Integro-Differential Operators as an Ore Algebra

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The simplest boundary problem

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Green's operator as integro-differential operator:

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- Integro(-differential) Weyl algebra as skew polynomials
- Algebraic properties
- Localization
- Fixing the integration constant gives integro-differential operators over $K[x]$

What is an Integro-Differential Algebra?

Definition

$(\mathcal{F}, \partial, \int)$ is an *integro-differential algebra* if $(\mathcal{F}, \partial)$ is a differential algebra such that $\partial \int = 1$ and the *differential Baxter axiom*

$$(\textstyle\int f')(\int g') = (\int f')g + f(\int g') - \int (fg)'$$

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"Integration gives one evaluation for free"

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We have an action $\bullet \colon \mathcal{F}[\partial, \int] \times \mathcal{F} \to \mathcal{F}$ given by:

$$\partial \colon \mathcal{F} \to \mathcal{F} \qquad \textstyle\int \colon \mathcal{F} \to \mathcal{F} \qquad \mathrm{E} = 1 - \textstyle\int \circ \partial$$

Algebra Structure

Think of the prototype models $\mathcal{F} = C^\infty(\mathbb{R})$ or $\mathcal{F} = K[x]$.

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$\mathtt{E}^2$	$=$	$\mathtt{E}$	$\partial \int$	$=$	1
$\mathtt{E}f$	$=$	$(\mathtt{E} \bullet f)\,\mathtt{E}$	$\partial\mathtt{E},\ \mathtt{E}\int$	$=$	0
$\int f \int$	$=$	$(\int \bullet f) \int - \int (\int \bullet f)$			
$\int f \partial$	$=$	$f - \int (\partial \bullet f) - (\mathtt{E} \bullet f)\,\mathtt{E}$			
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Subalgebras $\mathcal{F}[\partial], \mathcal{F}[\int], \mathcal{F}[\mathtt{E}] \leq \mathcal{F}[\partial, \int]$, morover $\mathcal{F}[\mathtt{E}] = (\mathtt{E})$

Algebra Structure

Think of the prototype models $\mathcal{F} = C^\infty(\mathbb{R})$ or $\mathcal{F} = K[x]$.

$$\partial \bullet f = \tfrac{df}{dx} \qquad \textstyle\int \bullet f = \int_c^x f(x)\,dx \qquad \mathtt{E} \bullet f = f(c)$$

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\hline
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$$\boxed{\mathcal{F}[\partial, \textstyle\int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\int] \dotplus (\mathtt{E})}$$

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"Operators as coefficients"

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$$\longrightarrow K[x][\textstyle\int] \cong \mathrm{A}_1(\ell).$$

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Right inverses in rings, approach based on representation theory

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$K\langle\partial,\ell\rangle$ is not Noetherian!

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$$\ell e_{ij} = e_{i+1,j} \quad \text{and} \quad \partial e_{ij} = e_{i-1,j}, \quad \partial e_{0j} = 0$$

Decomposition

$$\boxed{K\langle\partial,\ell\rangle = K[\partial] \dotplus K[\ell]\ell \dotplus (\mathtt{E})}$$

differential subrings (with, without unit), δ-ideal

Ideal Structure

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Every nonzero ideal in $K\langle\partial, \ell\rangle$ contains the evaluation ideal $(\mathtt{E})$. Moreover, $(\mathtt{E})$ is the only proper δ-ideal.

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$\mathrm{A}_1(\partial, \ell)$ is not simple.

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Theorem

If $\int$ is an integral operator for the standard derivation ∂ on $K[x]$, then

$$\mathrm{A}_1(\partial, \ell)/(\mathtt{E}\, x - c\, \mathtt{E}) \cong K[x][\partial, \textstyle\int]$$

with $c = \mathtt{E} \bullet x \in K$ as the constant of integration.

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Thank you.