq i < j \leq n$, then the toric ideal $I(d'_1, \ldots, d'_{n-1}) \subset k[x_1, \ldots, x_{n-1}]$ is a complete intersection, where $d'_r = d_r$ for $r \in \{1, \ldots, \widehat{i}, \ldots, j-1\}$, $d'_i = \gcd(d_i, d_j)$, and $d'_r = d_{r+1}$ for $r \geq j$.*

The theorem provides necessary but not sufficient conditions for $I(d_1, \ldots, d_n)$ to be a complete intersection as the following example shows:

Example 2. The toric ideal $I(45, 75, 70, 147, 98)$ is not a complete intersection since one gets using, e.g., SINGULAR [7] that 7 is the minimal number of quasi-homogeneous generators for $I(45, 75, 70, 147, 98)$ (see also Remark 5 below). Nevertheless,

$$c_1 = \min(\mathbb{Z}^+ 45 \cap \mathbb{N}\{75, 70, 147, 98\}) = 225 = \min(\mathbb{Z}^+ 75 \cap \mathbb{N}\{45, 70, 147, 98\}) = c_2$$

and setting $d'_1 := \gcd(d_1, d_2) = \gcd(45, 75) = 15$, $d'_2 := 70$, $d'_3 := 147$ and $d'_4 := 98$ we have that $I(d'_1, d'_2, d'_3, d'_4)$ is a complete intersection (see Example 4 below).

However, Theorem 1 can be refined in order to reach the following algorithm that has been implemented in C++:

Algorithm 3. Input: $\underline{d} = \{d_1, \ldots, d_n\}$
Output:
TRUE if $I(d_1, \ldots, d_n)$ is a complete intersection
FALSE otherwise

```
*boolean Complete_Intersection* ($\underline{d} = \{d_1, \ldots, d_n\}$) {
    $G_1 := \underline{d}$
    $V_i := \{d_i\}$ for $i \in \{1, \ldots, n\}$
    $c_i := \min(\mathbb{Z}^+ d_i \cap \sum_{\substack{j \in \{1,\ldots,n\} \\ j \neq i}} \mathbb{N} d_j)$
    FOR $i = 1$ TO $n - 1$ DO {
        IF $((\forall j, k,\ j \neq k / d_j, d_k \in G_i) \Rightarrow c_j \neq c_k)$ THEN {
            RETURN FALSE
        }
        IF $(i = n - 1)$ {
            RETURN TRUE
        }
        SET $j, k \ /\ j \neq k,\ d_j, d_k \in G_i$ and $c_j = c_k$
        $d_{n+i} := \gcd(d_j, d_k)$
        $V_{n+i} := V_j \cup V_k$
        $c_{n+i} := \min(\mathbb{Z}^+ d_{n+i} \cap \sum_{d_m \in\ G_i \setminus \{d_j, d_k\}} \mathbb{N} d_m)$
        IF $c_{n+i} \notin \sum_{d_j \in V_{n+i}} \mathbb{N} d_j$ THEN {
            RETURN FALSE
        }
        $G_{i+1} := (G_i \setminus \{d_j, d_k\}) \cup \{d_{n+i}\}$
    }
}
```

Example 4. The toric ideal $I(d_1, d_2, d_3, d_4)$, where $d_1 = 15, d_2 = 70, d_3 = 98$ and $d_4 = 147$, is a complete intersection. Indeed, set $G_1 := \{15, 70, 98, 147\}$, $V_1 := \{15\}$, $V_2 := \{70\}$, $V_3 := \{98\}$, and $V_4 := \{147\}$.

Now, for all $i \in \{1, \ldots, 4\}$ compute $c_i := \min\left(\mathbb{Z}^+ d_i \cap \sum_{\substack{j \in \{1,\ldots,4\} \\ j \neq i}} \mathbb{N} d_j\right)$.

We have that $c_1 = 210,\ c_2 = 210,\ c_3 = 294,\ c_4 = 294$ and observe that $c_1 = c_2$.

Then, we define

$$d_5 := \gcd(d_1, d_2) = \gcd(15, 70) = 5\,,$$

$$V_5 := V_1 \cup V_2 = \{15, 70\}\,, \text{ and}$$

$$c_5 := \min(\mathbb{Z}^+ d_5 \cap (\mathbb{N} d_3 + \mathbb{N} d_4)) = \min(\mathbb{Z}^+ 5 \cap (\mathbb{N}98 + \mathbb{N}147)) = 245\,.$$

Observe that

$$c_5 \in \mathbb{N}d_1 + \mathbb{N}d_2 = \mathbb{N}\{15, 70\} \ (245 = 7 * 15 + 2 * 70).$$

Set $G_2 := \{d_3, d_4, d_5\} = \{98, 147, 5\}$, compare c_3, c_4, c_5 and observe that $c_3 = c_4 = 294$. Then, we define

$$d_6 := \gcd(d_3, d_4) = \gcd(98, 147) = 49\,,$$

$$V_6 := V_3 \cup V_4 = \{98, 147\}\,, \text{ and}$$

$$c_6 := \min(\mathbb{Z}^+ d_6 \cap \mathbb{N}d_5) = \min(\mathbb{Z}^+ 49 \cap \mathbb{N}5) = 245.$$

Observe that

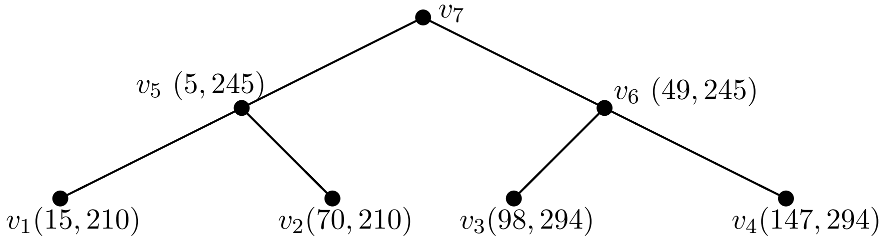
$$c_6 \in \mathbb{N}d_3 + \mathbb{N}d_4 = \mathbb{N}\{98, 147\} \ (245 = 1 * 98 + 1 * 147).$$

Finally, set $G_3 := \{d_5, d_6\} = \{5, 49\}$ and compare c_5, c_6.

Since $c_5 = c_6 = 245$, the toric ideal $I(15, 70, 98, 147)$ is a complete intersection.

Remark 5. Using the algorithm, let us now check that the ideal $I(45, 75, 70, 98, 147)$ in Example 2 is not a complete intersection. Indeed, setting $G_1 := \{45, 75, 70, 98, 147\}$, $d_1 := 45$, $d_2 := 75$, $d_3 := 70$, $d_4 := 98$, $d_5 := 147$ and $V_1 := \{45\}$, $V_2 := \{75\}$, $V_3 := \{70\}$, $V_4 := \{98\}$ and $V_5 := \{147\}$, we have that $c_1 = 225$, $c_2 = 225$, $c_3 = 210$, $c_4 = 294$ and $c_5 = 294$. We observe that $c_1 = c_2$ and define $d_6 := \gcd(d_1, d_2) = \gcd(45, 75) = 15$ and $V_6 := V_1 \cup V_2 = \{45, 75\}$. We get $c_6 = 210 \in \mathbb{N}\{45, 75\}$ $(210 = 3 * 45 + 1 * 75)$. Now, set $G_2 := \{d_3, d_4, d_5, d_6\} = \{70, 98, 147, 15\}$, observe that $c_3 = c_6 = 210$, and define $d_7 := \gcd(d_3, d_6) = \gcd(70, 15) = 5$ and $V_7 := V_3 \cup V_6 = \{70, 45, 75\}$. One has that $c_7 = 245 \notin \mathbb{N}\{70, 45, 75\}$. Thus, $I(45, 75, 70, 98, 147)$ is not a complete intersection.

Remark 6. Following the algorithm, we represent the complete intersection toric ideal $I(15, 70, 98, 147)$ in Example 4 by the following *binary tree labeled by* $\{(d_1, c_1), (d_2, c_2), (d_3, c_3), (d_4, c_4), (d_5, c_5), (d_6, c_6)\}$ called $\mathcal{G}$:



Observe that for all non-terminal vertex v_i of $\mathcal{G}$ with children v_j and v_k we have that $d_i = \gcd(d_j, d_k)$, $c_j = c_k$, and c_i belongs to the semigroup $\mathbb{N}\{d_s\,; v_s \in l[v_i]\}$ where $l[v_i]$ denotes the set of terminal vertices of the subtree of $\mathcal{G}$ whose root is v_i. For a toric ideal $I(d_1, \ldots, d_n)$, we have proved that the existence of a binary tree labeled by a subset $\{(d_1, c_1), \ldots, (d_{2n-2}, c_{2n-2})\}$ of $\mathbb{Z}^+ \times \mathbb{Z}^+$ verifying these properties is equivalent to saying that $I(d_1, \ldots, d_n)$ is a complete intersection. From this characterization the algorithm has been obtained.

The concept of binary tree labeled by $\{(d_1, c_1), \ldots, (d_{2n-2}, c_{2n-2})\}$ slightly modifies the concept of binary tree labeled by $\{d_1, \ldots, d_n\}$ introduced in Section 3 of [1]. When $I(d_1, \ldots, d_n)$ is a complete intersection, a binary tree labeled by $\{(d_1, c_1), \ldots, (d_{2n-2}, c_{2n-2})\}$ satisfying the above properties encodes the following information by [1, Remark 4.5]:

1. A system of quasi-homogeneous binomials $g_1, \ldots, g_{n-1}$ generating $I(d_1, \ldots, d_n)$.
2. The *Frobenius number* $g(S)$ of the numerical semigroup $S = \mathbb{N}\underline{d}$, i.e., the largest integer not in S. In fact,

$$g(S) = \sum_{1 \le i \le 2n-2} \frac{c_i}{2} - \sum_{1 \le i \le n} d_i\,.$$

For the toric ideal $I(15, 70, 98, 147)$, one gets $g_1 = x_1^{14} - x_2^3$, $g_2 = x_3^3 - x_4^2$, $g_3 = x_1^7 x_2^2 - x_3 x_4$, and the Frobenius number of the numerical semigroup $\mathbb{N}\{15, 70, 98, 147\}$ is 419.

As consequences of the algorithm, we have obtained the following results:

Corollary 7. *$I(d_1, d_2, d_3, d_4)$ is a complete intersection $\Longleftrightarrow$ after reindexing the d_i's if necessary, the following three conditions hold:*

1. $c_1 = c_2$,
2. *$I(d_3, d_4, d_5)$ is a complete intersection where $d_5 = \gcd(d_1, d_2)$, and*
3. *$c_3' = c_3$, $c_4' = c_4$ and $c_5' \in \mathbb{N}d_1 + \mathbb{N}d_2$, where $c_i' = \min\left(\mathbb{Z}^+ d_i \cap \sum_{\substack{j \in \{3,4,5\} \\ j \neq i}} \mathbb{N}d_j\right)$ for all $i \in \{3, 4, 5\}$.*

Example 2 shows that this characterization does not hold for $n > 4$. Indeed, as we have already seen, $c_1 = c_2 = 225$ and $I(70, 147, 98, 15)$ is a complete intersection. Moreover, one can check that $c_3' = c_3 = 210$, $c_4' = c_4 = 294$ and $c_5' = c_5 = 294$. Finally, as shown in Remark 5, $c_6' = \min\ (\mathbb{Z}^+ 15 \cap \mathbb{N}\{70, 147, 98\}) = 210 \in \mathbb{N}\{45, 75\}$.

Corollary 8. *([9, Theorem 3.5] & [6, Corollary 9]) Let* $\underline{d} = \{d_1, \ldots, d_n\}$ *be a set of positive integers with* $\gcd(\underline{d}) = 1$ *such that* $d_1, \ldots, d_n$ *is an arithmetic sequence. Then,*

$$I(d_1, \ldots, d_n) \textit{ is a complete intersection} \Longleftrightarrow \left\{ \begin{array}{c} n = 2 \\ \textit{or} \\ n = 3 \textit{ and } d_1 \textit{ even} \end{array} \right.$$

The next two results concerns with toric ideals $I(d_1, \ldots, d_n)$ where $d_1, \ldots, d_n$ is an almost arithmetic sequence. Recall from [9] that, for $n \geq 4$, $d_1, \ldots, d_n$ is an *almost arithmetic sequence* if $d_1, \ldots, d_{n-1}$ is an arithmetic sequence.

Corollary 9. *([9, Theorem 3.7]) Let* $\underline{d} = \{d_1, \ldots, d_n\}$ *be a set of positive integers with* $n > 4$ *and* $\gcd(\underline{d}) = 1$ *such that* $d_1, \ldots, d_n$ *is an almost arithmetic sequence. Then,* $I(d_1, \ldots, d_n)$ *is not a complete intersection.*

Corollary 10. *Let* $\underline{d} = \{d_1, d_2, d_3, d_4\}$ *be a set of positive integers with* $\gcd(\underline{d}) = 1$ *such that* d_1, d_2, d_3, d_4 *is an almost arithmetic sequence. Set* $e := \gcd(d_1, d_2, d_3)$ *and* $d_i' := \frac{d_i}{e}$ *for* $i \in \{1, 2, 3\}$*. Then,*

$$\left. \begin{array}{l} I(d_1, d_2, d_3, d_4) \textit{ is a} \\ \textit{complete intersection} \end{array} \right\} \Longleftrightarrow d_1' \textit{ is even and} \left\{ \begin{array}{ll} i) & d_4 \in \mathbb{N}\{d_1', d_2', d_3'\} \textit{ or} \\ ii) & d_4 \textit{ is even and } I(d_1, d_3, d_4) \\ & \textit{is a complete intersection} \end{array} \right.$$

In closing this extended abstract our focus will be on the methods for computing c_i which play a fundamental role in the efficiency of our algorithm. One idea is to use an Integer Linear Programming (ILP) formulation. For example, for c_1, this formulation is as follows:

$$\begin{aligned} \min \quad & x_1 && (1) \\ & d_1 + d_1 x_1 = d_2 x_2 + \cdots + d_n x_n && (2) \\ & x_1, x_2, \ldots, x_n \geq 0 && (3) \\ & x_1, x_2, \ldots, x_n \in \mathbb{Z} && (4) \end{aligned}$$

Clearly, an algorithm for computing c_i also decides whether a linear Diophantine equation admits a non-negative and non-trivial solution. Since this decision problem is NP-complete (see, e.g., [11]), then our optimization problem is NP-hard.

Despite the fact that this problem can be easily formulated as an ILP model, unfortunately modern ILP solvers (including CPLEX 10 and XPRESS 2006) fail to find even an integer solution for small instances. This justifies the research of new approaches to compute

c_i . After conducting several experiments based on different ideas, our best results have been obtained applying a Graph Theory method following the same spirit of Clausen and Fortenbacher [2]. The idea is to represent each element of the semigroup $(\mathbb{Z}^+ d_i \cap \sum_{\substack{j\in\{1,\ldots,n\}\\ j\neq i}} \mathbb{N} d_j)$ as the weight of a path between two vertices in a weighted graph. Then, the problem of computing c_i is transformed into the problem of finding the shortest weighted path in this graph.

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