

The number of real roots of a bivariate polynomial on a line

Martin Avendaño *

May 30, 2007

Abstract

We prove that a polynomial $f \in \mathbb{R}[x, y]$ with t non-zero terms, restricted on a real line $y = ax + b$, either has at most $6t - 4$ zeroes or vanishes over the whole line. As a consequence, we derive an alternative algorithm to decide whether a linear polynomial $y - ax - b \in K[x, y]$ divides a sparse polynomial $f \in K[x, y]$ with t terms in $[\log(H(f)H(a)H(b))][K : \mathbb{Q}] \log(\deg(f))t^{O(1)}$ bit operations, where K is a real number field.

1 Introduction

The famous Descartes' Rule of Signs, 1641, establishes that the number of positive real roots of a polynomial $f \in \mathbb{R}[x]$, counted with multiplicities, is bounded by the number of changes of signs in its ordered vector of coefficients, disregarding the zeros. As a direct consequence, the number of real roots of f is bounded by $2t - 1$, where t is its number of non-zero terms (here all roots are counted with multiplicities, except 0 which is counted at most once).

There are not yet natural generalizations of Descartes' Rule of Signs for the multivariate setting, but a lot of work has been and is being done for estimating the number of real isolated or non-degenerated roots (that is where the Jacobian does not vanish, condition that implies that the root is isolated) of multivariate square systems of real polynomials in the positive orthant, in terms of the number of variables and the number of non-zero terms that the system involves.

The main result in that direction is due to A. Khovanskii [4]. A simple version of it implies that a square system of n real polynomial equations in n indeterminates, which involves in total t non-zero terms has at most $(n+1)^t 2^{t(t-1)/2}$ non-degenerated real roots in the positive orthant. Improvements of Khovanskii's bound have afterwards been obtained by D. Perrucci [11] and T.Y. Li, J.M. Rojas and X. Wang [9], but for general systems the exponential dependence on the number of non-zero terms t can not be avoided yet.

In [9], T.Y. Li, M. Rojas and X. Wang studied particular cases of bivariate square systems and showed that the number of common isolated or non-degenerated roots of a trinomial and a polynomial with at most t non-zero terms, $t \geq 3$, is bounded by $2^t - 2$.

*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina. Research supported by grants CONICET PIP 2461/01 and UBACYT X-112. mavendar@dm.uba.ar

Furthermore, Kushnirenko's Conjecture, formulated in the mid-1970' (which says that a square system of n real polynomial equations in n indeterminates such that the k -th polynomial has t_k non-zero terms should have at most $(t_1 - 1) \cdots (t_n - 1)$ non-degenerate roots in the positive orthant) turned out to be false, by the counter-example provided by B. Haas in 2002 for a system of two trinomials in two variables [8].

The main result of this article is a refinement of the previous result for the particular case when the trinomial is a linear polynomial. Without loss of generality we can assume the linear polynomial is of the form $y - ax - b$ and we thus study the possible number of real roots of a bivariate polynomial on a line $y = ax + b$:

Theorem 1.1 *Let $f = \sum_{i=1}^t a_i x^{\alpha_i} y^{\beta_i} \in \mathbb{R}[x, y]$ be a polynomial with at most t non-zero terms, and let $a, b \in \mathbb{R}$. Set $g(x) = f(x, ax + b)$. Then either $g \equiv 0$ or g has at most $6t - 4$ real roots, counted with multiplicities except for the possible roots 0 and $-b/a$ that are counted at most once.*

To our knowledge, this is the first time a non-exponential bound is achieved, even for systems of very particular shape like this one. The tools we use are completely elementary, and we are now studying the possibility of extending the results for more general systems.

As a consequence of our result we derive an alternative algorithm for checking if a given linear form $y - ax - b$ divides a polynomial f in $K[x, y]$, where K is a real number field. The number of bit operations performed by the algorithm is polynomial in the degree $[K : \mathbb{Q}]$ of the field extension, in the number t of non-zero terms of f , in the logarithm of the degree of f and in the logarithmic height of a, b and f .

The first algorithm for this purpose can be deduced from a more general result by E. Kaltofen and P. Koiran [10]. They showed a polynomial-time algorithm for computing all linear factors of a sparse bivariate polynomial. This result has been further generalized in [12] and [13] to an algorithm that computes all the small degree factors of bi- and multi-variate sparse polynomials. All these algorithms use a version of the "gap theorem" introduced by F. Cucker, P. Koiran and S. Smale [5]. Instead of it, we reduce the problem to the univariate case by considering specializations $f(x, x^n)$ for small values of n .

Acknowledgements. The author thanks Teresa Krick for helping him to write this paper and Daniel Perruci for reading an earlier version of this work and for several useful discussions on fewnomial systems. Thanks also to J. Maurice Rojas and Frank Sottile for useful discussions.

2 Proof of Theorem 1.1

Definition 2.1 *Let $f = \sum_{i=0}^d a_i x^i \in \mathbb{R}[x]$ be a non-zero polynomial. We note by $V(f)$ the number of changes of signs in the ordered vector (a_d, \dots, a_0) of the coefficients of f , disregarding the zeroes. We also set $V(0) = -2$.*

Next result is our crucial ingredient in the proof of Theorem 1.1:

Lemma 2.2 *Let $f \in \mathbb{R}[x]$. Then $V((x + 1)f) \leq V(f)$.*

Proof : We proceed by induction in the number t of non-zero terms of f . The theorem is trivial for $t = 0$ and $t = 1$. Now let us suppose that it holds for all $t \leq n$. Let $f \in \mathbb{R}[x]$ with $n + 1$ non-zero monomials.

$$f = \sum_{i=1}^{n+1} a_i x^{\alpha_i} \text{ where } a_i \neq 0 \text{ for all } i \text{ and } 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n+1} = d = \deg(f)$$

Let $g = \sum_{i=1}^n a_i x^{\alpha_i}$. By inductive hypothesis we have $V((x+1)g) \leq V(g)$. First, we consider the case $\alpha_n < d-1$, i.e. when the terms of $(x+1)g$ do not overlap with those of $a_{n+1}x^d(x+1)$. There are two possibilities: if $a_n a_{n+1} > 0$, then $V((x+1)f) = V((x+1)g) \leq V(g) = V(f)$, and if $a_n a_{n+1} < 0$, then $V((x+1)f) = V((x+1)g) + 1 \leq V(g) + 1 = V(f)$. In both cases we have $V((x+1)f) \leq V(f)$. Now it only remains the case $\alpha_n = d-1$. Here $(x+1)f$ and $(x+1)g$ only differ in their terms of degree d and $d+1$, as shown in the following table.

	x^d	x^{d+1}
$(x+1)g$	a_n	0
$(x+1)f$	$a_n + a_{n+1}$	a_{n+1}

If $a_n a_{n+1} > 0$, then $V(f) = V(g)$, and according to the table, we have $V((x+1)f) = V((x+1)g)$. Therefore $V((x+1)f) \leq V(f)$. On the other hand, if $a_n a_{n+1} < 0$, then $V(f) = V(g) + 1$, but we have three different possibilities for the table, depending whether $|a_n|$ is greater, equal or less than $|a_{n+1}|$. Set $s = \text{sgn}(a_n)$.

	x^d	x^{d+1}
$(x+1)g$	s	0
$(x+1)f$	s	$-s$

Case $|a_n| > |a_{n+1}|$

	x^d	x^{d+1}
$(x+1)g$	s	0
$(x+1)f$	0	$-s$

Case $|a_n| = |a_{n+1}|$

	x^d	x^{d+1}
$(x+1)g$	s	0
$(x+1)f$	$-s$	$-s$

Case $|a_n| < |a_{n+1}|$

The tables above show that $V((x+1)f) \leq V((x+1)g) + 1$ for each of the three cases. Using the inductive hypothesis and $V(f) = V(g) + 1$, we conclude that $V((x+1)f) \leq V(f)$. \square

Remark 2.3 Let $f, g \in \mathbb{R}[x]$ and suppose that g has t terms. Then $V(f+g) \leq V(f) + 2t$.

Note that the value of $V(0)$ is not relevant for theorem 2.2. The only reason for setting $V(0) = -2$ is the previous remark.

Proposition 2.4 Let $f \in \mathbb{R}[x, y]$ be a polynomial with t non-zero terms. Then

$$V(f(x, x+1)) \leq 2t - 2.$$

Proof : We write $f = \sum_{i=1}^n a_i(x)y^{\alpha_i}$, where $0 \leq \alpha_1 < \dots < \alpha_n$ and $a_i(x) \in \mathbb{R}[x]$, and we set $t_i > 0$ the number of non-zero terms of a_i . It is clear that $t = t_1 + \dots + t_n$.

We define $f_k = \sum_{i=k}^n a_i(x)y^{\alpha_i - \alpha_k}$ for $k = 1, \dots, n$ and $f_{n+1} = 0$. Lemma 2.2 and Remark 2.3 imply that the polynomials f_k satisfy:

- $f_{n+1} = 0 \implies V(f_{n+1}(x, x+1)) = -2$
- $f_k = y^{\alpha_{k+1} - \alpha_k} f_{k+1} + a_k(x) \implies f_k(x, x+1) = (x+1)^{\alpha_{k+1} - \alpha_k} f_{k+1}(x, x+1) + a_k(x) \implies V(f_k(x, x+1)) \leq V(f_{k+1}(x, x+1)) + 2t_k$
- $f = y^{\alpha_1} f_1 \implies f(x, x+1) = (x+1)^{\alpha_1} f_1(x, x+1) \implies V(f(x, x+1)) \leq V(f_1(x, x+1))$.

Thus, we conclude that $V(f(x, x+1)) \leq -2 + 2(t_1 + \dots + t_n) = 2t - 2$. \square

Before finishing the proof of Theorem 1.1 we recall Descartes' Rule of Signs:

Theorem 2.5 (Descartes' rule of signs) Let $f \in \mathbb{R}[x]$ be a non-zero polynomial. Then f has at most $V(f)$ positive roots counted with multiplicities.

Proof of Theorem 1.1: If $a = 0$ or $b = 0$, then $g \in \mathbb{R}[x]$ is a polynomial with at most t non-zero terms. Descartes' rule of signs implies that, either $g \equiv 0$ or g has at most $2t - 1 \leq 6t - 4$ real roots (counted with multiplicities except for the possible root 0). In the case $a \neq 0$ and $b \neq 0$, the real roots of $f(x, ax + b)$ correspond one to one to the roots of $f(bx/a, b(x + 1)) = \widehat{f}(x, x + 1)$, where $\widehat{f} = \sum_{i=1}^t a_i a^{-\alpha_i} b^{\alpha_i + \beta_i} x^{\alpha_i} y^{\beta_i}$. Since this bijection preserves the multiplicity of the roots and maps the possible roots 0 and $-b/a$ of g to the roots 0 and -1 of $\widehat{f}(x, x + 1)$, it suffices to consider the case $a = b = 1$, i.e. $g = f(x, x + 1)$. Suppose that $g \not\equiv 0$. Descartes' rule of signs and proposition 2.4 imply that the number of positive roots of g counting with multiplicities is at most $2t - 2$. On the other hand, the roots of g in $(-\infty, -1)$ correspond to the positive roots of $0 \neq g(-1 - x) = f(-1 - x, -x) = f_1(x, x + 1)$, where $f_1 = \sum_{i=1}^t a_i (-1)^{\alpha_i + \beta_i} x^{\beta_i} y^{\alpha_i}$. Therefore the number of roots (with multiplicities) of g in $(-\infty, -1)$ is also bounded by $2t - 2$. Finally, the roots of g in $(-1, 0)$ correspond to the positives roots of

$$0 \neq (x + 1)^{\deg(g)} g\left(\frac{-x}{x + 1}\right) = (x + 1)^{\deg(g)} f\left(\frac{-x}{x + 1}, \frac{1}{x + 1}\right) = f_2(x, x + 1)$$

where $f_2 = \sum_{i=1}^t a_i (-1)^{\alpha_i} x^{\alpha_i} y^{\deg(g) - \alpha_i - \beta_i}$. Therefore there are at most $2t - 2$ of such roots. Taking into account the possible roots 0 and -1 , counted each one at most once, we conclude that g has at most $6t - 4$ real roots. \square

3 Checking linear factors of a bivariate polynomial

Proposition 3.1 Let $f = \sum_{i=1}^t a_i x^{\alpha_i} y^{\beta_i} \in \mathbb{R}[x, y]$. Let $a, b \in \mathbb{R}$ such that $|b| \neq |1 - a|$. Then $y - ax - b \mid f \Leftrightarrow x^n - ax - b \mid f(x, x^n)$ for at least $6t - 3$ odd integers $n \geq 3$.

Proof : (\Leftarrow) : Let $3 \leq n_1 < n_2 < \dots < n_{6t-3}$ the $6t - 3$ odd numbers for which $x^n - ax - b \mid f(x, x^n)$. Let $w_i \in \mathbb{R}$ be a root of $x^{n_i} - ax - b$ for each $1 \leq i \leq 6t - 3$. Then $f(w_i, aw_i + b) = f(w_i, w_i^{n_i}) = 0$ for all $1 \leq i \leq 6t - 3$. This means that $f(x, ax + b)$ has at least $6t - 3$ real roots. Applying theorem 1.1 we conclude that $f(x, ax + b) \equiv 0$, or simply $y - ax - b \mid f$. It only remains to proof that $w_i \neq w_j$ for all $i \neq j$. Actually, if $x^{n_i} - ax - b$ and $x^{n_j} - ax - b$ had a common root $w = w_i = w_j \in \mathbb{R}$, then $w^{n_i - n_j} = 1$ and therefore $w = \pm 1$. This would imply that $0 = w^{n_i} - aw - b = -b \pm (1 - a)$, in contradiction with the hypothesis $|b| \neq |1 - a|$. \square

Corollary 3.2 If $f \in \mathbb{R}[x, y]$ has t non-zero terms, then there is an odd integer $3 \leq n \leq 12t - 5$ such that $f(x, x^n) \neq 0$.

Proof : Otherwise every polynomial $y - ax - b \in \mathbb{R}[x, y]$ with $|b| \neq |1 - a|$ would divide f . \square

Note that if $(a, b) \neq (0, \pm 1)$, then either $|b| \neq |1 - a|$ or $|b| \neq |1 + a|$.

Algorithm TEST

Input : A sparse polynomial $f = \sum_{i=1}^t a_i x^{\alpha_i} y^{\beta_i} \in K[x, y]$ with t monomials, encoded as a list of vectors $(a_i, \alpha_i, \beta_i) \in K \times \mathbb{N}_0 \times \mathbb{N}_0$ representing the monomials of f , and two numbers $a, b \in K$.

Output : True or False depending whether $y - ax - b \mid f(x, y)$ or not.

Step 1 : If $(a, b) = (0, \pm 1)$, compute $f(x, b)$. If this polynomial is zero, return True. Otherwise return False.

Step 2 : If $|b| = |1 - a|$ then replace f by $f(-x, y)$ and a by $-a$.

Step 3 : For $n = 3, 5, 7, \dots, 12t - 5$ do

Step 3.1 : If $f(x, x^n) \neq 0$ then

Step 3.1.1 : Compute all the irreducible factors (with multiplicities) of $x^n - ax - b$ in $K[x]$ using a univariate dense factorization algorithm.

Step 3.1.2 : Compute all the irreducible factors (with multiplicities) of $f(x, x^n)$ in $K[x]$ with degree $\leq n$ using a univariate sparse factorization algorithm.

Step 3.1.3 : If there is an irreducible factor in the first list that either does not belong to the second list or belongs but with less multiplicity, then return False.

Step 4 : Return True.

The correctness of the algorithm is a consequence of the previous results. In order to estimate its complexity, we first state the following two famous results on the factorization of polynomials of univariate polynomials.

DenseFactor Given $f \in K[x]$ of degree d and absolute height H , it is possible to compute all its irreducible factors in $K[x]$ with multiplicities in $[d[K : \mathbb{Q}] \log H]^{O(1)}$ bit operations (see [1] for the rational case and [2] for the general case).

SparseFactor Given $f \in K[x]$ a sparse polynomial of degree d , with at most t monomials and absolute height H , it is possible to find all its irreducible factors (with multiplicities) in $K[x]$ of degree bounded by s in $[ts[K : \mathbb{Q}] \log d \log H]^{O(1)}$ bit operations (see [6]).

The complexity of algorithm TEST is clearly dominated by its main loop (step 3), where it performs $6t-3$ calls to DenseFactor and SparseFactor to factorize $x^n - ax - b$ completely and find all the factors of degree bounded by n of $f(x, x^n)$. We have that $\deg(x^n - ax - b) = n \leq 12t - 5$ and $H(x^n - ax - b) \leq H(a)H(b)$, therefore step 3.1.1 requires at most $[(6t - 3)(12t - 5)[K : \mathbb{Q}] \log(H(a)H(b))]^{O(1)}$ bit operations. On the other hand, we have that $f(x, x^n)$ is a sparse polynomial with at most t non-zero terms, of degree bounded by $nd \leq (12t - 5)d$ and absolute height bounded by $(2H)^t$ because the coefficients of $f(x, x^n)$ are sums of at most t coefficients of f . Thus, step 3.1.2 requires no more than $[(6t - 3)t(12t - 5)[K : \mathbb{Q}] \log(d(12t - 5)) \log(2H)^t]^{O(1)}$ bit operations. This proves that the total number of bit operations performed by the algorithm is polynomial in t , $\log(d)$, $[K : \mathbb{Q}]$ and $\log(H(a)H(b)H)$.

References

- [1] A.K. Lenstra, H.W. Lenstra, L. Lovasz: Factoring polynomials with rational coefficients. Math. Ann. 261 (1982), 515-534.
- [2] S. Landau: Factoring polynomials over algebraic number fields. SIAM J. Comput. 14 (1985), 184-195.
- [3] A.K. Lenstra: Factoring multivariate polynomials over algebraic number fields. SIAM J. Comput. Vol. 16 No. 3 (1987), 591-598.
- [4] A. Khovanskii: Fewnomials. AMS press, Providence, Rhode Island (1991).
- [5] F. Cucker, P. Koiran, S. Smale: A polynomial time algorithm for diophantine equations in one variable. JSC Vol. 27 No. 1 (1999), 21-29.

- [6] H.W. Lenstra: Finding small degree factors of lacunary polynomials. Number theory in progress, Vol. 1 (1999), 267–276.
- [7] H.W. Lenstra: On the factorization of lacunary polynomials. Number theory in progress, Vol. 1 (1999), 277–291.
- [8] B. Haas: A simple counter-example to Koušnirenko’s conjecture. Beiträge zur Algebra und Geometrie, Vol. 43 No. 1 (2002), 1–8.
- [9] T.Y. Li, J.M. Rojas, X. Wang: Counting real connected components of trinomial curves intersections and m -nomial hypersurfaces. Discrete and computational geometry, Vol. 30 No. 3 (2003), 379–414.
- [10] E. Kaltofen, P. Koiran: On the complexity of factoring bivariate supersparse (lacunary) polynomials. ISSAC (2005).
- [11] D. Perruci: Some bounds for the number of connected components of real zero sets of sparse polynomials. Discrete and computational geometry, Vol. 34 No. 3 (2005), 475–495.
- [12] M. Avendaño, T. Krick, M. Sombra: Factoring bivariate lacunary polynomials. Journal of Complexity (2006).
- [13] E. Kaltofen, P. Koiran: Finding small degree factors of multivariate supersparse (lacunary) polynomials over algebraic number fields. ISSAC (2006).