# PN-surfaces for blending applications

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**Abstract**

General theory of PN-surfaces (surfaces with rational offsets) is developed using dual Laguerre geometry and a universal rational parametrization of the Blaschke cylinder. This approach allows to construct new PN-blendings between two natural quadrics in many positions, and demostrate improved shape and lower parametrization degree than earlier proposed canal surface solutions. The construction can be extended via inversion to blendings between natural quadrics and Dupin cyclides as well.

## Introduction

In current CAD systems curves and surfaces are represented in a standard NURBS form, i.e. they are parametrized by rational B-splines. Unfortunately, offsets of rational surfaces arising in practical applications are in general not rational and need to be approximated. On the other hand traditional 3d modeling primitives like natural quadrics (sphere, circular cylinder/cone) or torus surfaces admit rational offsets. According to [12] about 99% of mechanical parts can be represented by combinations of planes and natural quadrics with the possibility of representing blends between them. Since rolling ball blends have rather intuitive shape, they are the most popular in practice, but in general they are irrational or admit rational parametrizations of high degree (see details in Section 1). Fortunately, in many cases the smoothness of the blend is more important than its exact shape.

This situation in CAD industry motivates us to search for rational surfaces with rational offsets of minimal parametrization degree that can serve as blending surfaces between natural quadrics.

A rational surface $F(t,u) = (F_1(t,u), F_2(t,u), F_3(t,u))$ with a unit normal

$$N(t,u) = \left(\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial u}\right) \Big/ \left\|\frac{\partial F}{\partial t} \times \frac{\partial F}{\partial u}\right\|$$

is called a *PN-surface* (PN = **P**itagorean **N**ormal) if the normal $N(t,u)$ is rational. Notice that an offset of such PN-surface in the distance $h$ can be parametrized as $F(t,u) + hN(t,u)$ and it is rational as well.

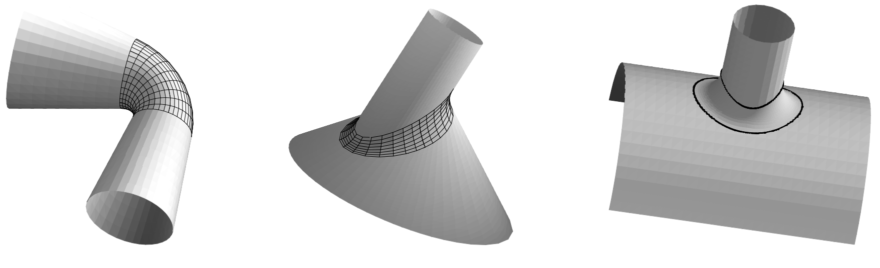

Figure 1: Blending constructions with canal surfaces.

# 1 Survey of PN-surface blending constructions

The majority of known blending constructions between natural quadrics under closer inspection appear to be rational canal surfaces.

A *canal surface* is an envelope of 1-parameter family of spheres in $\mathbb{R}^3$, defined by a *spine curve* $s(t) = (s_1(t), s_2(t), s_3(t))$ and a *radius function* $r(t)$. For the envelope to be real the condition on derivatives $\|\dot{s}(t)\|^2 - \dot{r}^2(t) \geq 0$ is necessary. Let us collect this data in a curve $\gamma(t) = (s(t), r(t)) \in \mathbb{R}^4$ and denote the canal surface by $C_\gamma$. Peternell and Pottmann [9] proved that a canal surface $C_\gamma$ defined by a rational curve $\gamma$ of degree $k$ can be rationally parametrized with bidegree $(5k-6, 2)$. Under a mild condition $\|\dot{s}(t)\|^2 - \dot{r}^2(t) > 0$ (for example, in case of constant radius) the degree bound of [9] was be improved to $(3k-2, 2)$ [4], which is in general optimal.

The first non-trivial blending application of canal surfaces was proposed by Pratt [11]. He used Dupin cyclides, that quartic canal surfaces defined by special conics $\gamma \in \mathbb{R}^4$. For example, any two circular cones with a common inscribed sphere can be blended by a part of Dupin cyclide bounded by two circles (see Fig. 1, left). Canal surfaces defined by general conics $\gamma \in \mathbb{R}^4$ can blend natural quadrics in more general positions (Fig. 1, middle) as was shown in [7].

Results on rational parametrizations [9, 4] enabled Kazakeviciute [2] to develop a theory of rational variable radius rolling ball blends between natural quadrics in arbitrary positions. Here we will consider just one important case.

**Example 1.1.** Let $Q_a$ and $Q_b$ be two cylinders defined by equations $x_1^2+x_2^2 = r_a^2$ and $x_2^1 + x_3^2 = r_b^2$, where $0 < r_a < r_b$ (Fig. 1, right). The conditions that a sphere touches both cylinders $Q_a$ and $Q_b$ define a quartic surface $V \subset \mathbb{R}^4$. Any curve on $V$ define a canal surface touching both cylinders, i.e. a rolling ball blend. Unfortunately a fixed radius case corresponds to irrational curve on $V$. Nevertheless, a certain rational quartic curve $\gamma \subset V$ can be found [6]. This construction generates a canal surface $C_\gamma$ of bidegree $(10, 2)$ which is minimal possible [4]. It is impossible to construct such a blending with a boundary circle on the cylinder $Q_a$, since the corresponding curve on $V$ is irrational.

## 2 Blaschke model of Laguerre geometry

The main idea of *Laguerre geometry* is to consider oriented spheres in $\mathbb{R}^3$ as points in $\mathbb{R}^4$ (see [10] for details). A sphere $c$ with a *center* $(x_1, x_2, x_3)$ and *radius* $x_4$ is mapped to a point $\zeta(c) = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ (the orientation is encoded by a sign of $x_4$). Each oriented plane $e$: $e_0+e_1x_1+e_2x_2+e_3x_3 = 0$, $e_1^2+e_2^2+e_3^2 = 1$, in $\mathbb{R}^3$ is mapped to the hyperplane $\zeta^*(e)$ in $(\mathbb{R}^4)^*$, defined as $e_0+e_1x_1+e_2x_2+e_3x_3 + x_4 = 0$, i.e. with homegeneous coordinates $(e_0, \dots, e_4, 1)$. A *Laguerre transformation* of $\mathbb{R}^4$ is an affine transformation $f(x) = \lambda A(x) + b$, where $\lambda \neq 0$ and $A$ is a linear transformation that preserves the *Minkowski metrics* $\langle v, w\rangle = v_1w_1 + v_2w_2 + v_3w_3 - v_4w_4$. In particular, offsetting corresponds to the simple Laguerre transformation – translation in the $x_4$-axis direction. Linear Laguerre transformations are in fact Lorenz transformations of the Minkowski space $\mathbb{R}^4$ up to scaling. The usual inversion in $\mathbb{R}^3$ can be naturally extended to $\mathbb{R}^4$ by the formula:

$$\mathrm{Inv}_p(x) = p + (x - p)/\langle x - p, x - p\rangle, \tag{1}$$

where $p \in \mathbb{R}^4$ is a center of the inversion.

Pottmann and Peternell [10] proposed to use the *Blaschke model* of Laguerre geometry for PN-surface modeling. The Blaschke map $\delta : (\mathbb{R}^4)^* \to \mathbb{R}^4$ maps hyperplanes to points. For an appropriate coordinate system $\delta(\zeta^*(e)) = (1, e_1, \dots, e_4, e_0)$. Therefore the image is lying in a quadric $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^4$ which is called the *Blaschke cylinder* $B \subset \mathbb{R}^4$.

The *main* observation of [10] is the following 1–1 correspondence:

$$\{\text{PN-surfaces in } \mathbb{R}^3\} \leftrightarrow \{\text{rational surfaces in the Blaschke cylinder } B\} \tag{2}$$

The unit sphere $S$: $x_1^2 + x_2^2 + x_3^2 = x_0^2$ in $\mathbb{R}P^3$ can be parametrized by homogeneous coordinates of $\mathbb{C}P^1$:

$$P_S(z_0, z_1) = (|z_0|^2 + |z_1|^2, 2\mathrm{Re}(z_0\bar{z}_1), 2\mathrm{Im}(z_0\bar{z}_1), |z_0|^2 - |z_1|^2). \tag{3}$$

The map $P_S : \mathbb{C}^2 \to \mathbb{R}^4$ is homogeneous in the sense that $P_S(\lambda z_0, \lambda z_1) = |\lambda|^2 P_S(z_0, z_1)$. Bézier curves and patches on $S$ can be lifted to $\mathbb{C}^2$ uniquely up to constant multiplier. Therefore $P_S$ is called a universal rational parametrization of $S$ (see [1, 5] for details).

From the projective point of view the Blaschke cylinder is a cone over a sphere, i.e. it is a toric threefold. Its universal rational parametrization is slightly more complicated (cf. [1]):

$$P_B(s, d, z_0, z_1) = (s, 2d\mathrm{Re}(z_0\bar{z}_1), 2d\mathrm{Im}(z_0\bar{z}_1), d(|z_0|^2 - |z_1|^2), d(|z_0|^2 + |z_1|^2)). \tag{4}$$

The map $P_B : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{R}^5$ is homogeneous, $P_B(\rho|\lambda|^2 s, \rho d, \lambda z_0, \lambda z_1) = \rho|\lambda|^2 P_B(s, d, z_0, z_1)$, and can be useful for studying rational surfaces in the Blaschke cylinder. According to (2), this is equivalent to studying NP-surfaces in $\mathbb{R}^3$.

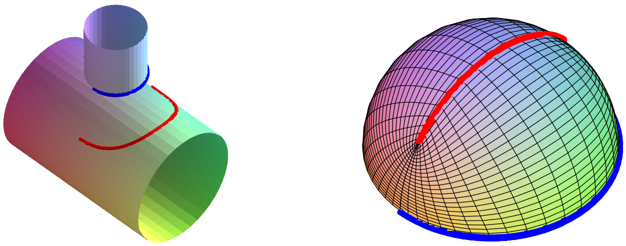

Figure 2: Boundary curves $C_a$, $C_b$ and their images on the unit sphere.

## 3 New blendings based on PN-surfaces

The general scheme of the proposed method consists of three steps: (1) construction of a *Gaussian map*; (2) definition of a *support function*; (3) convertion from *dual* to *point* representation.

Here we illustrate the new blending method in the case of two cylinders considered in Example 1.1.

**Construction 3.1.** Our goal is to generate a ring shaped PN-surface bounded by a circle $C_a$, $x_3 = h$, on the vertical cylinder $Q_a$ and certain rational curve $C_b$ (which will be determined later) on the upper side of the horizontal cylinder $Q_b$ as in Example 1.1, see Fig. 2(left).

*Step* 1. Normals along $C_a$ and $C_b$ define the following curves on the unit sphere: a circle $C'_a$ on the equator and a circular arc $C'_b$ on the plane section $x_1 = 0$, see Fig. 2(right). We do not fix endpoints of $C'_b$ yet: we keep them symmetric w.r.t. the plane $x_2 = 0$. In order to build a symmetric gaussian map it remains to find a Bézier representation of the spherical quarter, $x_1, x_3 \geq 0$, with the fixed quadratic and quartic parametrizations on the opposite boundary curves $C'_a$ and $C'_b$, respectively. This is a standard situation where a universal rational parametrization of a sphere can be used as described in [5] in order to obtain a unique gaussian map $N(t, u)$ of bidegree $(4, 2)$.

*Step* 2. Combining Eq. (3), Eq. (4) and taking $d = 1$, the following tangent plane formula for a PN-surface $x = x(t, u)$ with the gaussian map $N(t, u)$ has the following form (assume $x_4 = 0$)

$$S(t, u) + N_1(t, u)x_1 + N_2(t, u)x_2 + N_3(t, u)x_3 + N_0(t, u)x_4 = 0. \qquad (5)$$

Let us call $S(t, u)$ a *support function.* In fact usual support function is $S/N_0$ but since the gassian map is fixed, $N_0$ is not changing. A support function $S_a(t, u)$ of a sphere touching the vertical cylinder along the circle $C_a$ is derived from the same Eq. (5) by substituting $x = (0, 0, h, r_1)$. The other substitution $x = (0, 0, 0, r_2)$ will define a support function $S_b(t)$ of the horizontal cylinder along the curve $C_b$. Finally we define a support function for our blending surface as combination $S(t, u) = S_a(t, u) + u^2(S_b(t) - S_a(t, 1))$, which preserve $C^1$ contact with $S(t, u)$ along the boundary $u = 0$.

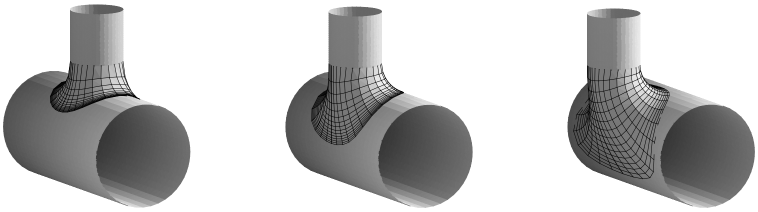

Figure 3: Blending two cylinders in three Laguerre-different positions.

*Step* 3. From the plane representation we obtain point representation by solving the following linear system (lower indices mean partial derivatives)

$$\begin{array}{lcl} S+N_1x_1+N_2x_2+N_3x_3 & = & 0,\\ S_t+(N_1)_tx_1+(N_2)_tx_2+(N_3)_tx_3 & = & 0,\\ S_u+(N_1)_ux_1+(N_2)_ux_2+(N_3)_ux_3 & = & 0. \end{array}$$

If bidegree of $(S, N_1, N_2, N_3)$ is $(d_t, d_u)$ then bidegree of the solution $(x_1, x_2, x_3)$ is $(3d_t - 2, 3d_u - 2)$ in general. Since $(d_t, d_u) = (4, 2)$ we can expect a solution of bidegree $(10, 4)$. Now it is time to remember that we still have one free parameter which controls endpoints of the arc $C'_b$. There exists a unique value of this parameter that enables us to drop bidegree down to $(6, 3)$.

Any circular cone/cylinder defines a line with positive signature in the Minkowski space $\mathbb{R}^4$. In general this gives at least three different Laguerre classes depending on possible signatures of the 2-plane generated by their directional vectors: $(+,+)$, $(+,0)$, and $(+,-)$. Since the middle case defines lower dimensional variety, the following theorem provides blendings in almost half of general positions. In particular, all positions of two cylinders satisfy this condition.

**Theorem 3.2.** *Construction 3.1 produces a blending PN-surface $X$ of bidegree $(6,3)$ in all positions of two circular cones/cylinders, when their directional vectors generate 2-plane of signature $(+,+)$ in the Minkowski space $\mathbb{R}^4$. The parametrization degrees are stable under Laguerre transformations.*

*Skech of the proof.* At first we check that Construction 3.1 gives blending surfaces $X$ of bidegree $(6, 3)$ for all Laguerre different subcases. They are classified in [3]: a full list of invariants consists of a signature of the 3-dimentional affine span of both lines in $\mathbb{R}^4$, i.e. $(+,+,+)$, $(+,+,0)$, and $(+,+,-)$ (see examples in Fig. 3) and angle between them. Degrees $(6, 3)$ are stable under Laguerre transformations, since the associated isotropic hypersurface $\text{iso}(X) \subset \mathbb{R}^4$ (defined by all tangent spheres) can be decomposed in two ways as a union of 1-parameter family of the following surfaces:

1. rulled cubic surfaces with directrices of deree 2 and 1;
2. rulled sextic surfaces with directrices of deree 4 and 2.

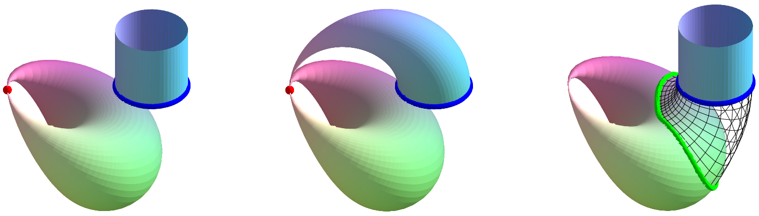

Figure 4: Blending construction between a cylinder and a Dupin cyclide.

□

Using linear directices of cubic ruled surfaces from the previous proof the corollary folows.

**Corollary 3.3.** *The blending surfce $X$ is envelope of a rational family of circular cones.*

**Construction 3.4.** Let a cylinder/cone $Q$ with a fixed circe on it and a Dupin cyclide $D$ are given. Then one can take any sphere $S$ inscribed in $D$ and find the other Dupin cyclide $D'$, which contains $S$ (as inscribed sphere) and is tangent to $Q$ along the circle. If $S$ corresponds to a point $s \in \mathbb{R}^4$ then we apply an inversion $\mathrm{Inv}_s$ to $D$ and $D'$ and get two cylinders/cones $\mathrm{Inv}_s(D)$ and $\mathrm{Inv}_s(D')$ which can be blended using some PN-surface $X$ according with Theorem 3.2 (if the required conditions are satisfied). Then we go the inversion $\mathrm{Inv}_s$ again and get certain blending $\mathrm{Inv}_s(X)$ between $Q$ and $D$. This construction is illustrated in Fig. 4, where $S$ is a point, i.e. sphere with zero radius.

**Theorem 3.5.** *Construction 3.4 produces a blending PN-surface $Y$ of bidegree $(8, 4)$ between a cylinder/cone $Q$ with a fixed circe on it and a Dupin cyclide $D$. The parametrization degrees are stable under Laguerre transformations.*

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