An algebraic version of a theorem of Whitney
[Extended abstract]

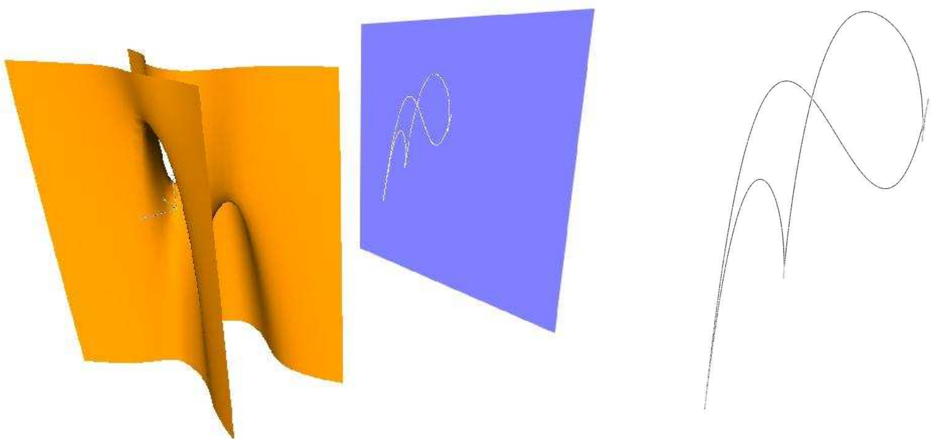
Laurent Busé & Bernard Mourrain
GALAAD, INRIA
BP 93, 06902 Sophia Antipolis, France
{lbuse,mourrain}@sophia.inria.fr

January 24, 2007

Consider the surface in $\mathbb{R}^3$ defined by $f(x,y,z) = z^4 - y^3 z + 2\,z^3 - yz^2 - y^2 - x\,z + 1$. Its discriminant in z is a polynomial of degree 12 and the discriminant in y of this discriminant factors as:

$$
\begin{aligned}
&5540271966595842048\\
&\quad \big(14348907\,x^{10} - 93002175\,x^9 + 273574017\,x^8 - 909290448\,x^7 + 2868603336\,x^6 - 5353192260\,x^5\\
&\quad + 9038030571\,x^4 - 17693165669\,x^3 + 17648229264\,x^2 - 4081683588\,x + 218938829\big)\,(x-1)^2\\
&\quad (125\,x - 173)^2\,\big(47832147\,x^4 + 147495688\,x^3 - 245928792\,x^2 - 212731008\,x + 230501936\big)^3 .
\end{aligned}
$$

Here is a picture of the situation, where we represent the surface $f = 0$ and the projection of its apparent contour[1] (the x-direction is pointing to the top of the image and z-direction to the left):



We observe that the first factor of degree 10 has 4 real roots corresponding to the smooth points of the surface with a tangent plane orthogonal to the x-direction. The second factor of multiplicity 2 corresponds to points of the polar variety which project in the (x,y)-plane onto the same point. Geometrically speaking, we have a double folding of the surface in the z-direction above these values. There are two such real points in our example. The last factor of multiplicity 3 corresponds to cusp points on the discriminant curve, which are the projection of a "fronce" or a pleat of the surface. There are 4 such real points. Notice that the branches of the discriminant curve between two of these

[1]The topology computation and visualization have been performed by the softwares AXEL (http://www-sop.inria.fr/galaad/axel/) and SYNAPS (http://www-sop.inria.fr/galaad/synaps/).

cusp points and one of the double folding points form a very tiny loop, which is difficult to observe at this scale.

This phenomena was analyzed from a singularity theory point of view. A well-known result in singularity theory, due to H. Whitney (see [Whi55, Mat73, AVGZ86]) asserts that the singularities of the projection of a generic surface onto a plane are of 3 types:

- a regular point on the contour curve corresponding to a fold of the surface,
- a cusp corresponding to a pleat,
- a double point corresponding to the projection of two transversal folds.

These are stable singularities, which remain by a small perturbation of the surface or of the direction of projection.

In this extended abstract we give a purely algebraic counterpart of this result for generic polynomials of a given degree. It answers conjectures of McCallum [McC99, McC01]. For more details, we refer the reader to [BM07]. Mention that another approach based on general elimination ideal constructions and geometric arguments has been recently worked out in [LM07].

Let $\mathbb{S}$ be a commutative ring, for any polynomial $P(X_1, X_2, X_3) \in \mathbb{S}[X_1, X_2, X_3]$, introducing a new indeterminate X_4 we will denote $\delta_{3,4}(P) := \frac{P(X_1,X_2,X_3)-P(X_1,X_2,X_4)}{X_3-X_4} \in \mathbb{S}[X_1, X_2, X_3, X_4]$ so that we have

$$P(X_1, X_2, X_4) = P(X_1, X_2, X_3) + (X_4 - X_3)\delta_{3,4}(P).$$

More generally, we define (over $\mathbb{Z}$ and then over $\mathbb{S}$ by specialization) $\delta^i_{3,4}P$ as the unique polynomial such that

$$P(X_1, X_2, X_4) = P(X_1, X_2, X_3) + (X_4 - X_3)\partial_3 P(X_3) + \cdots + \frac{1}{(i-1)!}(X_4 - X_3)^{i-1}\partial_3^{i-1}P(X_3) + (X_4 - X_3)^i\delta^i_{3,4}P$$

or equivalently by $\delta^i_{3,4}P = \sum_{k\geq i} \frac{1}{k!}(X_4 - X_3)^{k-1}\partial_3^k P(X_3)$.

Theorem 1 *Given a polynomial $f(\mathbf{x}, y, z)$ of the form*

$$f(\mathbf{x}, y, z) = \sum_{|\alpha|+i+j\leqslant d} a_{\alpha,i,j}\mathbf{x}^\alpha y^i z^j \in \mathbb{S}[\mathbf{x}][y, z],$$

where $\mathbf{x}$ denotes a set of variables $(x_1, \ldots, x_n)$ for some integer $n \geq 1$ and $\mathbb{S}$ is any commutative ring, the iterated discriminant $\mathrm{Disc}_y(\mathrm{Disc}_z(f)) \in \mathbb{S}[\mathbf{x}]$ has degree at most $d(d-1)(d^2-d-1)$ in $\mathbf{x}$. If the polynomial f is generic and $\mathbb{S}$ is an infinite field, we have

$$\mathrm{Disc}_y(\mathrm{Disc}_z(f)) = a_{0,0,d}\,\mathrm{Disc}_{y,z}(f)\,\mathfrak{F}(f)^3\,\mathfrak{U}(f)^2.$$

where

- *$\mathrm{Disc}_{y,z}(f)$ is an irreducible polynomial in $\mathbf{x}$ of degree $d(d-1)^2$,*
- *$\mathfrak{F}(f)$ is irreducible in $\mathbf{x}$ of degree $d(d-1)(d-2)$, such that $\mathrm{Res}_{y,z}(f, \partial_z f, \partial_z^2 f) = 2^{d(d-1)}\,a_{0,0,d}^2\,\mathfrak{F}(f)$.*
- *$\mathfrak{U}(f)$ is an irreducible polynomial in $\mathbf{x}$ of degree $\frac{1}{2}\,d(d-1)(d-2)(d-3)$, such that*

  $$\mathrm{Res}_{y,z,z'}(f, \delta_{z,z'}(f), \partial_z f, \delta_{z,z'}(\partial_z f)) = a_{0,0,d}^4\mathfrak{F}(f)^2\mathfrak{U}(f)^2$$

  and also

  $$\mathrm{Res}_{y,z,z'}(f, \partial_z f, \delta^2_{z,z'}(f), (\delta^2_{z,z'}\partial_z - 2\delta^3_{z,z'})f) = a_{0,0,d}^{2d(d-1)-6}\mathfrak{U}(f)^2.$$

To prove this theorem we proceeded by specialization of resultant-based constructions. In this way, the above theorem can be derived from the other following one:

Proposition 2 *Given two polynomials $f_k(\boldsymbol{x}, y, z), k = 1, 2,$ of the form*

$$f_k(\boldsymbol{x}, y, z) = \sum_{|\alpha|+i+j \leqslant d_k} a^{(k)}_{\alpha,i,j} \boldsymbol{x}^\alpha y^i z^j \in \mathbb{S}[\boldsymbol{x}][y, z],$$

where $\boldsymbol{x}$ denotes a set of variables $(x_1, \dots, x_n)$ for some integer $n \geq 1$ and $\mathbb{S}$ is any commutative ring, the iterated resultant $\mathrm{Disc}_y(\mathrm{Res}_z(f_1, f_2)) \in \mathbb{S}[\boldsymbol{x}]$ is of degree at most $d_1d_2(d_1d_2 - 1)$in $\boldsymbol{x}$ and factors, up to sign, as

$$\mathrm{Disc}_y(\mathrm{Res}_z(f_1, f_2)) = (-1)^{(d_2+1)(d_1d_2+d_1-1)} \mathrm{Disc}_{y,z}(f_1, f_2) \mathfrak{D}(f_1, f_2)^2.$$

Moreover, if the polynomials f_1 and f_2 are generic, then this iterated resultant has exactly degree $d_1d_2(d_1d_2 - 1)$ in $\boldsymbol{x}$ and the two terms in the right hand side of the above equality are irreducible and a square of an irreducible polynomial respectively.

Our systematic approach based on specialization of resultant computations can be applied in many circumstances and for other genericity conditions on the input polynomials. We also applied it to get the explicit factorizations of two times iterated resultants, formulas which complete the paper [McC99].

Proposition 3 *Given four polynomials $f_k(\boldsymbol{x}, y, z), k = 1, \dots, 4,$ of the form*

$$f_k(\boldsymbol{x}, y, z) = \sum_{|\alpha|+i+j \leqslant d_k} a^{(k)}_{\alpha,i,j} \boldsymbol{x}^\alpha y^i z^j \in \mathbb{S}[\boldsymbol{x}][y, z],$$

where $\boldsymbol{x}$ denotes a set of variables $(x_1, \dots, x_n)$ for some integer $n \geq 1$ and $\mathbb{S}$ is any commutative ring, then the iterated resultant $\mathrm{Res}_y(\mathrm{Res}_z(f_1, f_2), \mathrm{Res}_z(f_3, f_4)) \in \mathbb{S}[\boldsymbol{x}]$ is of degree at most $d_1d_2d_3d_4$ in $\boldsymbol{x}$ and we have

$$\mathrm{Res}_y(\mathrm{Res}_z(f_1, f_2), \mathrm{Res}_z(f_3, f_4)) = \mathrm{Res}_{y,z,z'}(f_1(\boldsymbol{x}, y, z), f_2(\boldsymbol{x}, y, z), f_3(\boldsymbol{x}, y, z'), f_4(\boldsymbol{x}, y, z')) \in \mathbb{S}[\boldsymbol{x}].$$

Moreover, if the polynomials f_1, f_2, f_3 and f_4 are generic then this iterated resultant is irreducible and has exactly degree $d_1d_2d_3d_4$ in $\boldsymbol{x}$.

Proposition 4 *Given three polynomials $f_k(\boldsymbol{x}, y, z), k = 1, \dots, 3,$ of the form*

$$f_k(\boldsymbol{x}, y, z) = \sum_{|\alpha|+i+j \leqslant d_k} a^{(k)}_{\alpha,i,j} \boldsymbol{x}^\alpha y^i z^j \in \mathbb{S}[\boldsymbol{x}][y, z],$$

where $\boldsymbol{x}$ denotes a set of variables $(x_1, \dots, x_n)$ for some integer $n \geq 1$ and $\mathbb{S}$ is any commutative ring, then the iterated resultant $\mathrm{Res}_y(\mathrm{Res}_z(f_1, f_2), \mathrm{Res}_z(f_1, f_3)) \in \mathbb{S}[\boldsymbol{x}]$ is of degree at most $d_1^2d_2d_3$ in $\boldsymbol{x}$ and we have

$$\begin{aligned}\mathrm{Res}_y(\mathrm{Res}_z(f_1, f_2), \mathrm{Res}_z(f_1, f_3)) = (-1)^{d_1^2d_2d_3} &\mathrm{Res}_{y,z}(f_1(\boldsymbol{x}, y, z), f_2(\boldsymbol{x}, y, z), f_3(\boldsymbol{x}, y, z)) \times \\ &\mathrm{Res}_{y,z,z'}(f_1(\boldsymbol{x}, y, z), f_2(\boldsymbol{x}, y, z), f_3(\boldsymbol{x}, y, z'), \delta_{z,z'}(f_1)).\end{aligned}$$

Moreover, if the polynomials f_1, f_2, f_3 are generic then this iterated resultant has exactly degree $d_1^2d_2d_3$ in $\boldsymbol{x}$ and both resultants on the right hand side of the above equality are distinct and irreducible.

Proposition 5 *Given three polynomials $f_k(\boldsymbol{x}, y, z), k = 1, 2, 3,$ of the form*

$$f_k(\boldsymbol{x}, y, z) = \sum_{|\alpha|+i+j \leqslant d_k} a^{(k)}_{\alpha,i,j} \boldsymbol{x}^\alpha y^i z^j \in \mathbb{S}[\boldsymbol{x}][y, z],$$

where $\boldsymbol{x}$ *denotes a set of variables* $(x_1,\ldots,x_n)$ *for some integer* $n\geq 1$ *and* $\mathbb{S}$ *is any commutative ring, then the iterated resultant* $\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Res}_z(f_2,f_3))\in\mathbb{S}[\boldsymbol{x}]$ *is of degree at most* $d_1(d_1-1)d_2d_3$ *in* $\boldsymbol{x}$ *and we have*

$$(a^{(1)}_{0,0,d})^{d_2d_3}\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Res}_z(f_2,f_3)) \quad=\quad \mathrm{Res}_{y,z}(f_1(\boldsymbol{x},y,z),\frac{\partial f_1}{\partial z}(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z),f_3(\boldsymbol{x},y,z)).$$

Moreover, if the polynomials f_1,f_2,f_3 *are generic then this iterated resultant has exactly degree* $d_1(d_1-1)d_2d_3$ *in* $\boldsymbol{x}$ *and the iterated resultant* $\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Res}_z(f_2,f_3))$ *is irreducible.*

Proposition 6 *Given two polynomials* $f_k(\boldsymbol{x},y,z),k=1,2,$ *of the form*

$$f_k(\boldsymbol{x},y,z)=\sum_{|\alpha|+i+j\leqslant d_k} a^{(k)}_{\alpha,i,j}\boldsymbol{x}^\alpha y^i z^j\in\mathbb{S}[\boldsymbol{x}][y,z],$$

where $d_1,d_2\geq 2$*,* $\boldsymbol{x}$ *denotes a set of variables* $(x_1,\ldots,x_n)$ *for some integer* $n\geq 1$ *and* $\mathbb{S}$ *is any commutative ring, then the iterated resultant* $\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Disc}_z(f_2))\in\mathbb{S}[\boldsymbol{x}]$ *is of degree at most* $d_1(d_1-1)d_2(d_2-1)$ *in* $\boldsymbol{x}$ *and we have*

$$(a^{(1)}_{0,0,d_1})^{d_2(d_2-1)}(a^{(2)}_{0,0,d_2})^{d_1(d_1-1)}\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Disc}_z(f_2)) = \mathrm{Res}_{y,z}(f_1(\boldsymbol{x},y,z),\frac{\partial f_1}{\partial z}(\boldsymbol{x},y,z)),f_2(\boldsymbol{x},y,z),\frac{\partial f_2}{\partial z}(\boldsymbol{x},y,z))).$$

Moreover, if the polynomials f_1,f_2,f_3 *are generic then this iterated resultant has exactly degree* $d_1(d_1-1)d_2(d_1-1)$ *in* $\boldsymbol{x}$ *and is irreducible.*

Proposition 7 *Given two polynomials* $f_k(\boldsymbol{x},y,z),k=1,2,$ *of the form*

$$f_k(\boldsymbol{x},y,z)=\sum_{|\alpha|+i+j\leqslant d_k} a^{(k)}_{\alpha,i,j}\boldsymbol{x}^\alpha y^i z^j\in\mathbb{S}[\boldsymbol{x}][y,z],$$

where $d_1,d_2\geq 2$*,* $\boldsymbol{x}$ *denotes a set of variables* $(x_1,\ldots,x_n)$ *for some integer* $n\geq 1$ *and* $\mathbb{S}$ *is any commutative ring, then the iterated resultant* $\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Res}_z(f_1,f_2))\in\mathbb{S}[\boldsymbol{x}]$ *is of degree at most* $d_1^2d_2(d_1-1)$ *in* $\boldsymbol{x}$ *and we have*

$$\mathrm{Res}_y(\mathrm{Disc}_z(f_1),\mathrm{Res}_z(f_1,f_2))=\mathrm{Res}_{y,z}(f_1(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z),\frac{\partial f_1}{\partial z}(\boldsymbol{x},y,z))^2\,\mathfrak{T}(f_1(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z))$$

where we recall that, in $\mathbb{S}[x]$*, we have* $\mathfrak{T}(f_1(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z))=1$ *if* $d_1=2$ *and otherwise*

$$(a^{(1)}_{0,0,d})^{d_1d_2}\mathfrak{T}(f_1(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z))=\mathrm{Res}_{y,z,z'}(f_1(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z),\frac{\partial f_1}{\partial z}(\boldsymbol{x},y,z'),\delta^{(2)}_{z,z'}f_1(\boldsymbol{x},y,z,z'))$$

with $\delta^{(2)}_{z,z'}f_1(\boldsymbol{x},y,z,z'):=\frac{\delta_{z,z'}(f_1)(\boldsymbol{x},y,z,z')-\frac{\partial f_1}{\partial z}(\boldsymbol{x},y,z)}{z'-z}\in\mathbb{S}[\boldsymbol{x}][y,z,z']$.

Moreover, if the polynomials f_1,f_2,f_3 *are generic then this iterated resultant has exactly degree* $d_1^2d_2(d_1-1)$ *in* $\boldsymbol{x}$ *and the resultant* $\mathrm{Res}_{y,z}(f_1(\boldsymbol{x},y,z),f_2(\boldsymbol{x},y,z),\frac{\partial f_1}{\partial z}(\boldsymbol{x},y,z))$ *and the polynomial* $\mathfrak{T}(f_1,f_2)$ *are both irreducible.*

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