Valuation of Mortgage-Backed Securities: A Closed-Form Approximation

RADON Workshop on Financial and Actuarial Mathematics for Young Researchers, Linz, 30/31-May-2007

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Outline

1. Introduction and motivation
2. The closed-form valuation formula
3. Parameter estimation and model calibration
4. Some model performance results
Introduction and motivation

What is a pass-through agency-MBS?

• Pass-through agency-MBS allow the investors to receive cash flows (principal and interest, less servicing and guarantee fee) from a pool of mortgages accumulated by a mortgage-agency (GNMA, FNMA, Freddie Mac).

• The agency provides a guarantee of payment to the investor.
Introduction and motivation

- Standard residential MBS in the US feature full prepayment flexibility.

- Thus, an investor in MBS is short a prepayment option.

- Careful modelling of homeowners’ prepayment behaviour is crucial for valuation and risk management of MBS.
Introduction and motivation
Categories of prepayment models

- Econometric approaches: Cash-Flow projection based on empirical prepayment models calibrated to historical data (e.g., Schwartz & Torous (1989))

- Option-based approaches: Based on the theory of the valuation of callable bonds (e.g., Stanton (1995), Kalotay, Young & Fabozzi (2004), Sharp, Newton & Duck (2006))

- Reduced-form approaches: Based on intensity models as used in credit risk modelling (e.g., Kau, Keenan & Smurov (2004), Goncharov (2005))
Introduction and motivation

Within a **reduced-form setting**, we develop a closed-form approximation formula for fixed-rate agency-MBS:

⇒ Reduces the computational burden of MBS valuation drastically

⇒ Useful for applications in risk and portfolio management
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The model set-up
Components of prepayment

- **Refinancing-related prepayment** (→ function of the spread between weighted average coupon and long-term - 10yr - interest rates)

\[ p_{\text{refi}}(t) = \beta \cdot \max(\min(\text{WAC} - R_{10}(t), \alpha), 0) \]

- **Baseline prepayment** (→ fit the second factor to macroeconomic data)

\[ dp_0(t) = (\theta_p + b_{pw} w(t) - a_p p_0(t))dt + \sigma_p d\tilde{W}_p(t) \]
\[ dw(t) = (\theta_w - a_w w(t))dt + \sigma_w d\tilde{W}_w(t) \]

- The overall prepayment intensity/speed

\[ p(t) = p_{\text{refi}}(t) + p_0(t) \]
The general valuation formula

Applying (discretised) results from the credit risk literature:

→ **Price of the MBS** at time 0

\[
V(0) = E_{\tilde{Q}} \left[ M \cdot \Delta t_1 \cdot e^{-\int_0^{t_1} r(s)ds} + A(t_1) \cdot p(t_1) \cdot \Delta t_1 \cdot e^{-\int_0^{t_1} (r(s)+p(s))ds} \right] \\
+ E_{\tilde{Q}} \left[ \sum_{k=2}^{K} (M \cdot \Delta t_k + A(t_k) \cdot p(t_k) \cdot \Delta t_k) \cdot e^{-\int_0^{t_k} (r(s)+p(s))ds} \right],
\]

where

- \( r(t) \): (non-defaultable) short-rate process (we use a 1-factor CIR model)
- \( p(t) \): (annualised) prepayment intensity/prepayment speed process
- \( M \): mortgage payment
- \( A(t_k) \): outstanding notional amount according to amortisation schedule
- \( \tilde{Q} \): risk-neutral pricing measure
The closed-form valuation formula

- $M$, $A(t_k)$ are deterministic and known
- We have to calculate:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\tilde{Q}} \left[ e^{-\int_0^{t_k} p_0(s) ds} \right]$</td>
<td>can directly be calculated using results from interest-rate theory</td>
</tr>
<tr>
<td>$E_{\tilde{Q}} \left[ p_0(t_k) \cdot e^{-\int_0^{t_k} p_0(s) ds} \right]$</td>
<td></td>
</tr>
<tr>
<td>$E_{\tilde{Q}} \left[ e^{-\int_0^{t_k} (r(s) + \text{refi}(s)) ds} \right]$</td>
<td>see next slides</td>
</tr>
<tr>
<td>$E_{\tilde{Q}} \left[ p_{\text{refi}}(t_k) \cdot e^{-\int_0^{t_k} (r(s) + \text{refi}(s)) ds} \right]$</td>
<td>follows from the previous expectation</td>
</tr>
</tbody>
</table>
The closed-form valuation formula

The calculation of $E_{\tilde{Q}} \left[ e^{-\int_0^{t_k} (r(s)+p_{\text{refi}}(s)) ds} \right]$ and $E_{\tilde{Q}} \left[ p_{\text{refi}}(t_k) \cdot e^{-\int_0^{t_k} (r(s)+p_{\text{refi}}(s)) ds} \right]$ can basically be reduced to calculating (with some approximation of the form $e^x \approx 1 + x$ for small $x$):

$$E_{\tilde{Q}} \left[ e^{-\int_0^{t_k} (r(s)+p_{\text{refi}}(s)) ds} \right] \approx C(t_k) \cdot E_{\tilde{Q}} \left[ e^{-\int_0^{t_k} c^* \cdot r(s) ds} \right]$$

$$-C(t_k) \cdot E_{\tilde{Q}} \left[ \max (r(t_k) - r_{\text{Cap}}, 0) \right]$$

$$+C(t_k) \cdot E_{\tilde{Q}} \left[ \max (r_{\text{Floor}} - r(t_k), 0) \right]$$

for some constants $C(t_k), c^*, r_{\text{Cap}}, r_{\text{Floor}}$. 
The closed-form valuation formula

- \( E_{\tilde{Q}} \left[ e^{-\int_0^{t_k} c^* \cdot r(s) ds} \right] \) can be calculated by adjusting the CIR bond-pricing formulas

- Moreover:

\[
E_{\tilde{Q}} \left[ \max (r(t_k) - r_{Cap}, 0) \right] =: \text{Caplet}(r, 0, t_k, r_{Cap}, \Delta t)
\]

\[
E_{\tilde{Q}} \left[ \max (r_{Floor} - r(t_k), 0) \right] =: \text{Floorlet}(r, 0, t_k, r_{Floor}, \Delta t),
\]

where
The closed-form valuation formula

\[ \hat{\text{Caplet}}(r, 0, T, r_X, \Delta t) := \Delta t \cdot \left[ \frac{q + 1 + u_k}{c_k} - \frac{u_k}{c_k} \cdot \chi^2(2c_kr_X, 2q + 6, 2u_k) \right. \\
- \left. \frac{q + 1}{c_k} \cdot \chi^2(2c_kr_X, 2q + 4, 2u_k) - r_X + r_X \cdot \chi^2(2c_kr_X, 2q + 2, 2u_k) \right] \]

\[ \hat{\text{Floorlet}}(r, 0, T, r_X, \Delta t) := \Delta t \cdot \left[ r_X \cdot \chi^2(2c_kr_X, 2q + 2, 2u_k) \right. \\
- \left. \frac{u_k}{c_k} \cdot \chi^2(2c_kr_X, 2q + 6, 2u_k) - \frac{q + 1}{c_k} \cdot \chi^2(2c_kr_X, 2q + 4, 2u_k) \right] \]

\[ c_k := \frac{2\hat{a}_r}{\sigma_r^2 \cdot (1 - e^{-\hat{a}_r \cdot k \cdot \Delta t})} \]

\[ u_k := c_k \cdot r(0) \cdot e^{-\hat{a}_r \cdot k \cdot \Delta t} \]

\[ q := \frac{2\theta_r}{\sigma_r^2} - 1 \]
The closed-form valuation formula

Putting everything together:

→ **Price of the MBS** at time 0:

\[
V(0) \approx S_1 + S_2 + S_3 - \Delta_1 + \Delta_2
\]

where, basically,

- \( S_1, S_2, S_3 \): reflect the terms coming from a purely linear interest-rate/refinancing-prepayment relationship
- \( \Delta_1, \Delta_2 \): are correction terms from the S-curve approximation
Outline

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3. Parameter estimation and model calibration

4. Some model performance results
Parameter estimation and model calibration

- CIR model parameters: Approximate Kalman filter with historically observed treasury rates

- Prepayment model parameters: Kalman filter with historically observed prepayment speeds

  - Measurement equation

  \[
  \begin{pmatrix}
  p_1(t_k) \\
  \vdots \\
  p_N(t_k)
  \end{pmatrix}
  =
  \begin{pmatrix}
  p_{1,\text{refi}}(t_k) \\
  \vdots \\
  p_{N,\text{refi}}(t_k)
  \end{pmatrix}
  +
  \begin{pmatrix}
  1 \\
  \vdots \\
  1
  \end{pmatrix}
  \cdot p_0(t_k) + \epsilon_k,
  \epsilon_k \sim N_N \left(0, h_p^2 \cdot I_N\right)
  \]

  - Transition equation

  \[
  p_0(t_{k+1}) = e^{-a_p \Delta t_{k+1}} \cdot p_0(t_k) + \frac{\theta_p + b_{pw}w(t_k)}{a_p} \cdot (1 - e^{-a_p \Delta t_{k+1}}) + \eta_{k+1},
  \eta_{k+1} \sim N_1 \left(0, \frac{\sigma_p^2}{2a_p} (1 - e^{-2a_p \Delta t_{k+1}})\right)
  \]
Parameter estimation and model calibration
Prepayment-risk-adjustment parameters

Under the pricing measure \( \tilde{Q} \) (\( \rightarrow \) Girsanov theorem):

\[
\begin{align*}
dp_0(t) &= (\theta_p + b_{pw}w(t) - \hat{a}_p p_0(t)) dt + \sigma_p d\tilde{W}_p(t), \\
dw(t) &= (\theta_w - \hat{a}_w w(t)) dt + \sigma_w d\tilde{W}_w(t),
\end{align*}
\]

where \( \hat{a}_i := a_i + \lambda_i \sigma_i^2 \), \( i = p, w \) and

\[
dp(t) = \mu \cdot (dp_{\text{refi}}(t) + dp_0(t)).
\]

\( \rightarrow \) clearly identifiable **prepayment-risk adjustment parameters** \( \mu, \lambda_p, \lambda_w \)
(accounting for two types of prepayment risk: turnover overstatement for discounts, refinancing understatement for premiums)

\( \rightarrow \) Calibrate prepayment-risk adjustment parameters to market prices of GNMA TBA securities (across different coupon levels).
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Some empirical/model-performance results
Historical market and model prices

Market and model prices for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006 when the prepayment-risk adjustment parameters are recalibrated once a year.
Some empirical/model-performance results
Effective duration and effective convexity for some benchmark securities

Modelling results for a series of 8 generic GNMA 30yr TBAs (Bloomberg ticker GNSF) on 12-Dec-2005
Some references

Thank you for your attention.