Homological description of Monomial Ideals using Mayer-Vietoris Trees

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Abstract

The goal of this paper is to provide homological descriptions of monomial ideals. The key concepts in these descriptions are the minimal free resolution, the Koszul homology and the multigraded Betti numbers. These three objects are strongly related, being Tor modules a simple way to describe this relation. We introduce a new tool, *Mayer-Vietoris trees* which provides a good way to compute the homological description of monomial ideals. They can be used to compute either minimal free resolutions, Koszul homology or multigraded Betti numbers of monomial ideals. Related with Mayer-Vietoris trees, we introduce the families of Mayer-Vietoris ideals, which include several well known families of monomial ideals as particular cases. Several examples and applications are also provided.

Introduction

Let $R = \mathbf{k}[x_1, \ldots, x_n]$ the ring of polynomials in n variables over a field $\mathbf{k}$ of characteristic 0, and $I \subseteq R$ a monomial ideal. Many homological invariants and properties of I can be read from the (multigraded) Betti numbers of it. These include, depth, dimension, Castelnuovo-Mumford regularity, etc. Being multigraded Betti numbers the ranks of the modules in a minimal resolution of I, they can be considered as the ranks of the multigraded $Tor(I, \mathbf{k})$ modules, and also as the ranks of the Koszul homology modules of I. These equivalences origin from equivalent ways to compute $Tor(I, \mathbf{k})$.

Definition 0.1. Let M be an R-module. The n-th left derived functor of the right-exact functor $M \otimes -$ is denoted by $Tor_n^R(M, -)$.

Following the definition of left derived functor, the computation of $Tor_\bullet^R(\mathbf{k}, I)$ goes as follows: Take a resolution $\mathbb{P}$ of I as an R-module, and tensor it with $\mathbf{k}$. Then, $Tor_i^R(I, \mathbf{k})$ is just the i-th homology module of the tensor complex $\mathbb{P} \otimes \mathbf{k}$. This homology is independent of the resolution taken. If $\mathbb{P}$ is a minimal resolution of I, then tensoring it with $\mathbf{k}$ yields a complex with zero differentials in every dimension, and the ranks of the homology modules are just the ranks of the modules in $\mathbb{P}$ (see [2] for example) i.e. the Betti numbers of I. Monomial ideals being multigraded modules of R, their minimal free resolutions are also multigraded and hence their Betti numbers and Tor modules. On the other hand, one can use a resolution of $\mathbf{k}$ as R-module, and tensor it with I, the homology of this product complex is again $Tor(I, \mathbf{k})$. The Koszul complex, let us denote it $\mathbb{K}$, provides a resolution of $\mathbf{k}$ [4], and $I \otimes \mathbb{K}$ is by definition the Koszul complex of I, which is no longer a resolution; the homology of this complex is called the *Koszul homology of I*. In this paper we use the following definition of Koszul homology: Let $\mathcal{V}$ be a n-dimensional $\mathbf{k}$-vector space. Let $S\mathcal{V}$ and $\wedge\mathcal{V}$ be the symmetric and exterior algebras of $\mathcal{V}$ respectively. We consider the basis of $\mathcal{V}$ given by $\{x_1, \ldots, x_n\}$; then we can identify $S\mathcal{V}$ and R and consider the following complex

$$\mathbb{K} : 0 \to R \otimes \wedge^n\mathcal{V} \xrightarrow{\partial} R \otimes \wedge^{n-1}\mathcal{V} \xrightarrow{\partial} \cdots R \otimes \wedge^1\mathcal{V} \xrightarrow{\partial} R \otimes \wedge^0\mathcal{V} \to \mathbf{k} \to 0$$

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Elements of $R\otimes\wedge^i\mathcal{V}$ are $\mathbf{k}$-linear combinations of elements of the form $x_1^{\mu_1}\dots x_n^{\mu_n}\otimes x_{j_1}\wedge\cdots\wedge x_{j_i}$ with $j_1<\cdots<j_i\in\binom{n}{i}$;then, the differentials have the form

$$\partial(x_1^{\mu_1}\dots x_n^{\mu_n}\otimes x_{j_1}\wedge\cdots\wedge x_{j_i})=\sum_{k=1}^{i}(-1)^{k+1}x_{j_k}\cdot x_1^{\mu_1}\dots x_n^{\mu_n}\otimes x_{j_1}\wedge\cdots\wedge\widehat{x_{j_k}}\wedge\cdots\wedge x_{j_i}$$

This differential satisfies $\partial^2=0$ and makes $\mathbb{K}$ a complex, which is called the *Koszul complex* [4, 6]. This complex is a minimal free resolution of $\mathbf{k}=R/\mathfrak{m}$, where $\mathfrak{m}=\langle x_1,\dots,x_n\rangle$, the maximal ideal in R. Given a graded module $\mathcal{M}$, its *Koszul complex* $(\mathbb{K}(\mathcal{M}),\partial)$ is the tensor product complex $\mathcal{M}\otimes_R\mathbb{K}$:

$$\mathbb{K}(\mathcal{M}):0\to\mathcal{M}\otimes\wedge^n\mathcal{V}\xrightarrow{\partial}\mathcal{M}\otimes\wedge^{n-1}\mathcal{V}\xrightarrow{\partial}\cdots\mathcal{M}\otimes\wedge^1\mathcal{V}\xrightarrow{\partial}\mathcal{M}\otimes\wedge^0\mathcal{V}\to\mathbf{k}\to 0$$

This complex is no longer acyclic, and we define the *Koszul homology* of $\mathcal{M}$ as the homology of $\mathbb{K}(\mathcal{M})$. It is clear that the Koszul differential preserves both the total degree and total multidegree. From the definitions, it is clear that we can identify the Koszul homology modules with $Tor_\bullet^R(\mathcal{M},\mathbf{k})$: We have a resolution of $\mathbf{k}$ (the Koszul complex) to which we have applied the functor $\mathcal{M}\otimes -$. The homology of the resulting complex is by definition $Tor_\bullet^R(\mathcal{M},\mathbf{k})$.

These equivalent ways to compute the Tor modules in the multigraded case is what allows us to compute the multigraded Betti numbers of a monomial ideal, and hence a *homological description* of the ideal without actually computing the minimal resolution of it. The computation of (the ranks of) the Koszul homology modules gives us the information we need. We see in the next section that efficient methods for these computations constitute an alternative to the minimal resolution approach.

We summarize the different ways of computing $Tor_\bullet^R(I,\mathbf{k})$ in the following equalities:

$$\beta_{i,\mathbf{a}}(I)=dim(Tor_{i,\mathbf{a}}^R(\mathbf{k},I))=dim(Tor_{i,\mathbf{a}}^R(I,\mathbf{k}))=dim_{\mathbf{k}}(H_{i,\mathbf{a}}(\mathbb{K}(I,\partial)))$$

for all $i\in\mathbb{N}$ and $\mathbf{a}\in\mathbb{N}^n$.

1 Mayer-Vietoris Trees

The Mayer-Vietoris sequence is a standard tool in algebraic topology used to explore the relations between the homology groups of two spaces A and B, their intersection $A\cap B$ and their union $A\cup B$. Given a monomial ideal I minimaly generated by $\{m_1,\dots,m_r\}$, we can construct an analogue of it in the following way:

Definition 1.1. For each $1\leq s\leq r$ denote $I_s:=\langle m_1,\dots m_s\rangle$, $\tilde{I}_s:=I_{s-1}\cap\langle m_s\rangle=\langle m_{1,s},\dots,m_{s-1,s}\rangle$, where $m_{i,j}$ denotes $lcm(m_i,m_j)$. Then, for each s we have

$$\begin{aligned}\cdots\longrightarrow H_{i+1}(\mathbb{K}(I_s))\xrightarrow{\Delta}H_i(\mathbb{K}(\tilde{I}_s)\longrightarrow\\ H_i(\mathbb{K}(I_{s-1})\oplus\mathbb{K}(\langle m_s\rangle))\longrightarrow H_i(\mathbb{K}(I_s))\xrightarrow{\Delta}\cdots\end{aligned}\qquad(1)$$

We also have a multigraded version of the sequence.

1.1 Mayer-Vietoris trees of I.

Using recursively these exact sequences for every $\mathbf{a}\in\mathbb{N}^n$ we could compute the Koszul homology of $I=\langle m_1,\dots,m_r\rangle$. The involved ideals can be displayed as a tree, the root of which is I and every node J has as *children* $\tilde{J}$ on the left and J' on the right (if J is generated by r monomials, $\tilde{J}$ denotes $\tilde{J}_r$ and J' denotes J_{r-1}). This is what we call a **Mayer-Vietoris Tree** of the monomial ideal I, and we will denote it $MVT(I)$.

Remark 1.2. Since by definition, in $MVT(I)$ every *father* has exactly two *children*, we can assign position indices to every node, in the following way: I has position 1 and if J has position p then $\tilde{J}$ has position $2p$ and J' has position $2p+1$.

1.2 $MVT(I)$ and Koszul Homology computations

Proposition 1.3. *If $H_{i,\mathbf{a}}(\mathbb{K}(I)) \neq 0$ for some i, then $x^{\mathbf{a}}$ is a generator of some node J in any Mayer-Vietoris tree $MVT(I)$.*

Thus, all the multidegrees of Koszul generators (equivalently Betti numbers) of I appear in $MVT(I)$. For a sufficient condition, we need the following notation: among the nodes in $MVT(I)$ we call *relevant nodes* to those in even position or in position 1.

Proposition 1.4. *If $x^{\mathbf{a}}$ appears only once as a generator of a relevant node J in $MVT(I)$ then there exists exactly one generator in $H_*(\mathbb{K}(I))$ which has multidegree $\mathbf{a}$.*

The dimension of the homology to which relevant multidegrees contribute, can also be read from their position in the tree.

2 Mayer-Vietoris trees and homological computations

2.1 Minimal free resolutions

A typical technique to build resolutions in a recursive way is using *mapping cone resolutions* [7]. For them, one must use the *algebraic mapping cone* of a map between two chain complexes. Two problems show up when using such a construction: first, it's is not always easy to build the chain complex map between the two small resolutions; and second, the mapping cone of two minimal resolutions needs not to be minimal. The first problem is solved using tecniques from *effective homology* [10]; the second can be treated in an efficient way using the same techniques.

Theorem 2.1. *Let $(A, \mathfrak{i}, \rho, B, \mathfrak{j}, \sigma, C)$ be an effective short exact sequence of effective chain-complexes:*

$$0 \longrightarrow A \underset{\rho}{\overset{\mathfrak{i}}{\rightleftarrows}} B \underset{\sigma}{\overset{\mathfrak{j}}{\rightleftarrows}} C \longrightarrow 0$$

where $\mathfrak{i}$ and $\mathfrak{j}$ are chain complex morphisms; ρ and σ are graded module morphisms and the following relations hold: $id_A = \rho\mathfrak{i}$, $id_B = \mathfrak{i}\rho + \sigma\mathfrak{j}$ and $id_C = \mathfrak{j}\sigma$, then, an algorithm constructs a canonical reduction (see [10]) between $Cone(\mathfrak{i})$ and C from the data.

The recursive application of this theorem builds a resolution of I supported on the Mayer-Vietoris tree of I.

2.2 Koszul homology

Non-repeated relevant multidegrees

First of all, the trivial case is that of the node in position one, i.e.generators of I, here

$$H_i(\langle m_1 \rangle) = \begin{cases} \mathbf{k} & i = 0, \\ 0 & otherwise. \end{cases}$$

and the generator of $H_0(\langle m_1 \rangle)$ can be identified with m_1 itself. Now we can begin with a recursive process starting with the node in which the multidegree we are interested in appears. The ingredient we need for this recursive process is a explicit formula to obtain preimages of the connecting morphism Δ.

Repeated multidegrees

We can associate to I and $\mathbf{a}$ a simplicial complex, called the *Koszul simplicial complex*, denoted by $\Delta^I_{\mathbf{a}}$. The singular chain complex associated to this complex is equal to the multidegree $\mathbf{a}$ piece of the Koszul complex of I, thus, the following proposition holds (see [9, 1]):

Proposition 2.2.

$$H_i(K(I)_{\mathbf{a}}) \simeq \widetilde{H}_{i-1}(\Delta^I_{\mathbf{a}}) \forall i$$

The isomorphism is explicit.

3 Mayer-Vietoris ideals

Let I be a monomial ideal and $MVT(I)$ a Mayer-Vietoris tree of I. Let $\mathbf{a} \in \mathbb{N}^n$; let $\overline{\beta_{i,\mathbf{a}}}(I) = 1$ if $\mathbf{a}$ is the multidegree of some non repeated generator in some relevant node of dimension i in $MVT(I)$ and $\overline{\beta_i}(I) = 0$ in other case. Let $\widehat{\beta_{i,\mathbf{a}}}(I)$ be the number of times $\mathbf{a}$ appears as the multidegree of some generator of dimension i in some relevant node in $MVT(I)$. Then for all $\mathbf{a} \in \mathbb{N}^n$ we have

$$\overline{\beta_{i,\mathbf{a}}}(I) \leq \beta_{i,\mathbf{a}}(I) \leq \widehat{\beta_{i,\mathbf{a}}}(I)$$

Definition 3.1. Let I be a monomial ideal.

- If there exists a Mayer-Vietoris tree of I such that there is no repeated generator in the ideals of the relevant nodes, then we say that I is a *Mayer-Vietoris ideal of type A*. In this case, $\overline{\beta_{i,\mathbf{a}}}(I) = \beta_{i,\mathbf{a}}(I) = \widehat{\beta_{i,\mathbf{a}}}(I)\ \forall i \in \mathbb{N}$, $\mathbf{a} \in \mathbb{N}^n$.
- If $\overline{\beta_{i,\mathbf{a}}}(I) = \beta_{i,\mathbf{a}}(I)$ for all $\mathbf{a} \in \mathbb{N}^n$ then we say that I is a *Mayer-Vietoris ideal of type B1.*
- If $\widehat{\beta_{i,\mathbf{a}}}(I) = \beta_{i,\mathbf{a}}(I)$ for all $\mathbf{a} \in \mathbb{N}^n$ then we say that I is a *Mayer-Vietoris ideal of type B2.*

Mayer-Vietoris ideals of type A

In [5], a link between monomial ideals and coherent systems from reliability theory was stablished. An important type of such systems are *consecutive k-out-of-n* systems. A *consecutive k-out-of-n* system is expresed by the ideal $\overline{I}_{k,n}$ generated by all k-tuples of consecutive variables in the polynomial ring on n variables.

Proposition 3.2. $\overline{I}_{k,n}$ *is a Mayer-Vietoris ideal of type A for all k and n.*

Mayer-Vietoris ideals of type B1

For every monomial ideal $I = \langle m_1, \ldots, m_r \rangle$ its *Scarf Complex*, Δ_I is the collection of all subsets of $\{m_1, \ldots, m_r\}$ whose least common multiple is unique. Let us call *Scarf monomial ideals* to those ideals I such that the chain complex supported on the Scarf complex Δ_I minimally resolves R/I.

Proposition 3.3. *If I is a Scarf monomial ideal, then I is a Mayer-Vietoris ideal of type $B1$.*

Corollary 3.4. *All generic monomial ideals are Mayer-Vietoris ideals of type $B1$*

3.0.1 Mayer-Vietoris ideals of type B2

Proposition 3.5. *If I is a monomial ideal minimally resolved by its Taylor resolution, then it is a Mayer-Vietoris ideal of type $B2$.*

Another family of ideals coresponding to coherent systems in reliabilty theory is that of *k-out-of-n* systems. These are represented by monomial ideals generated by products of any k variables in the polynomial ring of n variables.

Proposition 3.6. $I_{k,n}$ *is a Mayer-Vietoris ideal of type $B2$ for all k and n.*

4 Algorithm

A very intuitive algorithm performs the computation a Mayer-Vietoris tree of a monomial ideal I in a simple way. Every node is given by its position, dimension and generators. The complexity of this algorithm depends strongly on the number of generators of I and has only a weak dependence on the number of variables (the neccesary divisibility tests depend on the number of variables); it is independent on the degrees of the generators involved.

The following tables show the results of computations[1] in some examples. The fastest commands in Singular, compute nonminimal free resolutions, the size of which is compared to the size of the Mayer-Vietoris resolution in the second table. The size of a resolution is considered as the sum of the ranks of all modules in it; in oparticular, the size of the minimal resolution (column S in the first table) is the sum of the Betti numbers of I. We can see that Mayer-Vietoris trees constitute an alternative also to the computation of minimal free resolutions by usual algorithms.

Example	gens	S	CoCoA	mres	hres	sres	lres	Macaulay	Tree
$random(12)$	19	8977	980'71	89'53	202'30	2'52	0'75	2'68	0'53
$random(22)$	17	45431	?	2250'98	?	27'12	15'11	16'72	2'44
$random(55)$	17	106761	?	3839'89	?	119'49	170'42	?	6'81
$random(65)$	18	212514	?	?	?	?	660'11	?	13'17
$valla(10, 4, 2)$	715	178177	?	5487'85	?	277'19	301'59	?	15'18
$valla(11, 4, 2)$	1001	471041	?	?	?	2268'79	2773'21	?	34'12

Example	gens	minimal	sres	lres	Mayer-Vietoris
$random(12)$	19	8977	25915	11667	11667
$random(22)$	17	45431	63418	45431	55333
$random(55)$	18	106761	106761	106761	106761
$random(65)$	18	212514	?	212514	229239
$valla(10, 4, 2)$	715	178177	178177	178177	178177
$valla(11, 4, 2)$	1001	471041	471041	471041	471041

5 Conclusions and further work

We have introduced Mayer-Vietoris trees as a new tool for the computation of multigraded Betti numbers, minimal free resolutions and Koszul homology of monomial ideals. They are a description of a recursive process for such computations based on the Mayer-Vietoris sequence. As an algorithm, Mayer-Vietoris trees are comparatively efficient with respect to the usual approaches to the problem. Moreover, we have introduced the families of Mayer-Vietoris monomial ideals, which include several well known types of monomial ideals. Some applications to reliability theory have also been shown. Future work includes the efficient implementation of different versions of the Mayer-Vietoris tree algorithm, in order to adapt the application of Mayer-Vietoris trees to different goals. With respect to Mayer-Vietoris ideals, a deeper study of these ideals has to be done. Characterizations of Mayer-Vietoris ideals should also be given to obtain better knowledge of this family of monomial ideals. For ideals which are not Mayer-Vietoris, the construction of Mayer-Vietoris trees is a first step to the actual computations of their multigraded Betti numbers, which has to be combined with further algebraic, homological and/or topological techniques (see [12]).

[1] All computations were made on a Pentium IV processor (2.5GHz) running CoCoA 4.6, Singular 3.0.1 (commands *mres, hres, sres* and *lres*) and Macaulay2 0.9.8 under Linux (Mandriva 2007).

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