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Sylvester's Double Sums: the general case

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Abstract

In 1853 Sylvester introduced a *family* of double sum expressions for two finite sets of indeterminates and showed that some members of the family are essentially the polynomial subresultants of the monic polynomials associated with these sets. A question naturally arises: What are the *other* members of the family? This paper provides a complete answer to this question. The technique that we developed to answer the question turns out to be general enough to charactise *all* members of the family, providing a uniform method.

1. Introduction

Let A and B be finite lists (ordered sets) of distinct indeterminates. In (4), Sylvester introduced for each $0 \leq p \leq |A|, 0 \leq q \leq |B|$ the following *double-sum* expression in A and B:

$$\mathrm{Sylv}^{p,q}(A,B;x) := \sum_{\substack{A'\subset A,\, B'\subset B\\ |A'|=p,\, |B'|=q}} R(x,A')\,R(x,B')\,\frac{R(A',B')\,R(A\backslash A',B\backslash B')}{R(A',A\backslash A')\,R(B',B\backslash B')},$$

where

$$R(X,Y) := \prod_{x\in X, y\in Y} (x-y), \qquad R(x,Y) := \prod_{y\in Y}(x-y).$$

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Let now f, g be univariate polynomials such that

$$\begin{array}{rclcl} f & := & \prod_{\alpha\in A}(x-\alpha) & = & x^m + a_{m-1}x^{m-1} + \ldots + a_0 \\ g & := & \prod_{\beta\in B}(x-\beta) & = & x^n + b_{n-1}x^{n-1} + \ldots + b_0, \end{array}$$

where $m := |A|$ and $n := |B|$.

We want to give an expression for $\mathrm{Sylv}^{p,q}(A,B;x)$ in terms of $a_0,\ldots,a_{m-1}$, $b_0,\ldots,b_{n-1}$. Since this expression is polynomial in x and symmetric in the α's and β's, we know that for every pair (p,q), $0\le p\le m$, $0\le q\le n$, $\mathrm{Sylv}^{p,q}(A,B;x)$ can be expressed as a polynomial in x whose coefficients are rational functions in the a_i's and the b_j's. In (4), the rational expression for $\mathrm{Sylv}^{p,q}(A,B;x)$ is determined for the following values of (p,q) (see also (3)):

1. If $0\le d := p+q < \min\{m,n\}$, then

   $$\mathrm{Sylv}^{p,q}(A,B;x) = (-1)^{p(m-d)}\binom{d}{p}\mathrm{Sres}_d(f,g),$$

   where $\mathrm{Sres}_d(f,g)$ is the d-th subresultant of the polynomials f and g, which definition is recalled in Formula (3) next section (cf.(4, Art. 21) and also (3, Theorem 0.1)).

2. If $p+q = m < n$, then

   $$\mathrm{Sylv}^{p,q}(A,B;x) = \binom{m}{p}f(x),$$

   (cf.(4, Art. 21) and also (3, Proposition 2.9 (i))). In fact, the d-th subresultant is also well defined for $d = m < n$ as $\mathrm{Sres}_m(f,g) = f$. This implies that Case (2) can be seen as a special case of Case (1).

3. If $p+q = m = n$, then

   $$\mathrm{Sylv}^{p,q}(A,B;x) = \binom{m-1}{q}f(x) + \binom{m-1}{p}g(x)$$

   (cf.(4, Art. 22) and also (3, Proposition 2.9 (ii))).

4. If $m < p+q < n-1$, then $\mathrm{Sylv}^{p,q}(A,B;x) = 0$ (cf. (4, Arts. 23 & 24)).

5. If $m < p+q = n-1$, then $\mathrm{Sylv}^{p,q}(A,B;x)$ is a "numerical multiplier" of $f(x)$ (cf. (4, Art. 25)), but the ratio is not established.

The techniques used for proving each of these cases in both (4) and (3) are different. In (2) we used a simple matrix formulation that allowed us to deal with Cases (1) and (2). This note is a natural continuation of (2): we present a global matrix formulation that extends our previous construction such that it not only allows us to deal with all the known cases, but also to present in Theorem 2.10 below an explicit formula for $\mathrm{Sylv}^{p,q}(A,B;x)$ for *all* possible values of (p,q), i.e. for $0\le p\le m, 0\le q\le n$.

2. The global matrix formulation

As in (2), we define for a polynomial $p(t)$, a finite list $\Gamma := (\gamma_1, \dots, \gamma_u)$ of scalars and a non-negative integer v the (non necessarily square) matrix of size $v \times u$:

$$\langle p(t), \Gamma\rangle_v := \begin{array}{cc} \overset{u}{} & \\ \boxed{\begin{array}{ccc} p(\gamma_1) & \dots & p(\gamma_u) \\ \vdots & & \vdots \\ \gamma_1^{v-1}p(\gamma_1) & \dots & \gamma_u^{v-1}p(\gamma_u) \end{array}} & v \end{array} .$$

For instance, under this notation,

$$\langle x - t, \Gamma\rangle_v = (\gamma_j^{i-1}x - \gamma_j^i)_{1\le i\le v, 1\le j\le u},$$

and for $v = u$ we have the following equality for the Vandermonde determinant $\mathcal{V}(\Gamma)$ associated to Γ:

$$\mathcal{V}(\Gamma) := |(\gamma_j^{i-1})_{1\le i,j\le u}| = |\langle 1, \Gamma\rangle_u|.$$

For the rest of the paper $d \in \mathbb{N}$, $0 \le d \le m + n$ and $d' := m + n - d$. We take a new variable T and we denote by $U_d(x, T)$ the following square matrix of size $m + n = d' + d$:

$$U_d(x, T) := \begin{array}{c|c|c} n & m & \\ \hline \langle 1, B\rangle_{d'} & \langle T, A\rangle_{d'} & d' \\ \hline \langle x - t, B\rangle_d & \langle x - t, A\rangle_d & d \end{array} ,$$

where $\langle T, A\rangle_{d'} = (T\alpha^j)_{\alpha\in A,\, 0\le j\le d'-1}$. Finally we denote by $u_d(x, T)$ its determinant, that we develop in the powers of T:

$$u_d(x, T) := |U_d(x, T)| = u_{d,0}(x)T^m + \dots + u_{d,m-1}(x)T + u_{d,m}(x) \tag{1}$$

We are now ready to state our first result, that relates $\mathrm{Sylv}^{p,d-p}(A, B; x)$ to the coefficient $u_{d,p}(x)$:

THEOREM 2.1: *Let* $0 \le d \le m + n$, $0 \le p \le m$ *and define* $q := d - p$. *Following Notation (1), we have that if* $0 \le q \le n$ *then*

$$u_{d,p}(x) = (-1)^{q(m-p)}\mathcal{V}(A)\,\mathcal{V}(B)\ \mathrm{Sylv}^{p,q}(A, B; x)$$

while otherwise $u_{d,p}(x) = 0$.

Proof: We perform a Laplace expansion of the determinant of the matrix $U_d(x, T)$ on the last d rows and we get the following expression:

$$u_d(x, T) = \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'| + |B'| = d}} \sigma(B \setminus B' \cup A \setminus A', B \cup A)T^{m-|A'|}\mathcal{V}(B \setminus B' \cup A \setminus A')R(x, B')R(x, A')\mathcal{V}(B' \cup A'),$$

where, as in (2), "$\cup$" stands for list concatenation, "$\backslash$" means list subtraction and, for $S \subseteq T$ finite lists, $\sigma(S,T) := (-1)^j$, j being the number of transpositions needed to take T to $S \cup (T \backslash S)$.

We write $u_d(x,T)$ in powers of T, with $0 \le p \le m$ and $0 \le q := d-p \le n$ implying $\max\{0, d-n\} \le p \le \min\{d,m\}$:

$$u_d(x,T) = \sum_{p=\max\{0,d-n\}}^{\min\{d,m\}} \left(\sum_{\substack{A' \subset A,\, B' \subset B \\ |A'|=p,\ |B'|=q}} \sigma(B \setminus B' \cup A \setminus A', B \cup A) R(x,B') R(x,A') \mathcal{V}(B' \cup A') \mathcal{V}(B \setminus B' \cup A \setminus A') \right) T^{m-p}.$$

We recall the elementary fact that transposing a block of j columns with an adjacent block of i columns produces in the determinant a change of sign of order $(-1)^{ij}$. Hence, for $|A'| = p$ and $|B'| = q$,

$$\sigma(B \setminus B' \cup A \setminus A', B \cup A) = \sigma(A \setminus A', A)\sigma(B \setminus B', B)(-1)^{q(m-p)},$$

and we have for $\max\{0, d-n\} \le p \le \min\{d,m\}$:

$$u_{d,p}(x) = (-1)^{q(m-p)} \sum_{\substack{A' \subset A, B' \subset B \\ |A'|=p,\ |B'|=q}} \sigma(A \setminus A', A)\sigma(B \setminus B', B) R(x,A') R(x,B') \mathcal{V}(B' \cup A') \mathcal{V}(B \setminus B' \cup A \setminus A').$$

Now we apply repeatedly the elementary fact that

$$\mathcal{V}(X \cup Y) = \mathcal{V}(X)\,\mathcal{V}(Y)\,R(Y,X)$$

for any pair of finite lists X, Y:

$$\begin{aligned}
\mathcal{V}(B' \cup A')\,\mathcal{V}(B \setminus B' \cup A \setminus A') &= \mathcal{V}(A')\mathcal{V}(B')R(A',B')\mathcal{V}(A \setminus A')\mathcal{V}(B \setminus B')R(A \setminus A', B \setminus B') \\
&= \frac{\mathcal{V}(A \setminus A' \cup A')}{R(A', A \setminus A')} \frac{\mathcal{V}(B \setminus B' \cup B')}{R(B', B \setminus B')} R(A',B')R(A \setminus A', B \setminus B') \\
&= \sigma(A \setminus A', A)\sigma(B \setminus B', B)\mathcal{V}(A)\mathcal{V}(B) \frac{R(A',B')R(A \setminus A', B \setminus B')}{R(A', A \setminus A')R(B', B \setminus B')}.
\end{aligned}$$

We finally obtain that

$$\begin{aligned}
u_{d,p}(x) &= (-1)^{q(m-p)}\mathcal{V}(A)\mathcal{V}(B) \sum_{\substack{A' \subset A, B' \subset B \\ |A'|=p,\ |B'|=q}} R(x,A')R(x,B') \frac{R(A',B')R(A \setminus A', B \setminus B')}{R(A', A \setminus A')R(B', B \setminus B')} \\
&= (-1)^{q(m-p)}\mathcal{V}(A)\mathcal{V}(B)\,\mathrm{Sylv}^{p,q}(A,B;x).
\end{aligned}$$

$\square$

In view of Theorem 2.1, in order to produce a rational expression for $\mathrm{Sylv}^{p,q}(A,B;x)$ it is enough to give a rational expression for $u_{d,p}(x)$. To this aim we first observe the following straightforward factorization formula for $U_d(x,T)$ as a product of two rectangular matrices of sizes $(m+n) \times (m+n+1)$ and $(m+n+1) \times (m+n)$ respectively.

Lemma 2.2:

$$U_d(x,T) = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{d'} & \multicolumn{1}{c}{d+1} \\ \cline{2-3} d' & \mathbb{I}_{d'} & \mathbf{0} \\ \cline{2-3} d & \mathbf{0} & \begin{matrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & x & -1 \end{matrix} \\ \cline{2-3} \end{array} \; \begin{array}{|c|c|c} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\ \hline \langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & d' \\ \hline \langle 1,B\rangle_{d+1} & \langle 1,A\rangle_{d+1} & d+1 \\ \hline \end{array} \, . \qquad (2)$$

For the rest of the paper, we assume without loss of generality that $m \le n$. The previous factorization of $U_d(x,T)$ immediately yields

Proposition 2.3 (Arts. 23 & 24 (4)): *Let* $m \le n$. *If* $m < d < n-1$, *then* $u_d(x,T) = 0$.

Proof: The assumption implies $\max\{d', d+1\} < n$. Then the first n columns of the matrix at the right of (2) have deficient rank since all $n \times n$ minors vanish. A Binet-Cauchy expansion of $u_d(x,T)$ therefore implies that $u_d(x,T)$ vanishes as well. $\square$

Our goal now is to provide a factorization like in (2), but with square matrices, that allows to recover $u_d(x,T)$. To this aim we recall that for $0 \le k \le m < n$ or $0 \le k < m = n$, the k-th subresultant of the polynomials f and g with indeterminate coefficients is a well defined polynomial of degree k:

$$\mathrm{Sres}_k(f,g) := \det \begin{array}{|cccccc|c} \multicolumn{6}{c}{m+n-2k} & \\ \hline a_m & \cdots & & \cdots & a_{k+1-(n-k-1)} & x^{n-k-1}f(x) & \\ & \ddots & & & \vdots & \vdots & n-k \\ & & a_m & \cdots & a_{k+1} & f(x) & \\ \hline b_n & \cdots & & \cdots & b_{k+1-(m-k-1)} & x^{m-k-1}g(x) & \\ & \ddots & & & \vdots & \vdots & m-k \\ & & b_n & \cdots & b_{k+1} & g(x) & \\ \hline \end{array} \qquad (3)$$

with $a_\ell = b_\ell = 0$ for $\ell < 0$.

Expanding the determinant by the last column gives an expression

$$\mathrm{Sres}_k(f,g) = F_k(x)f(x) + G_k(x)g(x) \qquad (4)$$

with $\deg F_k \le n-k-1$ and $\deg G_k \le m-k-1$.

THEOREM 2.4: *Let $m \le n$. If $0 \le d \le m$ or $n-1 \le d \le m+n$, then there exist polynomials $P(x) := P_0 + \cdots + P_d x^d$ and $Q(x) := Q_0 + \cdots + Q_{d'-1}x^{d'-1}$ with $P \neq 0$, $k := \deg P \le d$ and $\deg Q \le d' - 1$ if $d' \neq 0$ such that we have the following matrix identity:*

$$\begin{array}{c|c|c|}
 & d' & d+1 \\ \hline
d' & \mathbb{I}_{d'} & \mathbf{0} \\ \hline
d & \mathbf{0} & \begin{array}{cccc} x & -1 & & 0 \\ & \ddots & \ddots & \\ 0 & \dots & x & -1 \end{array} \\ \hline
1 & Q_0 \quad \dots \quad Q_{d'-1} & P_0 \quad \dots \quad \dots \quad P_d \\ \hline
\end{array}
\;
\begin{array}{|c|c|c|c}
n & m & 1 & \\ \hline
\langle 1, B\rangle_{d'} & \langle T, A\rangle_{d'} & \mathbf{0} & d' \\ \hline
\langle 1, B\rangle_{d+1} & \langle 1, A\rangle_{d+1} & \mathbf{e}_k & d+1 \\ \hline
\end{array}
=
\begin{array}{|c|c|c}
m+n & 1 & \\ \hline
U_d(x,T) & * & m+n \\ \hline
\mathbf{0} & P_k & 1 \\ \hline
\end{array},$$

where $\mathbf{e}_k$ is defined as the vertical vector of size $d+1$ with a single non-zero entry 1 in position $k+1$ and P_k is the leading coefficient of P.

Moreover, $P(x)$ can be defined as

$$P(x) = \begin{cases} Sres_d(f,g) & \text{for } 0 \le d < m \quad \text{or} \quad d = m < n \\ f & \text{for } m < d = n-1 \\ F_{d'-1}f + TG_{d'-1}g & \text{for } n \le d < m+n \\ fg & \text{for } d = m+n \end{cases},$$

where $F_{d'-1}, G_{d'-1}$ are as in Identity (4) for $k = d'-1$.

REMARK 2.5: *We note that $P(x) = F_{d'-1}f + TG_{d'-1}g$ is the determinant of a matrix similar to the matrix (3) that defines $Sres_{d'}(f,g)$: we simply need to replace $g(x)$ by $Tg(x)$ in the last column of the matrix (3).*

Proof: To get the factorization stated in Theorem 2.4, we only need to look at the product of the last row of the first matrix by the second matrix in the right side of the equality. These are

$$\begin{array}{rcl} (Q_0 + \cdots + Q_{d'-1}\beta^{d'-1}) + (P_0 + \cdots + P_d\beta^d) & = & 0 \\ (TQ_0 + \cdots + TQ_{d'-1}\alpha^{d'-1}) + (P_0 + \cdots + P_d\alpha^d) & = & 0 \end{array}$$

for all $\beta \in B$, $\alpha \in A$. Equivalently, it is enough to produce polynomials $P(x) := P_0 + \cdots + P_d x^d$ and $Q(x) := Q_0 + \cdots + Q_{d'-1}x^{d'-1}$ with $P \neq 0$, $\deg P \le d$ and $\deg Q \le d' - 1$ if $d' \neq 0$ such that the following $m+n$ equations are satisfied:

$$\begin{cases} Q(\beta) + P(\beta) = 0, & \forall \beta \in B \\ TQ(\alpha) + P(\alpha) = 0, & \forall \alpha \in A. \end{cases} \tag{5}$$

For $0 \le d \le m$ if $m < n$ and $0 \le d < m$ if $m = n$, we define

$$\begin{cases} P(x) & := & \mathrm{Sres}_d(f,g) & = & F_d(x)f(x) + G_d(x)g(x) \\ Q(x) & := & -F_d(x)f(x) - \frac{1}{T}G_d(x)g(x) & & \end{cases}$$

where F_d, G_d are as in Identity (4) for $k := d$. Thus $\deg P = \deg \mathrm{Sres}_d(f,g) = d$ and $\deg_x Q \leq \max\{\deg(F_d f), \deg(G_d g)\} \leq d' - 1$. We look at Condition (5):

$$\begin{cases} Q(\beta) + P(\beta) & = & (1 - \frac{1}{T})G_d(\beta)g(\beta) & = & 0, & \forall \beta \in B \\ TQ(\alpha) + P(\alpha) & = & (1-T)F_d(\alpha)f(\alpha) & = & 0, & \forall \alpha \in A. \end{cases}$$

For $m < d = n-1$, we define

$$P(x) := f(x) \quad \text{and} \quad Q(x) := -f(x).$$

We have $\deg P = m < d$ and $\deg Q = m = m+n-d-1 = d'-1$ in this case. Condition (5) is trivially satisfied.

For $n \leq d < m+n$, we observe that $0 \leq d'-1 \leq m-1$. Thus $\mathrm{Sres}_{d'-1}(f,g)$ is well defined and we define

$$\begin{cases} Q(x) & := & -\mathrm{Sres}_{d'-1}(f,g) & = -F_{d'-1}(x)f(x) - G_{d'-1}(x)g(x) \\ P(x) & := & F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x) & \end{cases}$$

where $F_{d'-1}, G_{d'-1}$ are as in Identity (4) for $k := d'-1$. Thus $\deg Q = \deg \mathrm{Sres}_{d'-1}(f,g) = d'-1$ and $\deg_x P \leq \max\{\deg(F_{d'-1}f), \deg(G_{d'-1}g)\} \leq \max\{m+n-(d'-1)-1, n+m-(d'-1)-1\} = d$. Also $P \neq 0$ since the leading terms can not cancel each other. We look again at Condition (5):

$$\begin{cases} Q(\beta) + P(\beta) & = & (T-1)G_{d'-1}(\beta)g(\beta) & = & 0, & \forall \beta \in B \\ TQ(\alpha) + P(\alpha) & = & (1-T)F_{d'-1}(\alpha)f(\alpha) & = & 0, & \forall \alpha \in A. \end{cases}$$

For $d = m+n$, since $d' = 0$ in this case, we define $P(x) = f(x)g(x)$, which is of degree d, to satisfy Condition (5). □

Theorem 2.4 immediately implies that $u_d(x,T)$ can be computed as the determinant of two square matrices for the values of $d \leq m$ and $n-1 \leq d$. Our next goal is to compute P_k in each case, as well as the determinants of these square matrices. To this aim, for $0 \leq d \leq m < n$ or $0 \leq d < m = n$, we set $\Delta_k(f,g)$ for the leading coefficient of $\mathrm{Sres}_k(f,g)$, i.e. $\Delta_k(f,g)$ is the k-th scalar subresultant of f, g.

For $k = m = n$, we define for the coherence of the next results

$$\Delta_m(f,g) := 1.$$

LEMMA 2.6: *Let $m \leq n$. Following the notations of Theorem 2.4, we have*

$$\begin{cases} \deg P = d & \textit{and} & P_k = \Delta_d(f,g) & \textit{for} & 0 \leq d \leq m < n \textit{ or } d < m = n, \\ \deg P = m & \textit{and} & P_k = 1 & \textit{for} & m < d = n-1, \\ \deg P = d & \textit{and} & P_k = (-1)^{d-n}\Delta_{d'}(f,g)(T-1) & \textit{for} & m \leq n \leq d < m+n, \\ \deg P = d & \textit{and} & P_k = 1 & \textit{for} & d = m+n. \end{cases}$$

Proof: The first two cases and the last case are straightforward from the definition of P_k.
For $m \leq n \leq d < m+n$, we have that $P(x) = F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x)$. Thus $\deg_x P = \max\{\deg_x(F_{d'-1}f), \deg_x(G_{d'-1}g)\}$ since the leading terms can not cancel each other. A direct computation on the matrix in (3) that defines $\operatorname{Sres}_k(f,g)$ shows that —since for $k := d'-1 < m$, $n-k > 1$ and $m-k > 1$ hold— then $\deg_x F_k = n-k-1 = d-m$ and $\deg_x G_k = m-k-1 = d-n$. Therefore $\deg_x P = \max\{m+n-(d'-1)-1, n+m-(d'-1)-1\} = d$.
Finally, since f and g are monic, the leading coefficient of $F_k(x)$ equals

$$(-1)^{m+n-2k+1}(-1)^{n-k+1}\Delta_{k+1}(f,g) = (-1)^{d-n+1}\Delta_{d'}(f,g)$$

and the leading coefficient of $G_k(x)$ equals

$$(-1)^{m+n-2k+n-k+1}\Delta_{k+1}(f,g) = (-1)^{d-n}\Delta_{d'}(f,g).$$

Therefore $P_d = (-1)^{d-n}\Delta_{d'}(f,g)(T-1)$. □

Lemma 2.7:

$$\det \begin{array}{|c|c|l} \multicolumn{1}{c}{d'} & \multicolumn{1}{c}{d+1} & \\ \hline \mathbb{I}_{d'} & \mathbf{0} & d' \\ \hline \mathbf{0} & \begin{matrix} x & -1 & & 0 \\ & \ddots & \ddots & \\ 0 & \dots & x & -1 \end{matrix} & d \\ \hline Q_0 \ \ \dots \ \ Q_{d'-1} & P_0 \ \ \dots \ \ \dots \ \ P_d & 1 \\ \hline \end{array} \quad = P_0 + \cdots + P_d x^d =: P(x).$$

Proof: Because of the block triangular structure, this determinant equals

$$\det \begin{array}{|c|l} \multicolumn{1}{c}{d+1} & \\ \hline \begin{matrix} x & -1 & & 0 \\ & \ddots & \ddots & \\ 0 & \dots & x & -1 \end{matrix} & d \\ \hline P_0 \ \ \dots \ \ \dots \ \ P_d & 1 \\ \hline \end{array} \ .$$

We can permute the first d-block with the last row and expand the determinant by this new first row. We get

$$(-1)^d\left(P_0(-1)^d - P_1x(-1)^d + \cdots + (-1)^d P_d x^d\right).$$

□

Lemma 2.8: *Let $m \le n$. Then*

$$\det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & \mathbf{0} & d' \\ \hline \langle 1,B\rangle_{d+1} & \langle 1,A\rangle_{d+1} & \mathbf{e}_d & d+1 \\ \hline \end{array} =$$

$$= \begin{cases} (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_d(f,g)\,T^{m-d}(T-1)^d & \textit{for} \quad 0 \le d \le m, \\ (-1)^{m(d-1)+d}\mathcal{V}(A)\mathcal{V}(B)(T-1)^m & \textit{for} \quad m < d = n-1, \\ (-1)^{d'n}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_{d'}(f,g)\,(T-1)^{d'} & \textit{for} \quad n \le d < m+n, \\ \mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Res}(f,g) & \textit{for} \quad d = m+n. \end{cases}$$

Proof: First, let us recall (2, Lemma 2):

$$\mathrm{Sres}_k(f,g)\mathcal{V}(A) = \det \begin{array}{|c|l} \multicolumn{1}{c}{m} & \\ \hline \langle x-t, A\rangle_k & k \\ \hline \langle g(t), A\rangle_{m-k} & m-k \\ \hline \end{array} ,$$

which implies that its leading coefficient satisfies

$$\Delta_k(f,g)\mathcal{V}(A) = \det \begin{array}{|c|l} \multicolumn{1}{c}{m} & \\ \hline \langle 1, A\rangle_k & k \\ \hline \langle g(t), A\rangle_{m-k} & m-k \\ \hline \end{array} . \tag{6}$$

To simplify the notation of the proof, we will denote the matrix on the left side of the claim of the Lemma by M_d.

In case $0 \le d \le m$ or $n \le d < m+n$, $\deg P(x) = d$ by Lemma 2.6 and $\mathbf{e}_d := (0,\ldots,0,1)^t$. Therefore

$$|M_d| = \begin{array}{|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\ \hline \langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & d' \\ \hline \langle 1,B\rangle_{d} & \langle 1,A\rangle_{d} & d \\ \hline \end{array} .$$

For $d \le m$, we have that $d' \ge n \ge m \ge d$ holds and therefore row operations yield

$$
|M_d| \;=\; \begin{array}{|c|c|l}
\multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\
\cline{1-2}
\langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & d' \\
\cline{1-2}
\mathbf{0} & \langle 1-T,A\rangle_d & d \\
\cline{1-2}
\end{array}
$$

$$
= \begin{array}{|c|c|l}
\multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\
\cline{1-2}
\langle 1,B\rangle_{n} & \langle T,A\rangle_{n} & n \\
\cline{1-2}
\mathbf{0} & \langle Tg(t),A\rangle_{m-d} & m-d \\
\cline{1-2}
\mathbf{0} & \langle 1-T,A\rangle_d & d \\
\cline{1-2}
\end{array} \qquad \text{since } \forall\, \beta \in B, g(\beta) = 0
$$

$$
= \mathcal{V}(B) T^{m-d}(1-T)^d \det \begin{array}{|c|l}
\multicolumn{1}{c}{m} & \\
\cline{1-1}
\langle g(t),A\rangle_{m-d} & m-d \\
\cline{1-1}
\langle 1,A\rangle_d & d \\
\cline{1-1}
\end{array}
$$

$$
\begin{aligned}
&= \mathcal{V}(B)\, T^{m-d}(1-T)^d(-1)^{d(m-d)}\,\mathcal{V}(A)\,\Delta_d(f,g) \qquad \text{by (6)} \\
&= (-1)^{d(m-d+1)}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_d(f,g)\,T^{m-d}(T-1)^d \\
&= (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_d(f,g)\,T^{m-d}(T-1)^d.
\end{aligned}
$$

In case $d \ge n$, we have that $d' \le m \le d$ holds and therefore row operations yield

$$
|M_d| \;:=\; \begin{array}{|c|c|l}
\multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\
\cline{1-2}
\mathbf{0} & \langle T-1,A\rangle_{d'} & d' \\
\cline{1-2}
\langle 1,B\rangle_d & \langle 1,A\rangle_d & d \\
\cline{1-2}
\end{array}
$$

$$
= \begin{array}{|c|c|l}
\multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\
\cline{1-2}
\mathbf{0} & \langle T-1,A\rangle_{d'} & d' \\
\cline{1-2}
\langle 1,B\rangle_n & \langle 1,A\rangle_n & n \\
\cline{1-2}
\mathbf{0} & \langle g(t),A\rangle_{d-n} & d-n \\
\cline{1-2}
\end{array}
$$

$$
= (-1)^{d'n}\mathcal{V}(B)\,(T-1)^{d'} \det \begin{array}{|c|l}
\multicolumn{1}{c}{n} & \\
\cline{1-1}
\langle 1,A\rangle_{d'} & d' \\
\cline{1-1}
\langle g(t),A\rangle_{d-n} & d-n \\
\cline{1-1}
\end{array}
$$

$$
= (-1)^{d'n}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_{d'}(f,g)\,(T-1)^{d'}.
$$

In case $m < d = n-1$, $\deg P = m$ and $\mathbf{e}_d$ is the vertical vector with a single non-zero entry 1 in position $m+1$. Since $d+1=n$, $d'=m+1$ and $n \geq m+1$,

$$\begin{aligned}
|M_d| &= \det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \langle 1,B\rangle_{m+1} & \langle T,A\rangle_{m+1} & \mathbf{0} & m+1 \\ \hline \langle 1,B\rangle_n & \langle 1,A\rangle_n & \mathbf{e}_d & n \\ \hline \end{array} \\
&= \det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \mathbf{0} & \langle T-1,A\rangle_{m+1} & -\mathbf{e}_d & m+1 \\ \hline \langle 1,B\rangle_n & \langle 1,A\rangle_n & \mathbf{e}_d & n \\ \hline \end{array} \\
&= (-1)^{(m+1)n} \det\langle 1,B\rangle_n \det \begin{array}{|c|c|l} \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \langle T-1,A\rangle_{m+1} & -\mathbf{e}_d & m+1 \\ \hline \end{array} \\
&= -(-1)^{(m+1)n} \det\langle 1,B\rangle_n \det\langle 1,A\rangle_m (T-1)^m \\
&= (-1)^{m(d-1)+d}\mathcal{V}(A)\mathcal{V}(B)(T-1)^m.
\end{aligned}$$

Finally the case $d=m+n$ is straightforward since

$$|M_d| = \mathcal{V}(B \cup A) = \mathcal{V}(A)\mathcal{V}(B)\mathrm{Res}(f,g).$$

□

We are ready now to compute $u_d(x,T)$ for all values of d, $0 \leq d \leq m+n$, and to deduce $\mathrm{Sylv}^{p,q}(A,B;x)$ for all possible values of p and q.

THEOREM 2.9: *Let $m \leq n$. Then*

$$u_d(x,T) = \begin{cases} (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathit{Sres}_d(f,g)\,T^{m-d}(T-1)^d & \textit{for} \quad 0 \leq d < m \ \textit{or} \ d = m < n \\ 0 & \textit{for} \quad m < d < n-1 \\ (-1)^{\sigma}\mathcal{V}(A)\,\mathcal{V}(B)\,f(x)\,(T-1)^m & \textit{for} \quad m < d = n-1 \\ (-1)^{\sigma}\mathcal{V}(A)\,\mathcal{V}(B)\,\left(F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x)\right)\,(T-1)^{d'-1} & \textit{for} \quad n \leq d < m+n \\ \mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Res}(f,g)\,f(x)\,g(x) & \textit{for} \quad d = m+n \end{cases},$$

where $\sigma = (d'-1)n+d$, and $F_{d'-1}, G_{d'-1}$ are defined as in Identity (4) for $k := d'-1$.

Proof: If $m < d < n-1$ then by Proposition 2.3 we have that $u_d(x,T) = 0$. For the other cases of $0 \leq d \leq m+n$, we apply Theorem 2.4 and Lemma 2.7. We get

$$u_d(x,T) \cdot P_k = P(x) \cdot \det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & \mathbf{0} & d' \\ \hline \langle 1,B\rangle_{d+1} & \langle 1,A\rangle_{d+1} & \mathbf{e}_d & d+1 \\ \hline \end{array}.$$

Now for each of the following cases we also apply Lemmas 2.6 and 2.8:
For $0 \leq d < m$ or $d = m$ if $m < n$, $P(x) = \mathrm{Sres}_d(f,g)$ and $P_k = \Delta_d(f,g)$, therefore

$$\begin{aligned} u_d(x,T) &= \frac{1}{\Delta_d(f,g)} \left(\mathrm{Sres}_d(f,g)(-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_d(f,g)\,T^{m-d}(T-1)^d\right) \\ &= (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Sres}_d(f,g)\,T^{m-d}(T-1)^d. \end{aligned}$$

For $m < d = n-1$ we have that $P(x) = f(x)$ and $P_k = 1$, then

$$u_d(x,T) = f(x)(-1)^{m(d-1)+d}\mathcal{V}(A)\mathcal{V}(B)(T-1)^m,$$

and to get the sign $(-1)^\sigma$ as in the claim, we note that in this case $m = d'-1$ and thus $m(d-1)+d \equiv (d'-1)n + d \pmod 2$.

For $n \leq d < m+n$ we have that $P(x) = F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x)$ and $P_k = (-1)^{d-n}\Delta_{d'}(f,g)(T-1)$. We conclude

$$\begin{aligned} u_d(x,t) &= \frac{(F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x))\,(-1)^{nd'}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_{d'}(f,g)\,(T-1)^{d'}}{(-1)^{d-n}(T-1)\Delta_{d'}(f,g)} \\ &= (-1)^{n(d'-1)+d}\mathcal{V}(A)\,\mathcal{V}(B)\ (F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x))\ (T-1)^{d'-1}. \end{aligned}$$

The last case, $d = m+n$, is straightforward. $\square$

THEOREM 2.10: *Let $m \leq n, 0 \leq p \leq m, 0 \leq q \leq n$ and $d := p+q$. Then*

$$\mathrm{Sylv}^{p,q}(A,B;x) = \begin{cases} (-1)^{p(m-d)}\binom{d}{p}\,Sres_d(f,g) & \textit{for} \quad 0 \leq d < m \ \ \textit{or} \ \ d = m < n, \\ 0 & \textit{for} \quad m < d < n-1, \\ (-1)^{(m+q)(p+1)}\binom{m}{p} f(x) & \textit{for} \quad m < d = n-1, \\ (-1)^{\sigma}\left(\binom{d'-1}{m-p}F_{d'-1}(x)f(x) - \binom{d'-1}{n-q}G_{d'-1}(x)g(x)\right) & \textit{for} \quad n \leq d \leq m+n-1, \\ \mathrm{Res}(f,g)f(x)g(x) & \textit{for} \quad d = m+n, \end{cases}$$

where $\sigma := q(m-p)+n(d-m)+d+n-q-1$, and $F_{d'-1}$ and $G_{d'-1}$ are defined as in Identity (4) for $k := d'-1$.

Proof: We apply Theorems 2.1 and 2.9. By Theorem 2.1 we have that

$$u_d(x,T) = (-1)^{q(m-p)}\mathcal{V}(A)\,\mathcal{V}(B)\ \mathrm{Sylv}^{p,q}(A,B;x). \tag{7}$$

For $0 \leq d := p+q \leq m < n$ or for $0 \leq d < m = n$, we have by Theorem 2.9

$$u_d(x,T) = \sum_{p=0}^{d} u_{d,p}(x)T^{m-p} = (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Sres}_d(f,g)\,T^{m-d}(T-1)^d,$$

which implies that

$$u_{d,p}(x) \quad = \quad (-1)^{dm}(-1)^p\binom{d}{d-p}\,\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Sres}_d(f,g)$$

Therefore, using (7),

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B;x) &= (-1)^{dm+p-q(m-p)}\binom{d}{p}\mathrm{Sres}_d(f,g)\\ &= (-1)^{p(m-d)}\binom{d}{p}\mathrm{Sres}_d(f,g)\end{aligned}$$

since

$$dm+p-q(m-p)=pm+p+qp=p(m-d)+p(d+1+q)\equiv p(m-d)+p(p+1) \pmod 2.$$

For $m<d<n-1$, $\mathrm{Sylv}^{p,q}(A,B;x)=0$ since $u_d(x,T)=0$.
For $m<d:=p+q=n-1$,

$$u_d(x,T)=\sum_{p=0}^{m}u_{d,p}(x)T^{m-p}=(-1)^{(d'-1)n+d}\mathcal{V}(A)\mathcal{V}(B)(T-1)^m f(x)$$

which implies that

$$u_{d,p}(x,T) \;=\; (-1)^{(d'-1)n+q}\binom{m}{p}\mathcal{V}(A)\mathcal{V}(B)f(x).$$

Therefore, using (7), we get

$$\mathrm{Sylv}^{p,q}(A,B;x)=(-1)^{(m+q)(p+1)}\binom{m}{p}f(x),$$

since

$$(d'-1)n+q-q(m-p)=m(p+q-1)+q-qm+qp\equiv(m+q)(p+1) \pmod 2.$$

For $m\le n\le d:=p+q<m+n$,

$$u_d(x,T)=\sum_{p=d-n}^{m}u_{d,p}(x)T^{m-p}=(-1)^{n(d-m)+d}\mathcal{V}(A)\,\mathcal{V}(B)\,\left(F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x)\right)\,(T-1)^{d'-1},$$

which implies that for $d-n>p$, i.e. $d>p+n$, we have $u_{d,p}(x)=0$, while for $d-n\le p<m$ or $d-n<p\le m$,

$$\begin{aligned}u_{d,p}(x) &= (-1)^{n(d-m)+d}\big((-1)^{n-q-1}\binom{d'-1}{m-p}F_{d'-1}(x)f(x)+(-1)^{n-q}\binom{d'-1}{m-p-1}G_{d'-1}(x)g(x)\big)\,\mathcal{V}(A)\,V(B)\\ &= (-1)^{n(d-m)+d+n-q-1}\big(\binom{d'-1}{m-p}F_{d'-1}(x)f(x)-\binom{d'-1}{m-p-1}G_{d'-1}(x)g(x)\big)\,\mathcal{V}(A)\,V(B)\end{aligned}$$

Therefore, by (7), for $n\le d\le m+n-1$ we have

$$\mathrm{Sylv}^{p,q}(A,B;x) \;=\; (-1)^{q(m-p)+n(d-m)+d+n-q-1}\big(\binom{d'-1}{m-p}F_{d'-1}(x)f(x)-\binom{d'-1}{n-q}G_{d'-1}(x)g(x)\big),$$

since

$$\binom{d'-1}{m-p-1} = \binom{d'-1}{d'-m+p} = \binom{d'-1}{n-q}.$$

Finally for $d = m+n$, i.e. $p = m, q = n$ we have

$$u_d(x,T) = \mathcal{V}(A)\,\mathcal{V}(B)\ \mathrm{Sylv}^{m,n}(A,B;x) = \mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Res}(f,g)f(x)g(x)$$

which implies the claim. □

Finally we show how Theorem 2.10 implies Case (4) of the introduction:

COROLLARY 2.11 (ART. 22 (4) AND PROPOSITION 2.9(II)(3)): *For $d = m = n$ we have*

$$\mathrm{Sylv}^{p,q}(A,B;x) = \binom{m-1}{q} f(x) + \binom{m-1}{p} g(x).$$

Proof: We apply Theorem 2.10 for $d = m = n$. In this case

$$\mathrm{Sres}_{d'-1}(f,g) = \mathrm{Sres}_{m-1}(f,g) = g(x) - f(x).$$

Therefore $F_{m-1} = -1$ and $G_{m-1} = 1$. We also have $2m-d-1 = m-1$ and $m-p = d-p = q$ while $n-q = p$. Finally $p(m-q+1)+1 = p(p+1)+1 \equiv 1 \pmod 2$.

□

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