

Serre's Reduction of Linear Functional Systems: Theory, Implementation and Applications

Thomas Cluzeau

University of Limoges ; CNRS ; XLIM ; UMR 6172 & ENSIL



Joint work with **Alban Quadrat** (INRIA Sophia Antipolis)

ACA 2009 - AADIOS SESSION

É.T.S., Montréal, Québec, Canada

06/25/2009

Motivations

- Objective of this work:

Use computer algebra (algebraic manipulations) to **simplify systems** coming from mathematical physics, applied mathematics, engineering sciences or control theory

- Interest:

- Simplify the equations of the system
⇒ simplify the study of its structural properties
- Pre-conditioner to numerical analysis methods

Outline of the talk

- 1 **Review** of my 2008 ACA/AADIOS talk
- 2 **Serre's Reduction**
Boudellioua (Sultan Qaboos University, Oman), Quadrat
- 3 **Implementation**: the `SERRE` package (in progress)
- 4 **Going further**: the holonomic case (in progress)

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Review of my 2008 ACA/AADIOS talk

Decomposition problem

When can we decouple the equations of a linear functional system?

- Let D be a noncommutative polynomial ring of operators.
- Let $R \in D^{q \times p}$ be a matrix defining a linear functional system.

$$\exists W \in \text{GL}_p(D), V \in \text{GL}_q(D) : V R W = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} ?$$

Tools used

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D -module \mathcal{F} .
- Consider $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$.
- As in **number theory** or **algebraic geometry**, to $\ker_{\mathcal{F}}(R.)$ we associate the finitely presented left D -module:

$$M = D^{1 \times p} / (D^{1 \times q} R).$$

- **Method:**
 - 1 Compute $\text{end}_D(M)$, *i.e.*, solve the equation $RP = QR$;
 - 2 Compute idempotents in $\text{end}_D(M)$: $P^2 = P$;
 - 3 Check freeness of certain D -modules and compute bases (QUILENSUSLIN, STAFFORD).

Example: tank model (Petit-Rouchon, IEEE TAC 02)

- Consider $D = \mathbb{Q}(\alpha) [\partial, \delta]$ with $\partial = \frac{d}{dt}$, the matrix of the system

$$R = \begin{pmatrix} \partial & -\partial \delta^2 & \alpha \partial^2 \delta \\ \partial \delta^2 & -\partial & \alpha \partial^2 \delta \end{pmatrix} \in D^{2 \times 3}.$$

- The matrices $P = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ et $Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfy:

$$R P = Q R, \quad P^2 = P, \quad Q^2 = Q.$$

- Using **linear algebra**, we get:

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(D), \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(D),$$

$$\Rightarrow \bar{R} = V R U^{-1} = \begin{pmatrix} \partial(1-\delta)(1+\delta) & 0 & 0 \\ 0 & \partial(\delta^2+1) & 2\alpha\partial^2\delta \end{pmatrix}.$$

The OREMORPHISMS package

- Algorithms are implemented in a Maple package called **OREMORPHISMS** based on the library **OREMODULES** developed by Q. et Robertz:

<http://wwwb.math.rwth-aachen.de/OreModules>

- List of functions:
 - Morphisms, MorphismsConstCoeff, MorphismsRat,
 - Idempotents, IdempotentsConstCoeff, IdempotentsRat
 - IdempotentsMat, IdempotentsMatConstCoeffs, IdempotentsMatRat
 - KerMorphism(Rat), ImMorphism(Rat), CokerMorphism(Rat), CoimMorphism(Rat),
 - TestSurj(Rat), TestInj(Rat), TestIso(Rat).
- It is freely available with a library of examples at:

<http://www.ensil.unilim.fr/~cluzeau/OreMorphisms>

II

Serre's Reduction

(Boudellioua, Quadrat)

Serre's reduction problem

When can we reduce the number of equations and unknowns of a linear functional system?

- Let D be a noncommutative polynomial ring of operators.
- Let $R \in D^{q \times p}$ be a matrix defining a linear functional linear.

$$\exists W \in \text{GL}_p(D), V \in \text{GL}_q(D) : V R W = \begin{pmatrix} I_r & 0 \\ 0 & S \end{pmatrix} ?$$

- When $r = q - 1$, the system is equivalent to a **sole equation!**

String with an interior mass (Fließ et al, COCV 98)

$$\left\{ \begin{array}{l} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + a\phi_1(t) - a\psi_1(t) - b\phi_2(t) + b\psi_2(t) = 0, \\ \phi_1(t - 2h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2h_2) - v(t - h_2) = 0. \end{array} \right. \quad (\star)$$

$$\partial f(t) = \dot{f}(t), \quad \sigma_1 f(t) = f(t - h_1), \quad \sigma_2 f(t) = f(t - h_2).$$

$$V \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial + a & \partial - a & -b & b & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} W$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 + a + b & a\sigma_1 & b\sigma_2 \end{pmatrix}.$$

$$(\star) \Leftrightarrow \dot{z}_1(t) + (a + b)z_1(t) + az_2(t - h_1) + bz_3(t - h_2) = 0.$$

String with an interior mass (Fliess et al, COCV 98)

The **unimodular matrices** V and W are defined by:

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \sigma_1^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ (a - b - \partial)\sigma_1^2 + \partial + a & -1 & \partial - a + b & -2b \end{pmatrix} \in \text{GL}_4(D),$$

$$W = \begin{pmatrix} 1 & 0 & 0 & -1 & -\sigma_1 & 0 \\ 0 & -1 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \sigma_2 \\ 0 & -1 & -1 & -1 & 0 & -\sigma_2 \\ 0 & 0 & 0 & -\sigma_1 & -\sigma_1^2 + 1 & 0 \\ 0 & -\sigma_2 & -\sigma_2 & -\sigma_2 & 0 & -\sigma_2^2 + 1 \end{pmatrix} \in \text{GL}_6(D).$$

- The computations were obtained using the packages:

SERRE and **QUILLEN****SUSLIN**.

Generalization of a Serre's result

- **Theorem:** Let $R \in D^{q \times p}$ be a full row rank matrix, $\Lambda \in D^q$, $P = (R \quad -\Lambda)$ and the two left D -modules $M = D^{1 \times p} / (D^{1 \times q} R)$ and $E = D^{1 \times (p+1)} / (D^{1 \times q} P)$ defining an **extension of D by M** :

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0.$$

We have the equivalent assertions:

- 1 E is stably free of rank $p + 1 - q$: $E \oplus D^{1 \times q} \cong D^{1 \times (p+1)}$.
- 2 $P = (R \quad -\Lambda)$ admits a right-inverse over D .
- 3 $\text{ext}_D^1(E, D) = D^q / (P D^{p+1}) = 0$.
- 4 $\text{ext}_D^1(M, D) = D^q / (R D^p)$ is the cyclic right D -module generated by $\rho(\Lambda)$, where ρ denotes the projection:

$$\rho : D^q \longrightarrow \text{ext}_D^1(M, D) = D^q / (R D^p).$$

The previous equivalences **only depend on the residue class $\rho(\Lambda)$** .

Theorem

- Theorem:** Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that there exists $U \in \text{GL}_{p+1}(D)$ satisfying:

$$(R \quad -\Lambda) U = (I_q \quad 0).$$

Let us denote by

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \in \text{GL}_{p+1}(D),$$

where:

$$S_1 \in D^{p \times q}, S_2 \in D^{1 \times q}, Q_1 \in D^{p \times (p+1-q)}, Q_2 \in D^{1 \times (p+1-q)}.$$

Then, we have:

$$M = D^{1 \times p} / (D^{1 \times q} R) \cong L = D^{1 \times (p+1-q)} / (D Q_2)$$

The converse result also holds. These results only depend on:

$$\rho(\Lambda) \in \text{ext}_D^1(M, D) \triangleq D^q / (R D^p), \quad \rho : D^q \longrightarrow D^q / (R D^p).$$

Ring conditions

• **Proposition:** Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that $P = (R \quad -\Lambda) \in D^{q \times (p+1)}$ admits a **right-inverse over D** . Moreover, if D is either a

- 1 principal left ideal domain,
- 2 commutative polynomial ring with coefficients in a field,
- 3 Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - q \geq 1$,

then there exists $U \in \text{GL}_{p+1}(D)$ satisfying that $P U = (I_q \quad 0)$.

• The matrix U can be obtained by means of:

- 1 a Jacobson form (JACOBSON),
- 2 the Quillen-Suslin theorem (QUILLEN-SUSLIN),
- 3 Stafford's theorem (STAFFORD).

Example: Wind tunnel model

- The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a x_1(t) - k a x_2(t - h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2 x_2(t) + 2 \zeta \omega x_3(t) - \omega^2 u(t) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(a, k, \omega, \zeta)[\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial + a & -k a \delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2 \zeta \omega & -\omega^2 \end{pmatrix} \in D^{3 \times 4},$$

and the finitely presented D -module $M = D^{1 \times 4} / (D^{1 \times 3} R)$.

- The D -module $\text{ext}_D^1(M, D) = D^3 / (R D^4)$ is a $\mathbb{Q}(a, k, \omega, \zeta)$ -vector space of dimension 1 and $\rho((1 \ 0 \ 0)^T)$ is a basis.

Example: Wind tunnel model

- Let us consider $\Lambda = (1 \ 0 \ 0)^T$ and $P = (R \ -\Lambda)$.
- The matrix P admits the following **right-inverse** S :

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{\partial+2\zeta\omega}{\omega^2} & -\frac{1}{\omega^2} \\ -1 & 0 & 0 \end{pmatrix} \in D^{5 \times 3}.$$

- According to **Quillen-Suslin theorem**, $E = D^{1 \times 5} / (D^{1 \times 3} P)$ is **free** D -**module of rank 2**.

Example: Wind tunnel model

- Computing a basis of E , we obtain that $U \in \text{GL}_5(D)$,

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & 0 & \omega^2 \partial \\ 0 & -\frac{\partial + 2\zeta\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \partial^2 + 2\zeta\omega\partial + \omega^2 \\ -1 & 0 & 0 & -(\partial + a) & -\omega^2 k a \delta \end{pmatrix},$$

satisfies that $PU = (I_3 \ 0)$ (OREMODULES, QUILLEN/SUSLIN).

- The wind tunnel model is equivalent to the sole equation:

$$\begin{aligned} (\partial + a)\zeta_1 + \omega^2 k a \delta \zeta_2 &= 0 \\ \Leftrightarrow \dot{\zeta}_1(t) + a\zeta_1(t) + \omega^2 k a \zeta_2(t - h) &= 0. \end{aligned}$$

Equivalence

- Theorem:** If $\Lambda \in D^q$ admits a **left-inverse** $\Gamma \in D^{1 \times q}$, i.e., $\Gamma \Lambda = 1$, then Q_1 admits the left-inverse $T_1 + T_2 \Gamma R \in D^{(p+1-q) \times p}$ and the left D -module $\ker_D(\cdot Q_1)$ is stably free of rank $q - 1$:

$$\ker_D(\cdot Q_1) \oplus D^{1 \times (p+1-q)} \cong D^{1 \times p}.$$

If the left D -module $\ker_D(\cdot Q_1)$ is free, then $\exists Q_3 \in D^{p \times (q-1)}$ s.t.:

$$V = \begin{pmatrix} Q_3 & Q_1 \end{pmatrix} \in GL_p(D).$$

Then, we have $W = \begin{pmatrix} R & Q_3 & \Lambda \end{pmatrix} \in GL_q(D)$,

$$W^{-1} = \begin{pmatrix} Y_3 & S_1 \\ -S_2 + Q_2 & Y_1 & S_1 \end{pmatrix},$$

with $V^{-1} = (Y_3^T \quad Y_1^T)^T$, $Y_3 \in D^{(q-1) \times p}$, $Y_1 \in D^{(p-q+1) \times p}$ and:

$$W^{-1} R V = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Example: Wind tunnel model

- The vector $\Lambda = (1 \ 0 \ 0)^T$ admits the **left-inverse** $\Gamma = \Lambda^T$.
- We compute $Q_3 \in D^{2 \times 2}$ such that $V = (Q_3^T \ Q_1^T) \in GL_4(D)$:

$$V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & \omega^2 \partial \\ -\frac{1}{\omega^2} & -\frac{\partial + 2\zeta\omega}{\omega^2} & 0 & \partial^2 + 2\zeta\omega\partial + \omega^2 \end{pmatrix}.$$

- We have $W = (R Q_3 \ \Lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in GL_3(D)$ and:

$$W^{-1} R V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(\partial + a) & -\omega^2 k a \delta \end{pmatrix}.$$

III

Implementation

The SERRE package (in progress)

Algorithmic issue

- 1 Consider an **ansatz** $\Lambda \in D^q$ of a given order.
- 2 Compute a **Gröbner basis** of $D^q/(R D^p)$.
- 3 Compute the **normal form** $\bar{\Lambda} \in D^q$ of $\rho(\Lambda)$.
- 4 Compute the **obstructions to freeness** of the left D -module $\bar{E} = D^{1 \times (p+1)} / (D^{1 \times q} (R \quad - \bar{\Lambda}))$ (π -polynomials).
- 5 Solve the systems in the arbitrary coefficients obtained by making the obstructions vanish.
- 6 If a solution Λ_* exists, then compute $U \in \text{GL}_{p+1}(D)$ satisfying that $(R \quad - \Lambda) U = (I_q \quad 0)$ and return $Q_2 \in D^{1 \times (p+1-q)}$.

The SERRE package

- Algorithms are implemented in a Maple package called **SERRE** based on the library **OREMODULES** developed by Q. et Robertz:

<http://wwwb.math.rwth-aachen.de/OreModules>

- List of functions:
 - FreeExtension (distinct variants)
 - SerreReduction (distinct variants)
 - SerreEquivalence (distinct variants)
- It will be soon freely available with a library of examples at:

<http://www.ensil.unilim.fr/~cluzeau/Serre>

Maple demo

IV

Going further

The holonomic case (in progress)

Example: Stress tensor (elasticity)

- Let $D = \mathbb{Q}[\partial_x, \partial_y]$ and $M = D^{1 \times 3} / (D^{1 \times 2} R)$, where:

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix} \in D^{2 \times 3}.$$

- The D -module $\text{ext}_D^1(M, D) = D^2 / (R D^3)$ is a \mathbb{Q} -vector space of dimension 3 with basis $\{\rho((1 \ 0)^T), \rho((0 \ 1)^T), \rho((0 \ \partial_x)^T)\}$.

- Let us consider $\Lambda = (a \ b + c \partial_x)^T$, $P = (R \ -\Lambda)$.

- If we denote by $A = D[a, b, c]$ and $N = A^2 / (P A^4)$, then we get:

$$\text{ext}_A^1(N, A) = 0, \quad \text{ext}_A^2(N, A) = A / (\partial_x, \partial_y).$$

- Hence, $E = A^{1 \times 4} / (A^{1 \times 2} P)$ is never a projective A -module and

$$\begin{cases} \partial_x \sigma^{11} + \partial_y \sigma^{12} = 0, \\ \partial_x \sigma^{12} + \partial_y \sigma^{22} = 0, \end{cases}$$

cannot be defined by a sole equation!

Example: Stress tensor (elasticity)

- Let $D' = A_2(\mathbb{Q})$ be the Weyl algebra
- Remark that the $N' = D'^2 / (R D'^4)$ is a holonomic D' -module
- It is thus cyclic and a cyclic vector $C \in \mathbb{Q}[x, y]^2$ can be computed using A. Leykin algorithm
- We find that $C = (1 \quad x)^T$ is a cyclic vector which implies that $P' = (R \quad -C)$ admits a right-inverse S :

$$S = \begin{pmatrix} -x & 1 \\ -x^2 & x \\ -x^3 & x^2 \\ -2 - x \partial_x - x^2 \partial_y & \partial_x + x \partial_y \end{pmatrix}.$$

- The D' -module $E' = D'^{1 \times 4} / (D'^{1 \times 2} P')$ is thus free by Stafford theorem.

Example: Stress tensor (elasticity)

- Using SERRE or OREMORPHISMS, we find unimodular matrices U and V such that

$$V P' U = \begin{pmatrix} \partial_x x & -\partial_x + \partial_y x & -\partial_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- From these, we can easily deduce that

$$U' = \begin{pmatrix} -x & 1 & -\partial_y & -\partial_x + \partial_y x \\ -x^2 & x & -\partial_y x & 1 - \partial_x x + x^2 \partial_y \\ -x^3 & x^2 & 1 - x^2 \partial_y & -\partial_x x^2 + x^3 \partial_y \\ -2 - \partial_x x - x^2 \partial_y & \partial_x + \partial_y x & -\partial_x \partial_y - \partial_y^2 x & -\partial_x^2 + 2 \partial_y + \partial_y^2 x^2 \end{pmatrix}$$

satisfies $P' U' = (I_2 \ 0)$.

Example: Stress tensor (elasticity)

- As in Serre's reduction, we thus conclude that over $D' = A_2(\mathbb{Q})$, the stress tensor system defined by

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix}$$

is equivalent to

$$\bar{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial_x \partial_y + x \partial_y^2 & \partial_x^2 - 2 \partial_y - x^2 \partial_y^2 \end{pmatrix}.$$

- The stress tensor system can thus be defined by a sole equation!

General Result

- Theorem:** Let D be the **Weyl algebra** $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0.

Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $N = \text{ext}_D^1(M, D) = D^q / (R D^p)$.

If N is a **holonomic module** and $p - q \geq 1$, then there exists $Q \in D^{1 \times (p+1-q)}$ such that

$$M \cong D^{1 \times (p+1-q)} / (D Q).$$

If furthermore $q \geq 3$, then there exist two unimodular matrices U and V such that

$$V R U = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q \end{pmatrix}.$$