

# A normal form for 2-dimensional linear functional systems

Alban Quadrat

INRIA Sophia Antipolis,  
2004 route des lucioles, BP 93,  
06902 Sophia Antipolis cedex, France.

`Alban.Quadrat@sophia.inria.fr`

Applications of Computer Algebra (ACA 2009)  
Montreal (25-28/06/09)

- In this talk, we study the **integration of 2-dimensional linear functional systems** (e.g., PDEs in 2 variables).
- Normal forms for 1-dimensional linear functional systems with coefficients in a skew field  $K$  (over  $K[\partial; \alpha, \beta]$ ,  $\alpha$  injective) exist:

Smith and Jacobson normal forms.

- In this talk, we develop
  - **“a normal normal form” for 2-dimensional linear systems.**
- It is more intricate than Smith and Jacobson normal forms.
- We need to use **“sophisticated algebraic tools”** such as algebraic analysis, pure modules, filtration by purity,  $\text{ext}_D^i(\text{ext}_D^i(M, D), D)$ .

- The main idea is to integrate the system in “cascade”:

$$R\eta = 0 \quad \Leftrightarrow \quad \begin{cases} S\zeta = B\tau, \\ S''\tau = Av, \\ S'v = 0, \end{cases}$$

$$S'v = 0 \quad \text{0-dimensional}, \quad S''\tau = 0 \quad \text{1-dimensional}, \\ S\zeta = 0 \quad \text{2-dimensional}.$$

- In some particular cases, we can take  $A = 0$  and  $B = 0$

$\Rightarrow$  decouple the  $i^{\text{th}}$ -dimensional parts.

# Introduction

- How can we constructively find the 0-dimensional, 1-dimensional and 2-dimensional parts of a given linear functional system?

$$\begin{cases} \partial_2^2 \theta = 0, \\ \partial_1 \partial_2 \theta = 0. \end{cases}$$

- $z_1 = \partial_1 \theta$  satisfies  $\partial_2 z_1 = 0$ , i.e.,  $z_1 = f(x_1)$  : 1-dimensional.
- $z_2 = \partial_2 \theta$  satisfies:

$$\begin{cases} \partial_2 z_2 = 0, \\ \partial_1 z_2 = 0, \end{cases}$$

i.e.,  $z_2 = c$  is an arbitrary constant: 0-dimensional.

$$\Rightarrow \partial_2 \theta = c \Rightarrow \theta = c x_2 + f(x_1).$$

- 2-dimensional (underdetermined systems):

$$\partial_1 \sigma_1 + \partial_2 \sigma_2 = 0 \Rightarrow \sigma_1 = - \int_0^{x_1} \partial_2 \sigma_2(x_1, x_2) dx_1 + \phi(x_2).$$

# Notations

- Let  $D$  be a noetherian noncommutative polynomial ring.
- Let  $R \in D^{q \times p}$  be a  $q \times p$ -matrix with coefficients in  $D$ .
- $D^{1 \times q} = \{\lambda = (\lambda_1 \dots \lambda_q) \mid \lambda_i \in D, i = 1, \dots, q\}$ .
- $D^q = \{\eta = (\eta_1 \dots \eta_q)^T \mid \eta_i \in D, i = 1, \dots, q\}$ .
- Let us consider the **left** (resp., **right**)  $D$ -homomorphism:

$$\begin{array}{ccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & & D^p & \xrightarrow{R \cdot} & D^q \\ \lambda & \longmapsto & \lambda R, & & \eta & \longmapsto & R \eta. \end{array}$$

- Let us consider the left  $D$ -module  $M = D^{1 \times p} / (D^{1 \times q} R)$ .

$$0 \longrightarrow \ker_D(\cdot R) \longrightarrow D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0.$$

$$\forall \lambda \in \ker_D(\cdot R) : R \eta = \zeta \Rightarrow \lambda \zeta = (\lambda R) \eta = 0.$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\text{gld}(D) = 2 \quad \Rightarrow \quad \ker_D(\cdot R) = D^{1 \times r} R_2, \quad \ker_D(\cdot R_2) = 0,$$

i.e., the inhomogeneous linear system

$$R \eta = \zeta,$$

admits a **set of left  $D$ -linearly independent compatibility conditions**.

$$D = k[\partial_1, \partial_2], \quad A_2(k) = k[x_1, x_2][\partial_1, \partial_2], \quad B_2(k) = k(x_1, x_2)[\partial_1, \partial_2].$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{\cdot R} D^p$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{R \cdot} D^p \xleftarrow{Q \cdot} D^m$$

$$\ker_D(R \cdot) = Q D^m$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{R \cdot} D^p \xleftarrow{Q \cdot} D^m$$

$$D^r \xleftarrow{R_2 \cdot} D^q$$

$$\text{ext}_D^2(M, D) = \text{coker}_D(R_2 \cdot) = D^r / (R_2 D^q)$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccc} & & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{Q \cdot} & D^m \\ & & & & & & \\ D^r & \xleftarrow{R_2 \cdot} & D^q & \xleftarrow{R' \cdot} & D^{p'} & & \end{array}$$

$$\ker_D(R_2 \cdot) = R' D^{p'}$$

$$\begin{aligned} R_2 R = 0 & \Rightarrow \text{im}_D(R \cdot) = R D^p \subseteq \ker_D(R_2 \cdot) = R' D^{p'} \\ & \Rightarrow \exists R'' \in D^{p' \times p} : R = R' R'' \end{aligned}$$

$$\text{ext}_D^1(M, D) = \ker_D(R_2 \cdot) / \text{im}_D(R \cdot) = (R' D^{p'}) / (R D^p)$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccc} & & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{Q \cdot} & D^m \\ & & \parallel & & \downarrow R'' \cdot & & \\ D^r & \xleftarrow{R_2 \cdot} & D^q & \xleftarrow{R' \cdot} & D^{p'} & & \end{array}$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccccc}
 & & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{Q \cdot} & D^m \\
 & & \parallel & & \downarrow R'' \cdot & & \\
 D^r & \xleftarrow{R_2 \cdot} & D^q & \xleftarrow{R' \cdot} & D^{p'} & \xleftarrow{Q' \cdot} & D^{m'}
 \end{array}$$

$$\ker_D(R' \cdot) = Q' D^{m'}$$

# Duality & $\text{ext}_D^i(M, D)$

$$0 \longrightarrow D^{1 \times r} \xrightarrow{\cdot R_2} D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccc}
 & D^q & \xleftarrow{R \cdot} & D^p & \xleftarrow{Q \cdot} & D^m \\
 & \parallel & & \downarrow R'' \cdot & & \downarrow Q'' \cdot \\
 D^r & \xleftarrow{R_2 \cdot} & D^q & \xleftarrow{R' \cdot} & D^{p'} & \xleftarrow{Q' \cdot} & D^{m'}
 \end{array}$$

$$R' R'' Q = R Q = 0$$

$$\Rightarrow \text{im}_D((R'' Q) \cdot) = (R'' Q) D^p \subseteq \ker_D(R' \cdot) = Q' D^{m'}$$

$$\Rightarrow \exists Q'' \in D^{m \times m'} : R'' Q = Q' Q''.$$

# Biduality & $\text{ext}_D^j(\text{ext}_D^i(M, D), D)$

$$\begin{array}{ccc} D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} \\ \uparrow \cdot R'' & & \uparrow \cdot Q'' \\ D^{1 \times p'} & \xrightarrow{\cdot Q'} & D^{1 \times m'} \end{array}$$

$$R'' Q = Q' Q''$$

# Biduality & $\text{ext}_D^j(\text{ext}_D^i(M, D), D)$

$$\begin{array}{ccccc} D^{1 \times t} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} \\ \uparrow \cdot S'' & & \uparrow \cdot R'' & & \uparrow \cdot Q'' \\ D^{1 \times t'} & \xrightarrow{\cdot S'} & D^{1 \times p'} & \xrightarrow{\cdot Q'} & D^{1 \times m'} \end{array}$$

$$\ker_D(\cdot Q) = D^{1 \times t} S, \quad \ker_D(\cdot Q') = D^{1 \times t'} S', \quad S'' S = S' R''$$

# Biduality & $\text{ext}_D^j(\text{ext}_D^i(M, D), D)$

$$\begin{array}{ccccccc}
 D^{1 \times u} & \xrightarrow{\cdot T} & D^{1 \times t} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\cdot Q} & D^{1 \times m} \\
 \uparrow \cdot T'' & & \uparrow \cdot S'' & & \uparrow \cdot R'' & & \uparrow \cdot Q'' \\
 D^{1 \times u'} & \xrightarrow{\cdot T'} & D^{1 \times t'} & \xrightarrow{\cdot S'} & D^{1 \times p'} & \xrightarrow{\cdot Q'} & D^{1 \times m'}
 \end{array}$$

$$\ker_D(\cdot S) = D^{1 \times u} T, \quad \ker_D(\cdot S') = D^{1 \times u'} T', \quad T'' T = T' S''.$$

$$\begin{aligned}
 R Q = 0 &\Rightarrow \text{im}_D(\cdot R) = D^{1 \times q} R \subseteq \ker_D(\cdot Q) = D^{1 \times t} S \\
 &\Rightarrow \exists F \in D^{q \times t} : R = F S.
 \end{aligned}$$

$$\begin{aligned}
 R' Q' = 0 &\Rightarrow \text{im}_D(\cdot R') = D^{1 \times q} R' \subseteq \ker_D(\cdot Q') = D^{1 \times t'} S' \\
 &\Rightarrow \exists F' \in D^{q \times t'} : R' = F' S'.
 \end{aligned}$$

# Biduality & $\text{ext}_D^i(\text{ext}_D^i(M, D), D)$

- $t(M) = \{m \in M \mid \exists 0 \neq d \in D : dm = 0\}$  submodule of  $M$ .

$$\text{ext}_D^1(\text{ext}_D^1(M, D), D) \cong D^{1 \times t} / \left( D^{1 \times (t'+u)} \begin{pmatrix} S'' \\ T \end{pmatrix} \right),$$

$$\text{ext}_D^2(\text{ext}_D^2(M, D), D) \cong D^{1 \times t'} / \left( D^{1 \times (q+u')} \begin{pmatrix} F' \\ T' \end{pmatrix} \right),$$

$$t(M) \cong D^{1 \times t} / \left( D^{1 \times (q+u)} \begin{pmatrix} F \\ T \end{pmatrix} \right),$$

$$M/t(M) \cong D^{1 \times p} / (D^{1 \times t} S).$$

- We have the following short exact sequences ( $\text{gl}(D) = 2$ ):

$$0 \longrightarrow t(M) \xrightarrow{\iota} M \xrightarrow{p} M/t(M) \longrightarrow 0,$$

$$0 \longrightarrow \text{ext}_D^2(\text{ext}_D^2(M, D), D) \xrightarrow{\alpha} t(M) \xrightarrow{\beta} \text{ext}_D^1(\text{ext}_D^1(M, D), D) \longrightarrow 0.$$

- For  $D = k[x_1, x_2]$ ,  $k[\partial_1, \partial_2]$ ,  $B_2(k) = k(x_1, x_2)[\partial_1, \partial_2]$  ( $k$  field):

$$\left\{ \begin{array}{l} \dim_D(\text{ext}_D^2(\text{ext}_D^2(M, D), D)) = 0, \\ \dim_D(\text{ext}_D^1(\text{ext}_D^1(M, D), D)) = 1, \\ \dim_D(M/t(M)) = 2. \end{array} \right.$$

- For  $D = A_2(k) = k[x_1, x_2][\partial_1, \partial_2]$ ,  $A[\partial_1, \partial_2]$ , where  $A = k[[x_1, x_2]]$  or  $\mathbb{C}\{x_1, x_2\}$  and  $k$  a field of characteristic 0:

$$\left\{ \begin{array}{l} \dim_D(\text{ext}_D^2(\text{ext}_D^2(M, D), D)) = 2, \\ \dim_D(\text{ext}_D^1(\text{ext}_D^1(M, D), D)) = 3, \\ \dim_D(M/t(M)) = 4. \end{array} \right.$$

# First main theorem

- Theorem: We have  $t(M) \cong D^{1 \times (t+t')} / (D^{1 \times (t'+u+q+u')} U)$ ,

$$U = \begin{pmatrix} S'' & -A^{\natural} \\ T & -A^b \\ 0 & F' \\ 0 & T' \end{pmatrix}, \quad A^{\natural} \in D^{t' \times t'}, \quad A^b \in D^{u \times t'},$$

$$\text{with } \begin{pmatrix} A^{\natural} \\ A^b \end{pmatrix} = \begin{pmatrix} I_{t'} \\ 0 \end{pmatrix} + \begin{pmatrix} S'' \\ T \end{pmatrix} B + C \begin{pmatrix} F' \\ T' \end{pmatrix} \in D^{(t'+u) \times t'},$$

is any representative of the residue class of  $(I_{t'}^T \quad 0^T)^T$  in:

$$\Pi = D^{(t'+u) \times t'} / \left( \begin{pmatrix} S'' \\ T \end{pmatrix} D^{t \times t'} + D^{(t'+u) \times (q+u')} \begin{pmatrix} F' \\ T' \end{pmatrix} \right).$$

In particular, we can take  $A^{\natural} = I_{t'}$  and  $A^b = 0$ .

# First main theorem

- If there exist  $B \in D^{t \times t'}$  and  $C \in D^{(t'+u) \times (q+u')}$  such that

$$\begin{pmatrix} I_{t'} \\ 0 \end{pmatrix} + \begin{pmatrix} S'' \\ T \end{pmatrix} B + C \begin{pmatrix} F' \\ T' \end{pmatrix} = 0, \quad (\star)$$

then we can take  $A^{\natural} = 0$  and  $A^b = 0$  and get:

$$t(M) \cong \text{ext}_D^1(\text{ext}_D^1(M, D), D) \oplus \text{ext}_D^2(\text{ext}_D^2(M, D), D).$$

- If  $D = k[x_1, x_2]$  or  $D = k[\partial_1, \partial_2]$ , then  $(\star)$  is equivalent to

$$\text{row} \left( \begin{pmatrix} I_{t'} \\ 0 \end{pmatrix} \right) = -(\text{row}(B) \quad \text{row}(C)) \begin{pmatrix} \begin{pmatrix} S'' \\ T \end{pmatrix}^T \otimes I_{t'} \\ I_{t'+u} \otimes \begin{pmatrix} F' \\ T' \end{pmatrix} \end{pmatrix},$$

where  $\otimes$  denotes the Kronecker product (tensor product).

# Systemic interpretation

- **Corollary:** If we denote by  $V = (F^T \quad T^T)^T \in D^{(q+u) \times t}$  and  $\mathcal{F}$  is a left  $D$ -module, then  $t(M) \cong D^{1 \times t} / (D^{1 \times (q+u)} V)$  and

$$\begin{cases} F\theta = 0, \\ T\theta = 0, \end{cases} \Leftrightarrow \begin{cases} S''\tau - A^{\natural}v = 0, \\ T\tau - A^b v = 0, \\ F'v = 0, \\ T'v = 0, \end{cases}$$

$$\delta : \ker_{\mathcal{F}}(U.) \longrightarrow \ker_{\mathcal{F}}(V.)$$

$$\begin{pmatrix} \tau \\ v \end{pmatrix} \longmapsto \theta = \tau - Bv,$$

$$\delta^{-1} : \ker_{\mathcal{F}}(V.) \longrightarrow \ker_{\mathcal{F}}(U.)$$

$$\theta \longmapsto \begin{pmatrix} \tau \\ v \end{pmatrix} = \begin{pmatrix} \theta + BS''\theta \\ S''\theta \end{pmatrix}.$$

In particular, we can take  $A^{\natural} = I_{t'}$ ,  $A^b = 0$  and  $B = 0$ .

# Example I

- We consider  $D = \mathbb{Q}[\partial_1, \partial_2]$  and  $R = (\partial_2^2 \quad \partial_1 \partial_2)^T$ .

Let us integrate the following linear system of PDEs (*torsion*):

$$\begin{cases} \partial_2^2 \theta = 0, \\ \partial_1 \partial_2 \theta = 0. \end{cases}$$

- Denoting  $R_2 = (\partial_1 \quad -\partial_2)$ , we have the long exact sequence:

$$0 \longrightarrow D \xrightarrow{\cdot R_2} D^{1 \times 2} \xrightarrow{\cdot R} D \xrightarrow{\pi} M \longrightarrow 0.$$

- We can easily compute the different matrices

$$\begin{aligned} R' &= (\partial_2 \quad \partial_1)^T, \quad R'' = \partial_2, \quad Q = 0, \quad Q' = 0, \\ S &= 1, \quad S' = 1, \quad T = 0, \quad T' = 0, \quad F' = R', \quad S'' = R'', \end{aligned}$$

$$\Rightarrow \begin{cases} \text{ext}_D^2(\text{ext}_D^2(M, D), D) \cong D/(\partial_1, \partial_2), \\ \text{ext}_D^1(\text{ext}_D^1(M, D), D) \cong D/(D \partial_2). \end{cases}$$

## Example I

- Hence, we obtain the matrix

$$U = \begin{pmatrix} \partial_2 & -A^\natural \\ 0 & \partial_2 \\ 0 & \partial_1 \end{pmatrix},$$

where  $A^\natural = 1 + (B + C_{11})\partial_2 + C_{12}\partial_1$ , where  $B, C_{11}, C_{12} \in D$ ,

$$\begin{cases} \partial_2^2 \theta = 0, \\ \partial_1 \partial_2 \theta = 0, \end{cases} \Leftrightarrow \begin{cases} \partial_2 \tau - v = 0, \\ \partial_2 v = 0, \\ \partial_1 v = 0, \end{cases} \quad (*)$$

where  $\theta = \tau$  and  $v = \partial_2 \tau$ .

- We first integrate the last two PDEs to get  $v = c$ , where  $c$  is an arbitrary constant, and then the inhomogeneous PDE  $\partial_2 \tau = c$ ,

$$\theta = \tau = c x_2 + f(x_1),$$

where  $f$  is an arbitrary smooth function.

## Example II

- Let us consider  $D = \mathbb{Q}[\partial_1, \partial_2]$  and the matrix:

$$R = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 \\ \partial_1 & -\partial_1 & -2\partial_1 \end{pmatrix}.$$

- Denoting  $R_2 = (\partial_1 \quad -\partial_1 \quad \partial_2)$ , we have the exact sequence:

$$0 \longrightarrow D \xrightarrow{\cdot R_2} D^{1 \times 2} \xrightarrow{\cdot R} D \xrightarrow{\pi} M \longrightarrow 0.$$

## Example II

- We can easily compute the different matrices:

$$Q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad T = 0,$$

$$R' = \begin{pmatrix} 1 & 0 \\ 1 & -\partial_2 \\ 0 & -\partial_1 \end{pmatrix}, \quad Q' = 0, \quad S' = I_2, \quad T' = 0,$$

$$R'' = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 \\ -1 & 1 & 2 \end{pmatrix}, \quad S'' = \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ -1 & 1 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ \partial_2 & -\partial_1 \\ \partial_1 & -\partial_1 \end{pmatrix}, \quad F' = R'.$$

## Example II

- Hence, we get the following modules:

$$\left\{ \begin{array}{l} \text{ext}_D^1(\text{ext}_D^1(M, D), D) \cong D^{1 \times 2} / (D^{1 \times 2} S''), \\ \text{ext}_D^2(\text{ext}_D^2(M, D), D) \cong D^{1 \times 2} / (D^{1 \times 3} R'), \\ t(M) \cong D^{1 \times 2} / (D^{1 \times 3} F), \\ M/t(M) \cong D^{1 \times 3} / (D^{1 \times 2} S). \end{array} \right.$$

- In particular, we have:

$$\left\{ \begin{array}{l} \dim_D(\text{ext}_D^2(\text{ext}_D^2(M, D), D)) = 0, \\ \dim_D(\text{ext}_D^1(\text{ext}_D^1(M, D), D)) = 1, \\ \dim_D(M/t(M)) = 2. \end{array} \right.$$

## Example II

- Hence, we get the following matrix

$$U = \begin{pmatrix} 0 & \partial_2 - \partial_1 & -A_{11}^h & -A_{12}^h \\ -1 & 1 & -A_{21}^h & -A_{22}^h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\partial_2 \\ 0 & 0 & 0 & -\partial_1 \end{pmatrix},$$

where  $A_{11}^h$ ,  $A_{12}^h$ ,  $A_{21}^h$  and  $A_{22}^h$  are defined by

$$\begin{pmatrix} A_{11}^h & A_{12}^h \\ A_{21}^h & A_{22}^h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ -1 & 1 \end{pmatrix} B + C \begin{pmatrix} 1 & 0 \\ 1 & -\partial_2 \\ 0 & -\partial_1 \end{pmatrix},$$

for all  $B \in D^{2 \times 2}$  and all  $C \in D^{3 \times 3}$ .

$$B = -I_2, \quad C = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \Rightarrow A^h = 0.$$

$$\Rightarrow t(M) \cong \text{ext}_D^1(\text{ext}_D^1(M, D), D) \oplus \text{ext}_D^2(\text{ext}_D^2(M, D), D).$$

## Example II

$$\begin{cases} \partial_2 \theta_2 - \partial_1 \theta_2 = 0, \\ \partial_2 \theta_1 - \partial_1 \theta_2 = 0, \\ \partial_1 \theta_1 - \partial_1 \theta_2 = 0, \end{cases} \Leftrightarrow \begin{cases} \partial_2 \tau_2 - \partial_1 \tau_2 = 0, \\ -\tau_1 + \tau_2 = 0, \\ v_1 = 0, \\ v_1 - \partial_2 v_2 = 0, \\ -\partial_1 v_2 = 0, \end{cases}$$
$$\Leftrightarrow \begin{cases} \tau_1 = \tau_2, \\ \partial_2 \tau_2 - \partial_1 \tau_2 = 0, \\ v_1 = 0, \\ \partial_1 v_2 = 0, \\ \partial_2 v_2 = 0. \end{cases}$$

Hence,  $\tau_1 = \tau_2 = f(x_1 + x_2)$ ,  $v_1 = 0$  and  $v_2 = c$ , where  $f$  is an arbitrary smooth function and  $c$  an arbitrary constant.

## Example II

- Finally, the integration of the **sub-system** of  $R\eta = 0$

$$\begin{cases} \partial_2 \theta_2 - \partial_1 \theta_2 = 0, \\ \partial_2 \theta_1 - \partial_1 \theta_2 = 0, \\ \partial_1 \theta_1 - \partial_1 \theta_2 = 0, \end{cases}$$

corresponding to  $t(M)$ , gives the following **general solution**

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_1 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} f(x_1 + x_2) \\ f(x_1 + x_2) + c \end{pmatrix},$$

where  $f$  (resp.,  $c$ ) is an arbitrary smooth function (resp., constant).

## Second main theorem

- Theorem: We have  $M \cong D^{1 \times (p+t+t')} / (D^{1 \times (t+t'+u+q+u')} P)$ ,

$$P = \begin{pmatrix} S & -G^{\natural} & -G^b \\ 0 & S'' & -A^{\natural} \\ 0 & T & -A^b \\ 0 & 0 & F' \\ 0 & 0 & T' \end{pmatrix},$$

with  $(G^{\natural} \quad G^b) = (I_t \quad -B) + S J + K U \in D^{t \times (t+t')}$ ,

is any representative of the residue class of  $(I_t \quad -B)$  in

$$\Omega = D^{t \times (t+t')} / \left( S D^{p \times (t+t')} + D^{t \times (t'+u+q+u')} U \right).$$

In particular, we can take:

$$A^{\natural} = I_{t'}, \quad A^b = 0, \quad B = 0, \quad C = 0, \quad G^{\natural} = I_t, \quad G^b = 0.$$

## Second main theorem

- If there exist  $J \in D^{p \times (t+t')}$  and  $K \in D^{t \times (t'+u+q+u')}$  such that

$$(I_t \quad -B) + SJ + KU = 0, \quad (\star)$$

then we can take  $G^a = 0$  and  $G^b = 0$  and get:

$$M \cong t(M) \oplus M/t(M).$$

Then, we have:

$$R\eta = 0 \Leftrightarrow \eta = (Q \quad -J) \begin{pmatrix} \xi \\ \tau \\ v \end{pmatrix}, \quad U \begin{pmatrix} \tau \\ v \end{pmatrix} = 0, \quad \forall \xi \in \mathcal{F}^m.$$

- if  $D$  is a commutative ring, then  $(\star)$  is equivalent to:

$$\text{row}((I_t \quad -B)) = -(\text{row}(J) \quad \text{row}(K)) \begin{pmatrix} S^T \otimes I_{t+t'} \\ I_t \otimes U \end{pmatrix}.$$

- **Example:**  $D = k[x_1, x_2]$  or  $k[\partial_1, \partial_2]$ , where  $k$  is a field.

# Systemic interpretation

- Corollary: We have the following equivalence:

$$R\eta = 0 \quad \Leftrightarrow \quad \begin{cases} S\zeta - G^{\natural}\tau - G^b v = 0, \\ S''\tau - A^{\natural}v = 0, \\ T\tau - A^b v = 0, \\ F'v = 0, \\ T'v = 0. \end{cases}$$

In particular, we can take:

$$A^{\natural} = I_{t'}, \quad A^b = 0, \quad B = 0, \quad C = 0, \quad G^{\natural} = I_t, \quad G^b = 0.$$

# Systemic interpretation

- If  $\mathcal{F}$  is a left  $D$ -module, then we have:

$$\ker_{\mathcal{F}}(P.) \xrightarrow{\varpi} \ker_{\mathcal{F}}(R.)$$
$$\begin{pmatrix} \zeta \\ \tau \\ v \end{pmatrix} \mapsto \eta = \zeta - J \begin{pmatrix} \tau \\ v \end{pmatrix},$$

$$\ker_{\mathcal{F}}(R.) \xrightarrow{\varpi^{-1}} \ker_{\mathcal{F}}(P.)$$
$$\eta \mapsto \begin{pmatrix} \zeta \\ \tau \\ v \end{pmatrix} = \begin{pmatrix} I_p + J \begin{pmatrix} (I_t + B S'') S \\ S'' S \end{pmatrix} \\ (I_t + B S'') S \\ S'' S \end{pmatrix} \eta.$$

## Example II (continued)

$$\begin{cases} \partial_2 \eta_2 - \partial_1 \eta_2 + \partial_2 \eta_3 - \partial_1 \eta_3 = 0, \\ \partial_2 \eta_1 - \partial_1 \eta_2 - \partial_2 \eta_3 - \partial_1 \eta_3 = 0, \\ \partial_1 \eta_1 - \partial_1 \eta_2 - 2 \partial_1 \eta_3 = 0, \end{cases}$$
$$\Leftrightarrow \begin{cases} \zeta_1 - \zeta_3 - G_{11}^{\flat} \tau_1 - G_{12}^{\flat} \tau_2 - G_{11}^{\flat} v_1 - G_{12}^{\flat} v_2 = 0, \\ \zeta_2 + \zeta_3 - G_{21}^{\flat} \tau_1 - G_{22}^{\flat} \tau_2 - G_{21}^{\flat} v_1 - G_{22}^{\flat} v_2 = 0, \\ \partial_2 \tau_2 - \partial_1 \tau_2 = 0, \\ -\tau_1 + \tau_2 = 0, \\ v_1 = 0, \\ v_1 - \partial_2 v_2 = 0, \\ -\partial_1 v_2 = 0. \end{cases}$$

- We can take:

$$G_{11}^{\flat} = 1, \quad G_{12}^{\flat} = G_{21}^{\flat} = G_{22}^{\flat} = G_{11}^{\flat} = G_{12}^{\flat} = G_{21}^{\flat} = G_{22}^{\flat} = 0,$$

## Example II (continued)

- If we consider the matrices

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then  $(G^{\natural} \quad G^{\flat}) = (I_t \quad -B) + S J + K U = 0$  and:

$$M \cong \text{ext}_D^1(\text{ext}_D^1(M, D), D) \oplus \text{ext}_D^2(\text{ext}_D^2(M, D), D) \oplus M/t(M).$$

$$\begin{cases} \zeta_1 - \zeta_3 = 0, \\ \zeta_2 + \zeta_3 = 0, \\ \partial_2 \tau_2 - \partial_1 \tau_2 = 0, \\ -\tau_1 + \tau_2 = 0, \\ v_1 = 0, \\ v_1 - \partial_2 v_2 = 0, \\ -\partial_1 v_2 = 0, \end{cases} \Leftrightarrow \begin{cases} \zeta_1 = \zeta_3, \\ \zeta_2 = -\zeta_3, \\ \tau_1 = \tau_2, \\ \partial_2 \tau_2 - \partial_1 \tau_2 = 0, \\ v_1 = 0, \\ \partial_2 v_2 = 0, \\ \partial_1 v_2 = 0, \end{cases} \Leftrightarrow \begin{cases} \zeta_1 = \zeta_3, \\ \zeta_2 = -\zeta_3, \\ \tau_1 = f(x_1 + x_2), \\ \tau_2 = f(x_1 + x_2), \\ v_1 = 0, \\ v_2 = c. \end{cases}$$

## Example II (continued)

- A general solution of the following linear PD system

$$\begin{cases} \partial_2 \eta_2 - \partial_1 \eta_2 + \partial_2 \eta_3 - \partial_1 \eta_3 = 0, \\ \partial_2 \eta_1 - \partial_1 \eta_2 - \partial_2 \eta_3 - \partial_1 \eta_3 = 0, \\ \partial_1 \eta_1 - \partial_1 \eta_2 - 2 \partial_1 \eta_3 = 0, \end{cases}$$

$$\text{is } \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \zeta_3 \\ -\zeta_3 \\ \zeta_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(x_1 + x_2) \\ f(x_1 + x_2) \\ 0 \\ c \end{pmatrix}$$
$$= \begin{pmatrix} \zeta_3 \\ -\zeta_3 + 2f(x_1 + x_2) \\ \zeta_3 - f(x_1 + x_2) + c \end{pmatrix},$$

where  $\zeta_3$  (resp.,  $f$ ) is an arbitrary smooth function of  $\mathbb{R}^2$  (resp.,  $\mathbb{R}$ ) and  $c$  an arbitrary real constant.

# A few more results

- The left  $D$ -module  $M/t(M)$  is **torsion-free**. Since  $\text{gl}(D) = 2$ :
  - ①  $M/t(M)$  is either purely torsion-free,
  - ② or purely projective, i.e., stably free,
  - ③ or free.
- If  $D$  is a commutative polynomial ring with coefficients in a field  $k$ , then projective modules are free (**Quillen-Suslin theorem**).
- If  $D = A_2(k)$  or  $B_2(k)$ ,  $k$  a field of characteristic 0, then projective modules of rank at least 2 are free (**Stafford theorem**).
- **Corollary:** If  $M/t(M)$  is a **projective** left  $D$ -module, namely, there exist  $X \in D^{p \times t}$  and  $Y \in D^{t \times u}$  such that  $SX + YT = I_t$ , then:

$$R\eta = 0 \quad \Leftrightarrow \quad \eta = (X \quad Q) \begin{pmatrix} \tau - Bv \\ \xi \end{pmatrix}, \quad \forall \xi \in \mathcal{F}^m.$$

## Example II (continued)

- We have  $M = D^{1 \times 3} / (D^{1 \times 2} S) \cong D^{1 \times 3} Q$ , where:

$$S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

- $S$  admits a **right-inverse**  $X$ , i.e.,  $SX = I_2$ , defined by:

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

- $B = -I_2$ ,  $\tau = (f(x_1, x_2) \quad f(x_1, x_2))^T$ ,  $\tau = (0 \quad c)^T$ , we get:

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(x_1, x_2) \\ f(x_1, x_2) + c \\ \xi \end{pmatrix} = \begin{pmatrix} \xi \\ 2f(x_1, x_2) + c - \xi \\ -f(x_1, x_2) + \xi \end{pmatrix}.$$