

# Tropical plane geometric constructions (extended abstract)

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## Abstract

We present a general method to compute sets of algebraic curves with prescribed incidence relations and tropicalizations. It generalizes the geometric constructions presented in Tabera, 2005 to the non linear case.

*Keywords:* tropical geometry, geometric constructions

## 1 Introduction

One of the most interesting aspects of tropical geometry is the relationship between results in algebraic and tropical geometry. A good example is Mikhalkin's correspondence Theorem [4], where the relation of Gromov-Witten invariants in the plane is discussed or, less ambitious, the relation between solutions of linear systems of equations and its application to prove a constructive version of Pappus Theorem presented in [8]. In this paper, we go further than [8] and extend the results there to the nonlinear case.

The main problem we are studying is the following: suppose given a finite set of plane tropical curves and points. Suppose also that we have a set of incidence relations between these objects of the form: Point  $a$  belongs to curve  $C$ . In many cases, given a set of tropical objects realizing a set of incidence conditions, it is not possible to compute a set of algebraic objects

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realizing the same incidence conditions. Hence, we want to detect the cases that this is possible and compute such algebraic objects. Moreover, we are interested in computing sufficient conditions for a set of algebraic objects that determines if they will tropicalize correctly onto the tropical data and, what is perhaps more important, to find families of incidence conditions such that, for every tropical realization of the incidence conditions, there is at least one algebraic realization that projects correctly.

The basic notions and notations are the following. Let  $\mathbb{K}$  be an algebraically closed field provided with a non trivial rank one valuation  $v : \mathbb{K} \rightarrow \Gamma$ ,  $\mathbb{Q} \subseteq \Gamma \subseteq \mathbb{R}$ . We denote by  $k$  the residual field of  $\mathbb{K}$  under the valuation. Our method works for every characteristic and residual characteristic, so we will not restrict ourselves to the characteristic zero case. It is also assumed that we have fixed a multiplicative subgroup  $\Gamma' \subseteq \mathbb{K}^*$  which is isomorphic to  $\Gamma$  by the valuation map. The element  $t^\gamma$  represents the element of  $\Gamma'$  whose valuation is  $\gamma$ . Any element  $x$  of  $\mathbb{K}^*$  can be uniquely written as  $x_0 t^\gamma$ , where  $v(x_0) = 0$ . We will denote the principal coefficient of an element  $x$  of  $\mathbb{K}$  by  $Pc(x) = \bar{x}_0 \in k^*$  and  $Pc(0) = 0$ . This notion depends on the concrete group  $\Gamma'$ , but it is well defined once we fix the subgroup  $\Gamma'$ . We denote the principal term of an element by  $Pt(x) = \bar{x}_0 t^\gamma$ . In some cases, if we can identify  $k$  with a subfield of  $\mathbb{K}$  compatible with the residual projection, we may think that the principal term is an element of  $\mathbb{K}$ . But, in general, the principal term is just a notation, not an element of any field. If  $y$  is an element of  $\mathbb{K}^*$ ,  $Pt(x) = Pt(y)$  if and only if  $v(x) = v(y) < v(x - y)$ .

The tropicalization map is minus the valuation,  $T(x) = -v(x)$ . The tropical semiring  $\mathbb{T}$  is the group  $\Gamma$  with the operations of tropical addition “ $a + b$ ” =  $\max\{a, b\}$  and tropical product “ $ab$ ” =  $a + b$ . Let  $f = “\sum_{i \in I} a_i x^i” = \max_{i \in I} \{a_i + ix\} \in \mathbb{T}[x_1, \dots, x_n]$  be a polynomial of support  $I$ , where  $x = x_1, \dots, x_n$ ,  $i = i_1, \dots, i_n$ ,  $ix = i_1 x_1 + \dots + i_n x_n$ . The set  $\mathcal{T}(f)$  of zeroes of  $f$  is the set of points in  $\mathbb{T}^n$  such that the maximum of the piecewise affine function  $\max_{i \in I} \{a_i + ix\}$  is attained for at least two different indices. It is known (Kapranov’s Theorem, [3]) that if  $\tilde{f} = \sum_{i \in I} \tilde{a}_i x^i$  is any polynomial in  $\mathbb{K}[x_1, \dots, x_n]$  such that  $T(\tilde{a}_i) = a_i$ , then  $T(\{\tilde{f}(x) = 0\} \cap (\mathbb{K}^*)^n)$  is exactly the set of zeroes of  $f$ . Moreover, if  $q \in \mathbb{T}^n$  is a point, let  $J \subseteq I$  be the set of indices where the value  $f(q)$  is attained and  $\alpha_i = Pc(\tilde{a}_i)$ . We define the residual polynomial of  $\tilde{f}$  over  $q$  as:

$$\tilde{f}_q(x_1, \dots, x_n) = \sum_{i \in J} \alpha_i x^i = Pc(\tilde{f}(x_1 t^{-q_1}, \dots, x_n t^{-q_n})) \in k[x_1, \dots, x_n]$$

Then, it happens that:

**Theorem 1.**  $\tilde{f} \in \mathbb{K}[x_1, \dots, x_n]$  and  $(\tilde{b}_1, \dots, \tilde{b}_n) \in (\mathbb{K}^*)^n$  any point, then there is a root  $(\tilde{c}_1, \dots, \tilde{c}_n)$  of  $\tilde{f}$  such that  $Pt(\tilde{c}_i) = Pt(\tilde{b}_i)$ ,  $1 \leq i \leq n$ , if and only if  $b = (T(\tilde{b}_1), \dots, T(\tilde{b}_n))$  is a zero of the tropical polynomial  $f$  and  $(Pc(\tilde{b}_1), \dots, Pc(\tilde{b}_n))$  is a root of  $f_b$  in  $(k^*)^n$ .

For a constructive proof of this Theorem we refer to [10].

## 2 Incidence structures

A very useful tool to deal with this problem is the notion of incidence configurations. It parametrizes a set of incidence conditions over the set of points and curves, classifying different kind of algebraic restrictions.

**Definition 1.** An incidence structure is a triple  $G = (\mathfrak{p}, \mathfrak{B}, \mathfrak{I})$ , where  $\mathfrak{p}$ ,  $\mathfrak{B}$ ,  $\mathfrak{I}$  are finite sets such that

$$\mathfrak{p} \cap \mathfrak{B} = \emptyset, \quad \mathfrak{I} \subseteq \mathfrak{p} \times \mathfrak{B}$$

The elements in  $\mathfrak{p}$  are called points, those in  $\mathfrak{B}$  are curves (or blocks) and the elements of  $\mathfrak{I}$  are called flags or incidence conditions.

This is the definition of incidence structure in the classical context of finite geometries, see for example [2]. The usual approach is to interpret blocks as lines of a finite geometry. We want to interpret them as curves, but not as arbitrary curves. In the algebraic torus  $(\mathbb{K}^*)^n$ , a hypersurface defined by a polynomial is invariant by multiplication of the polynomial by a monomial. The support of a curve  $I$  is the support of a defining polynomial modulo translation by an integer vector in  $\mathbb{Z}^2$ . We denote by  $\mathcal{P}^f(\mathbb{Z}^2)$  the finite subsets of  $\mathbb{Z}^2$  and  $\sim$  the equivalence relation  $a \sim b$  if there is a  $c \in \mathbb{Z}^2$  such that  $a = b + c$ . We suppose that we have the support map on the blocks of a finite structure:

$$Sup : \mathfrak{B} \rightarrow \mathcal{P}^f(\mathbb{Z}^2) / \sim$$

that associates each curve with the support of a defining polynomial. Given a support  $I \subseteq \mathbb{Z}^2$  we denote by  $\delta(I)$  the number of elements of  $I$  and its convex hull in  $\mathbb{R}^2$ ,  $\Delta = cv(I)$ , the Newton Polygon of the curve. Note that  $\delta$  is invariant by translations, so it is well defined and  $\Delta$  is well defined modulo translations.

Every incidence structure  $G = (\mathfrak{p}, \mathfrak{B}, \mathfrak{I})$  is identifiable with a graph, the Levi graph of the incidence structure. This is a bipartite graph whose vertices are of colors  $\mathfrak{p}$  and  $\mathfrak{B}$ . Its edges are the elements of  $\mathfrak{I}$ . We will use, indistinctly, either the given definition or its interpretation as a graph.

**Definition 2.** Let  $G = (\mathfrak{p}, \mathfrak{B}, \mathfrak{I})$  be an incidence structure. Denote by  $n_{\mathfrak{p}}$ ,  $n_{\mathfrak{B}}$  the cardinals of  $\mathfrak{p}$ ,  $\mathfrak{B}$  respectively. For each  $y \in \mathfrak{B}$ , let  $\delta_y = \delta(\text{Sup}(y))$  be the cardinal of the associated support. The algebraic (resp. tropical) support of  $G$  is the space

$$S_G = \prod_{x \in \mathfrak{p}} (\mathbb{K}^*)^2 \times \prod_{y \in \mathfrak{B}} (\mathbb{K}^*)^{\delta_y - 1}, \quad S_G^t = \prod_{x \in \mathfrak{p}} \mathbb{T}^2 \times \prod_{y \in \mathfrak{B}} \mathbb{T}^{\delta_y - 1}.$$

We identify the space  $(\mathbb{K}^*)^{\delta_y - 1}$  (resp.  $\mathbb{T}^{\delta_y - 1}$ ) with the space of algebraic curves (resp. tropical curves) of support  $\text{Sup}(y)$  (for example, dehomogenizing the equation of the curve with respect to a fixed monomial of its support). The dimension of  $S_G$  is  $2n_{\mathfrak{p}} + \sum_{y \in \mathfrak{B}} (\delta_y - 1)$ .

An algebraic realization (resp. tropical realization) of  $G$  is a point

$$(x_1, \dots, x_{n_{\mathfrak{p}}}; y_1, \dots, y_{n_{\mathfrak{B}}}) \in S_G (S_G^t)$$

such that, for every edge  $(x_i, y_j) \in \mathfrak{I}$  we have that  $x_i \in y_j$ , identifying  $y_j$  with the (tropical) plane curve it represents. The set of algebraic realizations of  $G$  is the algebraic set  $R_G$  of  $S_G$  (resp.  $R_G^t \subseteq S_G^t$ ).

So, the original problem could be stated as studying the comparison of  $R_G$  and  $R_G^t$ . It is clear that  $T(R_G) \subseteq R_G^t$ , but this is not an equality in general. So, we are interested in determine when does the equality hold,  $T(R_G) = R_G^t$ . This question could be approached using the notion of tropical basis (cf. [1]). But we do not only want to determine  $T(R_G)$ . Given  $x \in R_G^t$ , we want to determine if it is contained in  $T(R_G)$  and, in the affirmative case, compute a lift  $\tilde{x}$  in  $R_G$ . Furthermore, we are interested in how much of this information can be derived from the graph structure of  $G$ . A first result is the following:

**Theorem 2.** *Let  $G$  be an incidence structure such that its graph is acyclic. Then,  $R_G = R_G^t$  and, for all tropical realization of the graph, we can compute an algebraic realization that projects onto the tropical one.*

*Proof.* (Sketch). We may suppose that  $G$  is connected. Let  $x_0$  be any vertex of  $G$  and  $\tilde{x}_0$  be any lift to the algebraic space of  $x_0$ . Then, we lift the rest

of the vertices of  $G$  inductively. If  $y$  is an adjacent vertex to an element  $x$  representing a curve that has already been lifted to an algebraic curve  $\tilde{x}$ , then we have to lift a point  $y \in x$  to the curve  $\tilde{x}$ . This is done by Theorem 1. In the case where  $x$  is a point, we have to compute a curve  $\tilde{y}$  of given support passing through  $\tilde{x}$ . The coordinate vector  $(a_i)$  of  $y$  is in the hypersurface defined by  $\sum_{i \in I} a_i x_1^{i_1} x_2^{i_2}$ . The coordinates of a curve  $\tilde{y}$  passing through  $\tilde{x}$  and projecting onto  $y$  can be computed as a lift of  $(a_i)$  to the hypersurface defined by  $\sum_{i \in I} a_i \tilde{x}_1^{i_1} \tilde{x}_2^{i_2}$ .  $\square$

This Theorem allows, for example, to construct a lift of two curves with a prescribed non stable intersection point. Take the two curves  $C_1, C_2$  and a non stable intersection point  $p$ . The way to proceed is lifting  $C_1$ , then lifting  $p$  belonging to  $C_1$  and, finally, compute a lift of  $C_2$  passing through  $p$ . That is, this Theorem does not need the, natural but restrictive, notion of stability in order to achieve the result. It is also remarkable that it works for every realization of the incidence conditions. The drawback is that it is not able to work with more than one intersection point. As long as two curves  $C_1, C_2$  are not both lines, there is a cycle in the incidence graph  $[c_1, p_1, c_2, p_2, c_1]$ , with  $P_1, P_2$  two different intersection points. For the case of working with more than one intersection point, we propose to work with the more meaningful notion of stable intersection. Given two tropical curves, there is always a well defined finite set of intersection points that can be continuously translated as both curves are perturbed. We refer to [5] for a definition of stable intersection of curves and to [11] for a study of its relationship with the intersection of algebraic curves. Analogously, given a support  $I$ , and  $\delta(I) - 1$  points, there is always a well defined curve of support  $I$  passing through the set of points that can be continuously perturbed as we perturb the set of points. When dealing with these notions of stability, it is better to work with tropical constructions instead of just incidence relations.

**Definition 3.** A geometric construction is an abstract procedure consisting in:

- Input elements: two finite subsets  $\mathfrak{p}_0, \mathfrak{B}_0$  such that  $\mathfrak{p}_0 \cap \mathfrak{B}_0 = \emptyset$  and a support map

$$Sup : \mathfrak{B}_0 \rightarrow \mathcal{P}^f(\mathbb{Z}^2) / \sim$$

The set of incidence relations is the empty set  $\mathfrak{I} = \emptyset$ .

- Steps of the construction, a finite sequence of different steps:

- Given a support  $I$  with  $\delta(I) = n$  and  $n - 1$  points  $\{q_1, \dots, q_{n-1}\}$  we add a new curve  $C$  of support  $I$  to  $\mathfrak{B}$ , we also add new oriented incidence conditions  $q_i \rightarrow C$ ,  $1 \leq i \leq n - 1$ .
- Given two curves  $C_1, C_2$  of support  $I_1, I_2$  and Newton Polygons  $\Delta_1, \Delta_2$  respectively, we add  $M = \mathcal{M}(\Delta(I_1), \Delta(I_2))$  new points  $q_1, \dots, q_M$ . We add the oriented incidence conditions  $C_1 \rightarrow q_i, C_2 \rightarrow q_i, 1 \leq i \leq M$ .

- Output: an incidence graph  $G$  provided with an orientation.

An algebraic (resp. tropical) realization of a geometric construction  $\mathfrak{C}$  is an algebraic (resp. tropical) realization of its associated graph  $G$  such that:

- If  $x \in \mathfrak{B}$  is a curve and it is not an input element, let  $I$  be its support and let  $\{y_1, \dots, y_{\delta(I)-1}\}$  be the direct predecessors of  $x$ . Then,  $x$  is the unique curve (resp. the stable curve) of support  $I$  that passes through the points  $\{y_1, \dots, y_{\delta(I)-1}\}$ .
- If  $x \in \mathfrak{p}$  and it is not an input point, let  $y_1, y_2$  be the direct predecessors of  $x$  and let  $\{x_1, \dots, x_n\}$  be the common direct successors of  $y_1$  and  $y_2$ . Then, the algebraic curves  $y_1, y_2$  intersect exactly in the finite set of points  $\{x_1, \dots, x_n\}$  where the points are counted with multiplicities (resp. the intersection of the tropical curves  $y_1, y_2$  need not be finite, but their stable intersection is the set  $\{x_1, \dots, x_n\}$  counted with multiplicities).

The idea of a geometric construction associated to an incidence structure  $G$  is that of an algorithm that provides a way to compute realizations of  $G$  starting from the input elements. Moreover, the construction provides an orientation on the graph  $G$  and it is clear that  $G$  does not have oriented cycles. This allows to reason by induction on the depth of the constructions. Being the input elements precisely the elements of depth 0.

This notion of geometric construction behaves well with tropicalization. It provides a natural way to compute a lift of a tropical realization of a construction. Next Theorem provides sufficient conditions for a tropical realization to be lifted to the algebraic case.

**Theorem 3.** *Let  $\{C_1, \dots, C_n, p_1, \dots, p_m\}$  be the input elements of a geometric construction  $\mathfrak{C}$ , curve  $C_i$  of support  $I_i$ , point  $p_j \in \mathbb{T}^2$ . Take  $N =$*

$2m + \sum_{i=1}^n (\delta(I_i) - 1)$  and let  $\{\tilde{f}_1, \dots, \tilde{f}_n, \tilde{p}_1, \dots, \tilde{p}_m\}$  be a set of lifts of a concrete tropical instance of the input,  $f_i = \sum_{(k,l) \in I_i} \tilde{a}_{(k,l)}^i x^k y^l$ ,  $\tilde{p}_j = (\tilde{b}_1^j, \tilde{b}_2^j)$ . Then, we present an algorithm that takes such a construction and computes a constructible set  $\mathfrak{S} \subseteq (k^*)^N$ , not always empty, such that if the vector

$$(Pc(\tilde{a}_{(k,l)}^1), \dots, Pc(\tilde{a}_{(k,l)}^n), Pc(\tilde{b}_1^1), \dots, Pc(\tilde{b}_2^m)) \in (k^*)^N$$

of principal coefficients lies in  $\mathfrak{S}$ , then the algebraic construction is well defined and the result projects onto the tropical construction.

*Proof.* (Sketch). The construction of the set  $\mathfrak{S}$  is provided. First, we define an auxiliary set  $T$ . This set is defined adding the residual restrictions that ensure that each step of the construction is compatible with tropicalization. Let

$$f_i = \sum_{(k,l) \in I_i} a_{(k,l)}^i x^k y^l, 1 \leq i \leq n,$$

$$p_j = (b_1^j, b_2^j), 1 \leq j \leq m$$

be the tropical input elements. Take a generic lift of the input

$$\tilde{f}'_i = \sum_{(k,l) \in I_i} \tilde{a}_{(k,l)}^i x^k y^l, 1 \leq i \leq n,$$

$$\tilde{p}'_j = (\tilde{b}_1^j t^{-b_1^j}, \tilde{b}_2^j t^{-b_2^j}), 1 \leq j \leq m$$

where  $Pt(\tilde{a}_{(k,l)}^i) = \alpha_{(k,l)}^i t^{-a_{(k,l)}^i}$ ,  $Pt(\tilde{b}_i^j) = \beta_i^j t^{-b_i^j}$  and  $V_0 = \{\alpha_{(k,l)}^i, \beta_r^j\}$  is a set of indeterminates over  $k$ . These indeterminates will describe  $\mathfrak{S}$ . Perform the construction with this data as follows.

First,  $T = (k^*)^N = \{x \in k^N | \alpha_{(k,l)}^i \neq 0, \beta_r^j \neq 0, 1 \leq i \leq n, 1 \leq j \leq m\}$  and  $V = V_0$ . We are going to redefine  $T$  and  $V$  inductively at each step of the construction. Suppose computed the constructible set  $T \subseteq (k^*)^K$  and a set of  $K$  variables  $V$  for the construction up to a construction step. Define  $T$  after the step as follows: For the case of the computation of the curve  $C$  of support  $I$  passing through  $\delta(I) - 1$  points, we have to solve a system of linear equations. The coefficients of  $\tilde{C}$  are rational functions  $h_i(s)/g_i(s)$  of the variables  $V$ . In [8], sufficient conditions in the variables  $V$  for the system being compatible with tropicalization are provided. We add  $\delta(I) - 1$  new variables  $s_1, \dots, s_{\delta-1}$  to  $V$  and we consider  $T \subseteq (k^*)^{K+\delta-1}$ . We add the

conditions  $h_i, g_i \neq 0$  to the definition of  $T$  and the equations  $s_i - h_i(s)/g_i(s)$ . We follow the construction with  $\tilde{C}$  among our available objects.

Suppose now that our construction step consists in the (stable) intersection of two curves  $\tilde{f}, \tilde{g}$  of support  $I_f, I_g$  respectively. This stable intersection can be determined using the techniques presented in [11]. That is, there are three resultants  $\tilde{R}_x(x) = \text{Res}_y(\tilde{f}, \tilde{g}), \tilde{R}_y(y) = \text{Res}_x(\tilde{f}, \tilde{g}), \tilde{R}_z(z) = \text{Res}_y(\tilde{f}(zy^a, y), \tilde{g}(zy^a, y))$  (where  $a$  is an appropriate natural number and  $zy^a = x$ ) such that they correspond with the tropical resultants in the generic case. If  $t_r$  are the variables of  $V$  corresponding with the principal coefficients of  $\tilde{f}, \tilde{g}$ , we add the conditions  $\tilde{h}(t_r) \neq 0$  for the resultants being compatible as in [11]. There are  $M = \mathcal{M}(\Delta_f, \Delta_g)$  stable intersection points  $b^j = (b_1^j, b_2^j)$ . We add  $2M$  new variables  $s_1^j, s_2^j, 1 \leq j \leq M$  to  $V$ . Consider  $T$  contained in  $(k^*)^{K+2M}$ . For each tropical point  $b_j$ , let  $s^{j_1}, \dots, s^{j_n}$  be the algebraic points projecting onto  $b_j$ . We take the following equalities:

$$(\tilde{R}_x)_{b_1^j} = \prod_{r=1}^n (x - s_1^{j_r}), \quad (\tilde{R}_y)_{b_2^j} = \prod_{r=1}^n (y - s_2^{j_r}),$$

$$(\tilde{R}_z)_{b_1^j(b_2^j)^{-a}} = \prod_{r=1}^n (z - s_1^{j_r}(s_2^{j_r})^{-a}).$$

We identify in this way the coefficients of the resultants  $\tilde{R}_x, \tilde{R}_y, \tilde{R}_z$  with the symmetric functions in the variables  $(s_1^{j_r}, s_2^{j_r})$  and we add them to the definition of  $T$ . In this way, we ensure that there is a bijection between the roots of the resultants and the variables  $s^j$ . We also add the residual conditions of the curves over the intersection points  $\tilde{f}_{b^j}(s_1^j, s_2^j) = 0, \tilde{g}_{b^j}(s_1^j, s_2^j) = 0$ , and the conditions of the points being in the torus  $s_1^1 s_2^1 \neq 0$ . We continue the construction with the points  $(s_i^1 t^{-b_i^1}, s_i^2 t^{-b_i^2})$ . After the whole construction, we have defined a constructible set  $T$  that characterizes the possible principal coefficient of every element in the construction. Let  $\mathfrak{S}$  be the projection of the set defined by  $T$  into the space of variables  $V_0$ . The constructible set  $\mathfrak{S}$  is the set of valid principal coefficients of the input elements.  $\square$

In this Theorem, it is not claimed that there is always a possible lift, as Theorem 2 does. It is possible that the constructible set  $\mathfrak{S}$  is empty. In this case, the Theorem does not yield to any conclusion. A sufficient condition for  $\mathfrak{S}$  being non empty for every instance of the construction is the notion of admissibility. This is a generalization of the notion of admissibility of [8].

**Definition 4.** Let  $\mathfrak{C}$  be a geometric construction. Let  $G$  be its incidence graph with the orientation induced by the construction. We call the construction admissible if, for every two nodes  $A, B$  of  $G$ , there is at most one oriented path from  $A$  to  $B$ . We will denote the case where there are at least two paths from  $A$  to  $B$  by  $A \rightrightarrows B$ .

It is easy to see that if the construction consists in just one step, then it can always be lifted. If the input elements are lifted to generic algebraic elements, each step is well defined and it is compatible with tropicalization (cf. [8], [11]). The idea of admissibility is that, inductively for each step, we may consider that the input elements of the step are generic. But, in order to apply induction, we need to prove that the output of a single step is generic.

**Theorem 4.** *Let  $I$  be a support,  $\delta = \delta(I)$ . Let  $P = \{p_1, \dots, p_{\delta-1}\}$  be a set of tropical points. Let  $C$  be the stable tropical curve of support  $I$  passing through  $P$ . If the principal coefficients of the lifts  $\tilde{p}_i$  of  $p_i$  range over a dense constructible set of  $k^{2\delta-2}$ . Then, the possible principal of  $\tilde{C}$  dehomogenized with respect to a monomial  $i_0$  contains a dense constructible subset of  $k^{\delta-1}$ . That is, if the principal coefficients of the points  $\tilde{p}_i$  are generic, so they are the principal coefficients of  $\tilde{C}$ .*

*Proof.* (Sketch). Write  $p_l = (b_1^l, b_2^l)$ ,  $C = \sum_{ij} a_{ij} x^i y^j$ . Then,  $C$  is the curve defined by the stable solution of:

$$\sum_{(i,j) \in I} A_{ij} (b_1^l)^i (b_2^l)^j, 1 \leq l \leq N$$

and the lifts of  $C$  verify the relations

$$\sum_{(i,j) \in I} \tilde{a}_{ij} (\tilde{b}_1^l)^i (\tilde{b}_2^l)^j = 0, 1 \leq l \leq N$$

Take the equations

$$\tilde{f}_l = \sum_{(i,j) \in I} \tilde{a}_{ij} x^i y^j t^{-a_{ij} - i b_1^l - j b_2^l}, 1 \leq l \leq N.$$

Which correspond to a (tropical) translation of the problem to the point 0. We dehomogenize this equation imposing  $\tilde{\alpha}_{i_0 j_0} = 1$ . The conditions on the principal terms are:

$$f_l = \sum_{J_l} \alpha_{ij} (\beta_1^l)^i (\beta_2^l)^j, 1 \leq l \leq N$$

Where  $J_l \subseteq I$  are the monomials such that  $-a_{ij} - ib_1^l - jb_2^l$  is minimized. Write  $\alpha = \{\alpha_{ij} | (i, j) \neq (i_0, j_0)\}$ ,  $\beta_1 = \{\beta_1^1, \dots, \beta_1^{\delta-1}\}$ ,  $\beta_2 = \{\beta_2^1, \dots, \beta_2^{\delta-1}\}$ . Each  $f_l$  is affine in the set of variables  $\alpha$ , and the coefficients of these affine equations are monomials in  $\{\beta_{l1}, \beta_{l2}\}$ . We saturate each  $f_l$  with respect to the coordinate hyperplanes (eliminate redundant  $\beta$ ). We still denote the saturated polynomials by  $f_l$ . Thus, we have a system of equations in  $3\delta - 3$  unknowns. The solutions are rational functions of the variables  $\beta_1, \beta_2$ . Hence we have the rational variety defined as the image of:

$$\begin{aligned} k^{2\delta-2} &\longrightarrow k^{3\delta-3} \\ (\beta_1, \beta_2) &\mapsto (\beta_1, \beta_2, \text{Cramer}(\beta_1, \beta_2)) \end{aligned}$$

It is clear that this is a birational map between the space  $k^{2\delta-2}$  and its image. Let  $\mathcal{I}$  be the ideal of the image.  $\mathcal{I}$  is a prime ideal that contains  $(f_1, \dots, f_{\delta-1})$  in  $k[\alpha, \beta_1, \beta_2]$ . By construction, the field of rational functions of the variety is

$$\mathbb{L} = \text{Frac} \left( \frac{k[\beta_1, \beta_2, \alpha]}{\mathcal{I}} \right) = k(\beta_1, \beta_2)$$

In particular,  $\beta_1, \beta_2$  is a transcendence basis of  $k \subseteq \mathbb{L}$  and the dimension of  $\mathbb{L}$  is  $2\delta - 2$ .

For each  $f_l$ , if the variable  $\beta_1^l$  does not appear in  $f_l$ , then  $\beta_2^l$  is an element of  $\mathbb{L}$  which is algebraic over  $k(\alpha, \beta_1^l)$ . Analogously, if  $\beta_2^l$  does not appear in  $f_l$ , then  $\beta_1^l$  is algebraic over  $k(\alpha, \beta_2^l)$ . If both variables appear in  $f_l$ , then just choose  $\beta_j^l$  algebraic over  $k(\alpha, \beta_{3-j}^l)$ . In this way, we define a set  $\alpha \cup \{\beta_{3-j}^l, 1 \leq l \leq \delta - 1\}$  which is a transcendence basis of  $\mathbb{L}$  and  $\alpha$  is algebraically independent over  $k$ . This means that:

$$\mathcal{I} \cap k[\alpha] = \mathcal{I} \cap k[\beta_1, \beta_2] = 0 \tag{1}$$

Turn back to the original problem. Suppose that the possible coefficients  $\beta$  belong to a dense definable set  $F(\beta_1, \beta_2) \neq 0$ . Without loss of generality, we may suppose that  $F$  contains as factors every variable  $\beta_{ij}$  and every pseudodeterminant associated to the system. The fact that the image set is dense is equivalent to claim that the system

$$\{f_1 = 0, \dots, f_{\delta-1} = 0, F(\beta) \neq 0, G(\alpha) \neq 0\}$$

has a solution whenever  $F$  and  $G$  are nonzero polynomials. If this system does not have any solution, then  $f_i = 0 \rightarrow F(\beta)G(\alpha) = 0$ , this means that  $F(\beta)G(\alpha) \in \mathcal{I}$  and, as it is prime,  $F(\beta) \in \mathcal{I}$  or  $G(\alpha) \in \mathcal{I}$  which contradicts the equation (1).  $\square$

**Theorem 5.** *Suppose given two tropical curves  $C_1, C_2$  with support  $I_1$  and  $I_2$  respectively. Let  $q_1, \dots, q_N$  be the intersection points counted with multiplicities.  $N = \mathcal{M}(\Delta_1, \Delta_2)$ . Let  $\tilde{C}_1, \tilde{C}_2$  be two lifts of the curves whose principal coefficients are generic. Choose  $q$  one stable intersection point. Then, the principal coefficients of  $\tilde{q}$  are generic.*

*Proof.* The proof mimics Theorem 4. Let

$$f_1 = \text{“} \sum_{(i_1, i_2) \in I_1} a_i x^{i_1} y^{i_2} \text{”} \quad f_2 = \text{“} \sum_{(j_1, j_2) \in I_2} b_j x^{j_1} y^{j_2} \text{”}$$

be the tropical polynomials defining the curves and let

$$\tilde{f}_1 = \sum_{(i_1, i_2) \in I_1} \tilde{a}_i x^{i_1} y^{i_2} \quad \tilde{f}_2 = \sum_{(j_1, j_2) \in I_2} \tilde{b}_j x^{j_1} y^{j_2}$$

be the lifts of the curves. We may suppose that these lifts are dehomogenized with respect to one monomial. Let  $g_1, g_2$  be the residual polynomials over the point  $q$ .

$$g_1 = \sum_{J_1} \alpha_i x^{i_1} y^{i_2} \quad g_2 = \sum_{J_2} \beta_j x^{j_1} y^{j_2}$$

and let  $(\gamma_1, \gamma_2)$  be the principal coefficients of  $\tilde{q}$ . In order to ensure that the root exists,  $(\alpha, \beta)$  must not be a zero of some residual polynomials obtained by resultants as described in [11]. Denote the product of all these conditions by  $F'(\alpha, \beta) \neq 0$ . Saturate the polynomials  $g_1, g_2$  with respect to the coordinate hyperplanes. Let  $\alpha_0, \beta_0$  be two variables corresponding to a monomial, not the monomial of dehomogenization, in  $g_1, g_2$  respectively. Let  $\alpha = \{\alpha_i \mid i \neq 0\}, \beta = \{\beta_j \mid j \neq 0\}$  be the rest of the variables. The values of  $\alpha_0, \beta_0$  are a rational expression of the values of  $\alpha, \beta, \gamma$ . Consider the rational variety given by the image of

$$\begin{array}{ccc} k^{\delta_1 + \delta_2 - 2} : & \longrightarrow & k^{\delta_1 + \delta_2} \\ (\alpha, \beta, \gamma) & \longrightarrow & (\alpha, \beta, \gamma, \alpha_0, \beta_0) \end{array}$$

The variety defined by this parametrization is an irreducible variety of dimension  $\delta_1 + \delta_2 - 2$  (recall that we have dehomogenized both polynomials, so there are two variables  $\alpha_i, \beta_j$  that do not appear). Let  $\mathbb{L}$  be the field of rational functions of the variety. It is clear that  $\gamma_1, \gamma_2$  are algebraically independent over  $k$ . It is also clear that  $\gamma_1, \gamma_2$  are algebraic over  $\alpha, \beta, \alpha_0, \beta_0$

(the equations of the algebraic relation can be computed using resultants). And hence  $\alpha, \beta, \alpha_0, \beta_0$  are also algebraically independent over  $k$ . Now the proof goes as in Theorem 4.  $\square$

Now, we present the main Theorem.

**Theorem 6.** *Let  $\mathfrak{C}$  be an admissible geometric construction. Then, for every tropical instance of the construction, the set  $\mathfrak{S}$  defined in Theorem 3 is nonempty and dense in  $(k^*)^N$ . Moreover, for every element  $X$  of the construction, its values, as the input elements belong to  $\mathfrak{S}$  contains a dense open subset of its support space.*

*Proof.* (Sketch). The Theorem is proved by induction in the depth of the construction. If the construction is of depth 0, then there is nothing to prove, because the set of steps is empty and  $\mathfrak{S} = (k^*)^N$  which is dense and the values of each element are dense in their respective space of configurations. Suppose the Theorem proved for constructions of depth smaller or equal to  $i$ . Let  $\mathfrak{C}$  be any construction of depth  $i + 1$ . For each element  $X$  of depth  $i + 1$ , let  $Y_1, \dots, Y_n$  be the elements needed to define  $X$ . By induction hypothesis, every element  $Y_i$  contains a dense open set in its space of configurations. As the construction is admissible, the set of elements recursively needed to define  $Y_i$  is a disjoint set from the elements recursively needed to define  $Y_j$ , if  $i \neq j$ , because if both elements had a common predecessor, there would be a double path  $A \rightrightarrows X$ . Hence, the coefficients  $Y_1, \dots, Y_n$  are completely independent. That is, the possible tuples  $(Y_1, \dots, Y_n)$  are precisely the product of possible values of the coefficients. By Theorems 4 and 5, as the elements  $Y_j$  are generic, so it is  $X$ . That is, the possible values of  $X$  contains a dense open set of its support space. The conditions imposed by the definition of  $X$  to the auxiliary set  $T$  in Theorem 3 are a set of inequalities in the tuples  $(Y_1, \dots, Y_n)$  that are verified on an open set. Furthermore, the restrictions in the elements  $Y_j$  impose other restrictions to the elements that have been used to define the  $Y_j$ . Again, these restrictions are verified on an open set.  $\square$

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